EVERY LARGE POINT SET CONTAINS MANY COLLINEAR POINTS OR AN EMPTY PENTAGON

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Abstract. We prove the following generalised empty pentagon theorem: for every integer $\ell \geq 2$, every sufficiently large set of points in the plane contains ℓ collinear points or an empty pentagon. As an application, we settle the next open case of the "big line or big clique" conjecture of Kára, Pór, and Wood [*Discrete Comput. Geom.* 34(3):497–506, 2005].

1. Introduction

While the majority of theorems and problems about sets of points in the plane assume that the points are in general position, there are many interesting theorems and problems about sets of points with collinearities. The Sylvester-Gallai Theorem and the orchard problem are some examples; see [6]. The main contribution of this paper is to extend the 'empty pentagon' theorem about point sets in general position to point sets with collinearities.

1.1. Definitions.

We begin with some standard definitions. Let P be a finite set of points in the plane. We say that P is in *general position* if no three points in P are collinear. Let $\text{conv}(P)$ denote the convex hull of P. We say that P is in *convex position* if every point of P is on the boundary of $\text{conv}(P)$. A point $v \in P$ is a *corner* of P if $\text{conv}(P - v) \neq \text{conv}(P)$. We say that P is in *strictly convex position* if each point of P is a corner of P. A *strictly convex k-gon* is the convex hull of k points in strictly convex position. If $X \subseteq P$ is a set of k points in strictly convex position and $\text{conv}(X) \cap P = X$, then $\text{conv}(X)$ is called a *k-hole* (or an *empty strictly convex k-gon*) of P. A 4-hole is called an *empty quadrilateral*, a 5-hole is called an *empty pentagon*, a 6-hole is called an *empty hexagon*, etc.

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For distinct points a, b, c in the plane, let $\Delta[a, b, c]$ be the closed triangle determined by a, b, c, and let $\Delta(a, b, c)$ be the open triangle determined by a, b, c. For integers $n \leq m$, let $[n, m] := \{n, n+1, \ldots, m\}$ and $[n] := [1, n]$.

1.2. **Erdős-Szekeres Theorem.** The Erdős-Szekeres Theorem [16] states that for every integer k there is a minimum integer $\mathrm{ES}(k)$ such that every set of at least $\mathrm{ES}(k)$ points in general position in the plane contains k points in convex position (which are therefore in strictly convex position). See [2, 6, 28, 34] for surveys of this theorem. The following generalisation of the Erdős-Szekeres Theorem for point sets with collinearities is easily proved by applying a suitable perturbation of the points (see Section 3):

Theorem 1. *For every integer k every set of at least $\mathrm{ES}(k)$ points in the plane contains k points in convex position.*

The Erdős-Szekeres Theorem generalises for points in strictly convex position as follows:

Theorem 2. *For all integers $\ell \geq 2$ and $k \geq 3$ there is a minimum integer $\mathrm{ES}(k, \ell)$ such that every set of at least $\mathrm{ES}(k, \ell)$ points in the plane contains:*

- *ℓ collinear points, or*
- *k points in strictly convex position.*

Of course, the conclusion in Theorem 2 that there is a large set of collinear points is unavoidable, since a large set of collinear points only contains two points in strictly convex position. Theorem 2 is known (it is Exercise 3.1.3 in [26]), but as far as we are aware, no proof of it has appeared in the literature and no explicit bounds on $\mathrm{ES}(k, \ell)$ have been formulated. To illustrate various proof techniques in geometric Ramsey theory, we present three proofs of Theorem 2. The first proof finds a large subset of points in general position and then applies the standard Erdős-Szekeres theorem (see Lemma 9). The second proof first applies Theorem 1 to obtain a large subset in convex position, in which a large subset in strictly convex position is found (see Lemma 8). The third proof is based on Ramsey's Theorem for hypergraphs (see Section 4).

1.3. **Empty Polygons.** Attempting to strengthen the Erdős-Szekeres Theorem, Erdős [12] asked whether for each fixed k every sufficiently large set of points in general position contains a k-hole. Harborth [20] answered this question in the affirmative for $k \leq 5$, by showing that every set of at least ten points in general position contains a 5-hole. On the other hand, Horton [21] answered Erdős' question in the negative for $k \geq 7$, by constructing arbitrarily large sets of points in general position that contain no 7-hole. The remaining case of $k = 6$ was recently solved independently by Gerken [18] and Nicolás [29], who proved that every sufficiently large set of points in general position contains a 6-hole. See [3, 4, 7, 8, 10, 22, 24, 25, 30, 31, 32, 35, 36, 37, 38, 39, 40, 41] for more on empty convex polygons.

The above results do not immediately generalise to sets with a bounded number of collinear points by simply choosing a large subset in general position as in the first proof of the Erdős-Szekeres Theorem (since the deleted points might 'fill a hole'). Nevertheless, we prove the following 'generalised empty pentagon' theorem, which is the main contribution of this paper (proved in Section 5).

Theorem 3. *For every integer $\ell \geq 2$, every finite set of at least* $\mathrm{ES}(\frac{(2\ell-1)^{\ell}-1}{2\ell-2})$ *points in the plane contains*

- *ℓ collinear points, or*
- *a 5-hole.*

Note that Eppstein [11] characterised the point sets with no 5-hole in terms of the acyclicity of an associated *quadrilateral graph*. However, it is not clear how this result can be used to prove Theorem 3. Earlier, Rabinowitz [33] defined a set of points with no 5-hole to have the *pentagon property*.

1.4. **Big Line or Big Clique Conjecture.** Theorem 3 has an important ramification for the following "big line or big clique" conjecture by Kára et al. [23]. Let P be a finite set of points in the plane. Two distinct points $v, w \in P$ are *visible* with respect to P if $P \cap \overline{vw} = \{v, w\}$, where $\overline{vw}$ denotes the closed line segment between v and w. The *visibility graph* of P has vertex set P, where two distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P.

Conjecture 4 ([23])**.** For all integers k and ℓ there is an integer n such that every finite set of at least n points in the plane contains:

- ℓ collinear points, or
- k pairwise visible points (that is, the visibility graph contains a k-clique).

Conjecture 4 has recently attracted considerable attention [1, 23, 27]. It is trivially true for $\ell \leq 3$ and all k. Kára et al. [23] proved it for $k \leq 4$ and all ℓ. Addario-Berry et al. [1] proved it in the case that $k = 5$ and $\ell = 4$. Here we prove the next case of Conjecture 4 for infinitely many values of ℓ.

Theorem 5. *Conjecture 4 is true for $k = 5$ and all ℓ.*

Proof. By Theorem 3, every sufficiently large set of points contains ℓ collinear points (in which case we are done) or a 5-hole H. Let H' be a 5-hole contained in H with minimum area. Then the corners of H' are five pairwise visible points (otherwise there is a 5-hole contained in H with less area, as illustrated in Figure 1). □

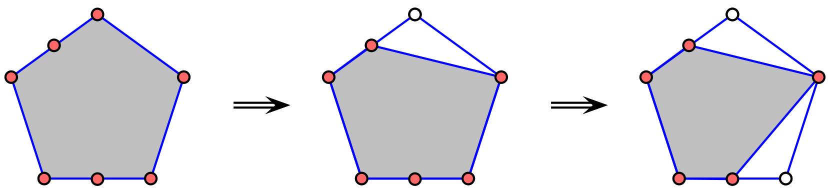


FIGURE 1. Every 5-hole contains five pairwise visible points.

2. Points in Convex Position

In this section we consider the following problem (which will be relevant to the proofs of Lemma 8 and Theorem 3 to come): given a set P of points in convex position, choose a large subset of P in strictly convex position. For integers $k \geq 1$ and $\ell \geq 1$, let $q(k, \ell)$ be the minimum integer such that every set of at least $q(k, \ell)$ points in the plane in convex position contains ℓ collinear points or k points in strictly convex position. Trivially, if $k \leq 2$ or $\ell \leq 2$ then $q(k, \ell) = \min\{k, \ell\}$. Since every set of points with no three points in strictly convex position is collinear, $q(3, \ell) = \ell$ for all $\ell \geq 1$. Since every set of points in convex position with no three collinear points is in strictly convex position, $q(k, 3) = k$ for all $k \geq 1$.

Lemma 6. *For all $\ell \geq 3$ and $k \geq 3$,*

$$q(k,\ell) = \begin{cases} \frac{1}{2}(\ell-1)(k-1)+1 & \text{, if } k \text{ is odd} \\ \frac{1}{2}(\ell-1)(k-2)+2 & \text{, if } k \text{ is even.} \end{cases} \tag{1}$$

Proof. Let $f(k, \ell)$ denote the right-hand-side of (1).

We first prove the lower bound on $q(k, \ell)$ for odd $k \geq 5$, the case $k = 3$ having been proved above. As illustrated in Figure 2(a), let P be a set consisting of $\ell - 1$ points on every second side of a convex $(k-1)$-gon. Thus P has $\frac{1}{2}(k-1)(\ell-1)$ points with no ℓ collinear points and no k in strictly convex position (since at most two points from each side are in strictly convex position). Hence $q(k, \ell) > \frac{1}{2}(\ell-1)(k-1)$, which is an integer. Thus $q(k, \ell) \geq \frac{1}{2}(\ell-1)(k-1) + 1 = f(k, \ell)$.

Now we prove the lower bound on $q(k, \ell)$ for even $k \geq 4$. For $k = 4$, a set of $\ell - 1$ collinear points plus one point off the line has no four points in strictly convex position; hence $q(4, \ell) \geq \ell + 1$. Now assume $k \geq 6$. As illustrated in Figure 2(b), let P be a set consisting of $\ell - 1$ points on every second side of a convex $(k-2)$-gon, plus one more point not collinear with any two other points. Thus P has $\frac{1}{2}(\ell-1)(k-2)+1$ points with no ℓ collinear points and no k in strictly convex position (since at most two points from each 'long' side are in strictly convex position plus the one extra point). Hence $q(k, \ell) > \frac{1}{2}(\ell-1)(k-2)+1$, which is an integer. Thus $q(k, \ell) \geq \frac{1}{2}(\ell-1)(k-2)+2 = f(k, \ell)$.

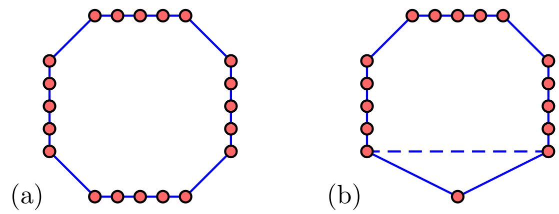


FIGURE 2. Extremal examples for $\ell = 6$ and (a) $k = 9$ and (b) $k = 8$.

We now prove the upper bound $q(k, \ell) \leq f(k, \ell)$ for $\ell \geq 3$ and $k \geq 1$. We proceed by induction on $k \geq 1$. The cases $k \in \{1, 2, 3\}$ or $\ell = 3$ follow from the discussion at the start of the section. Now assume that $k \geq 4$ and $\ell \geq 4$. Let P be a set of at least $f(k, \ell)$ points in convex position with no ℓ collinear points and no k points in strictly convex position. Let $v_1, \dots, v_m$ be the corners of P in clockwise order, where $v_{m+1} := v_1$ and $v_0 := v_m$. Let $P_i := P \cap \overline{v_i v_{i+1}}$ for each $i \in [m]$. Thus $|P_i| \in [2, \ell - 1]$ for each $i \in [m]$.

Suppose that $|P_i| \geq 4$ for some $i \in [m]$. Thus $|P - P_i| \geq f(k, \ell) - (\ell - 1) = f(k-2, \ell)$. By induction, $P - P_i$ has a subset S of $k - 2$ points in strictly convex position (since P and thus $P - P_i$ has no ℓ collinear points). Thus S plus two internal points on P_i form a subset of k points in strictly convex position, which is a contradiction. Now assume that $|P_i| \leq 3$ for all $i \in [m]$.

Suppose that $|P_i| = 2$ for some $i \in [m]$. Say t, u, v, w, x, y are the consecutive points on the boundary of $\text{conv}(P)$, where $P_i = \{v, w\}$. Since $\{u, v, w, x\}$ are in strictly convex position, assume that $k \geq 5$. Thus $|P - \{t, u, v, w, x, y\}| \geq f(k, \ell) - 6 \geq f(k - 4, \ell)$. By induction, $P - \{t, u, v, w, x, y\}$ has a subset S of $k - 4$ points in strictly convex position (since P and thus $P - \{t, u, v, w, x, y\}$ has no ℓ collinear points). Since $|P_{i-1}| \leq 3$ and $|P_{i+1}| \leq 3$, it follows that $S \cup \{u, v, w, x\}$ is a set of k points in strictly convex position, which is a contradiction.

Now assume that $|P_i| = 3$ for all $i \in [m]$. Thus $|P| = 2m$. As illustrated in Figure 3, let S consist of each of the m non-corner points of P, plus every second corner point (where in the case of odd m, we omit two consecutive corners from S). Thus S is a set of at least $\frac{1}{2}(3m - 1)$ points in strictly convex position. We have $|P| \geq f(k, \ell)$, which, since $\ell \geq 4$, is at least $\frac{3}{2}k - 1$. Since no k points are in strictly convex position, $|S| \leq k - 1$ and

$$8(k - 1) \geq 8|S| \geq 12m - 4 = 6|P| - 4 \geq 6(\tfrac{3}{2}k - 1) - 4 = 9k - 10\ ,$$

implying $k \leq 2$, which is a contradiction. □

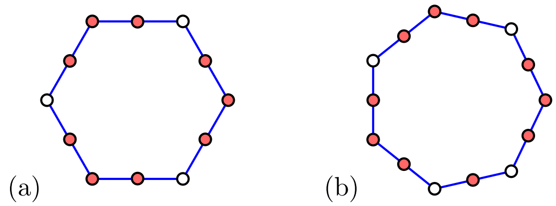


FIGURE 3. The case $|P_i| = 3$ for all $i \in [m]$ where (a) $m = 6$ and (b) $m = 7$. Dark points are in S.

3. Generalisations of the Erdős-Szekeres Theorem

In this section we prove Theorems 1 and 2, which generalise the Erdős-Szekeres Theorem for points in general position. If P' is a perturbation of a finite set P of points in the plane, let $v' \in P'$ denote the image of a point $v \in P$, and let $S' := \{v' : v \in S\}$ for each $S \subseteq P$. If $\text{dist}(v, v') \leq \epsilon$ for each $v \in P$ then P' is an ϵ-*perturbation*. Observe that Theorem 1 follows from the next lemma and the Erdős-Szekeres Theorem for points in general position (applied to P').

Lemma 7. *For every finite set P of points in the plane, there is a general position perturbation P' of P, such that if S' is a subset of P' in convex position, then S is in (non-strict) convex position.*

Proof. For each non-collinear ordered triple (u, v, w) of points in P there exists $\mu > 0$ such that every ϵ-perturbation of P will not change the orientation[1] of (u, v, w) whenever $0 < \epsilon < \mu$. Since there are finitely many such triples there is a minimal such μ. Let P' be a μ-perturbation of P in general position.

Let S' be a subset of P' in convex position. Consider S' in anticlockwise order. Thus each ordered triple of consecutive points in S' has positive orientation. Now consider S in the corresponding order as S'. Since the perturbation preserved negatively oriented triples, each ordered triple of consecutive points in S has non-negative orientation. That is, S is in (non-strict) convex position, as desired. □

We now prove two lemmas, each of which shows how to force k points in strictly convex position, thus proving Theorem 2.

Lemma 8. *For all $k \geq 3$ and $\ell \geq 3$, if k is odd then*

$$\text{ES}(k, \ell) \leq \text{ES}(\tfrac{1}{2}(k-1)(\ell-1)+1) \ ,$$

and if k is even then

$$\text{ES}(k, \ell) \leq \text{ES}(\tfrac{1}{2}(k-2)(\ell-1)+2) \ .$$

[1]The *orientation* of an ordered triple of points (u, v, w) is 0 if u, v, w are collinear; otherwise it is *positive* or *negative* depending on whether we turn left or right when going from u to w via v.

Proof. For odd k, let P be a set of at least $\mathrm{ES}(\frac{1}{2}(k-1)(\ell-1)+1)$ points with no ℓ points collinear. Thus P contains $\frac{1}{2}(k-1)(\ell-1)+1$ points in convex position by Theorem 1. Thus P contains k points in strictly convex position by Lemma 6. The proof for even k is analogous. $\square$

Lemma 9. *For all $k \geq 3$ and $\ell \geq 3$,*

$$\mathrm{ES}(k,\ell) \leq (\ell-3)\binom{\mathrm{ES}(k)-1}{2} + \mathrm{ES}(k) \ .$$

Proof. It is well known [5, 13, 14, 15, 17] and easily proved[2] that every set of at least $(\ell-3)\binom{k-1}{2}+k$ points in the plane contains ℓ collinear points or k points in general position. Thus every set P of at least $(\ell-3)\binom{\mathrm{ES}(k)-1}{2}+\mathrm{ES}(k)$ points in the plane contains ℓ collinear points or $\mathrm{ES}(k)$ points in general position. In the latter case, P contains k points in strictly convex position. $\square$

The best known upper bound on $\mathrm{ES}(k)$, due to Tóth and Valtr [34], is

$$\mathrm{ES}(k) \leq \binom{2k-5}{k-2} + 1 \in O\left(\frac{2^{2k}}{\sqrt{k}}\right) \ .$$

Thus Lemma 8 implies that if k is odd then

$$\mathrm{ES}(k,\ell) \in O\left(\frac{2^{(k-1)(\ell-1)}}{\sqrt{k\ell}}\right) \ , \tag{2}$$

and if k is even then

$$\mathrm{ES}(k,\ell) \in O\left(\frac{2^{(k-2)(\ell-1)}}{\sqrt{k\ell}}\right) \ . \tag{3}$$

Similarly, Lemma 9 implies that

$$\mathrm{ES}(k,\ell) \in O\left(\frac{\ell \cdot 2^{4k}}{k}\right) \ . \tag{4}$$

Note that the bound in (4) is stronger than the bounds in (2) and (3) for $\ell \geq 6$ and sufficiently large k. For $\ell \leq 5$ the bounds in (2) and (3) are stronger.

[2] Let P be a set of points in the plane with at most $\ell-1$ points collinear and at most $k-1$ points in general position. Let $S \subseteq P$ be a maximal set of points in general position. Thus every point in $P-S$ is collinear with two points in S. The set S determines $\binom{|S|}{2}$ lines, each with at most $\ell-3$ points in $P-S$. Thus $|P| \leq \binom{|S|}{2}(\ell-3)+|S| \leq \binom{k-1}{2}(\ell-3)+k-1$. That is, if $|P| \geq \binom{k-1}{2}(\ell-3)+k$ then P contains ℓ collinear points or k points in general position.

4. Empty Quadrilaterals

Point sets with no 4-hole are characterised as follows.

Theorem 10 ([9, 11])**.** *The following are equivalent for a finite set of points P:*

(a) P contains no 4-hole,

(b) the visibility graph of P is crossing-free,

(c) P has a unique triangulation,

(d) at least one of the following conditions hold:

- *all the points in P, except for at most one, are collinear; see Figures 4(a) and (b),*
- *there are two points $v, w \in P$ on opposite sides of some line L, such that $P - \{v, w\} \subseteq L$ and the intersection of $\operatorname{conv}(P - \{v, w\})$ and $\overline{vw}$ either is a point in $P - \{v, w\}$ or is empty; see Figures 4(c) and (d),*
- *P is a set of six points with the same order type as the set illustrated in Figure 4(e).*

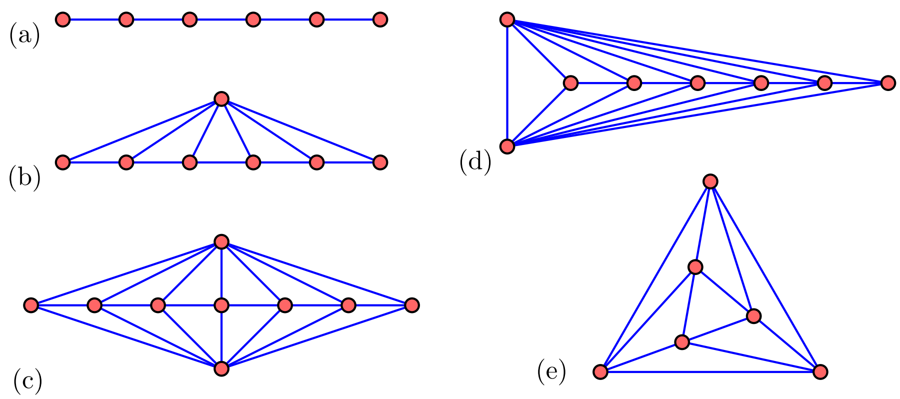


Figure 4. The point sets with no 4-hole.

Corollary 11. *For every integer $\ell \geq 2$, every set of at least $\max\{7, \ell+2\}$ points in the plane contains ℓ collinear points or a 4-hole.*

Corollary 11 enables a third proof of Theorem 2: By the 2-colour Ramsey Theorem for hypergraphs (see [19]), for every integer t there is an integer n such that for every 2-colouring of the edges of any complete 4-uniform hypergraph on at least n vertices, there is a set X of t vertices such that the edges induced by X are monochromatic. Apply this result with $t := \max\{7, k, \ell+2\}$. We claim that $\text{ES}(k, \ell) \leq n$. Let P be a set of at least n points in the plane with no ℓ collinear points. Let G be the complete 4-uniform hypergraph with vertex set P. For each 4-tuple T of vertices, colour the edge T *blue* if T forms a strictly convex quadrilateral, and *red* otherwise. Thus there is a set X of t points such that the edges induced by X are monochromatic. If all the edges induced by X are red, then no 4-tuple of points in X forms a strictly convex quadrilateral, which contradicts Corollary 11 since $|X| \geq \max\{7, \ell + 2\}$. Otherwise, all the edges induced by X are blue. That is, every 4-tuple of vertices in X forms a strictly convex quadrilateral. This implies that X forms a strictly convex t-gon (for otherwise some non-corner in X would be in a triangle of corners of X, implying there is a 4-tuple of points in X that do not form a strictly convex quadrilateral). Since $t \geq k$ we are done.

5. Empty Pentagons

In this section, we prove our main result, Theorem 3. The proof loosely follows the proof of the 6-hole theorem for points in general position by Valtr [38], which in turn is a simplification of the proof by Gerken [18].

Proof of Theorem 3. Fix $\ell \geq 3$ and let $k := \frac{(2\ell-1)^\ell - 1}{2\ell - 2}$, which is an integer.

Let P be a set of at least $\mathrm{ES}(k)$ points in the plane. By Theorem 1, P contains k points in convex position. Suppose for the sake of contradiction that P contains no ℓ collinear points and no 5-hole.

A set X of at least k points in P in convex position is said to be *k-minimal* if there is no set Y of at least k points in P in convex position, such that $\mathrm{conv}(Y) \subsetneq \mathrm{conv}(X)$.

As illustrated in Figure 5, let A_1 be a k-minimal subset of P. Let $A_2, \dots, A_{\ell-1}$ be the convex layers inside A_1. More precisely, for $i = 2, \dots, \ell - 1$, let A_i be the set of points in P on the boundary of the convex hull of $(P \cap \mathrm{conv}(A_{i-1})) - A_{i-1}$. Let $A_\ell := (P \cap \mathrm{conv}(A_{\ell-1})) - A_{\ell-1}$.

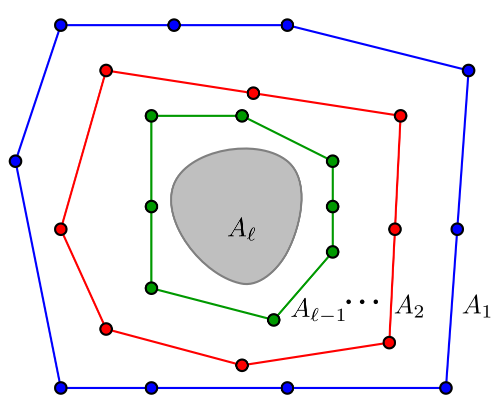


FIGURE 5. Definition of $A_1, \dots, A_\ell$.

By Lemma 6 with $k = 5$, for each $i \in [2, \ell]$, any $2\ell - 1$ consecutive points of A_{i-1} contains five points in strictly convex position. Thus the convex hull of any $2\ell - 1$ consecutive points of A_{i-1} contains a point in A_i, as otherwise it would contain a 5-hole. Now A_{i-1} contains $\left\lfloor \frac{|A_{i-1}|}{2\ell-1} \right\rfloor$ disjoint subsets, each consisting of $2\ell - 1$ consecutive points, and the convex hull of each subset contains a point in A_i. Since the convex hulls of these subsets of A_{i-1} are disjoint,

$$|A_i| \geq \left\lfloor \frac{|A_{i-1}|}{2\ell - 1} \right\rfloor > \frac{|A_{i-1}|}{2\ell - 1} - 1 \ ,$$

implying

$$|A_{i-1}| < (2\ell - 1)(|A_i| + 1) \ . \tag{5}$$

Suppose that $A_i = \varnothing$ for some $i \in [2, \ell]$. By (5), $|A_{i-1}| < 2\ell - 1$ and $|A_{i-2}| < (2\ell - 1)^2 + (2\ell - 1)$, and by induction,

$$|A_1| < \sum_{j=1}^{i-1} (2\ell - 1)^j < \frac{(2\ell - 1)^i - 1}{2\ell - 2} \leq \frac{(2\ell - 1)^\ell - 1}{2\ell - 2} = k \ ,$$

which is a contradiction. Now assume that $A_i \neq \varnothing$ for all $i \in [\ell]$. Fix a point $z \in A_\ell$.

Note that if $|A_i| \leq 2$ for some $i \in [\ell - 1]$ then $A_{i+1} = \varnothing$. Thus we may assume that $|A_i| \geq 3$ for all $i \in [\ell - 1]$. Consider each such set A_i to be ordered clockwise around $\mathrm{conv}(A_i)$. If x and y are consecutive points in A_i with y clockwise from x then we say that the oriented segment $\overrightarrow{xy}$ is an *arc* of A_i.

Let $\overrightarrow{xy}$ be an arc of A_i for some $i \in [\ell - 2]$. We say that $\overrightarrow{xy}$ is *empty* if $\Delta(x, y, z) \cap A_{i+1} = \varnothing$, as illustrated in Figure 6(a). In this case, the intersection of the boundary of $\mathrm{conv}(A_{i+1})$ and $\Delta(x, y, z)$ is contained in an arc $\overrightarrow{pq}$. We call $\overrightarrow{pq}$ the *follower* of $\overrightarrow{xy}$.

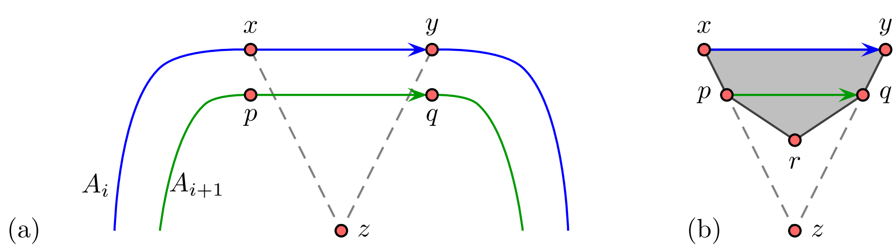


FIGURE 6.

Claim 12. *If $\overrightarrow{pq}$ is the follower of an empty arc $\overrightarrow{xy}$, then $\{x, y, p, q\}$ is a 4-hole and $\overrightarrow{pq}$ is empty.*

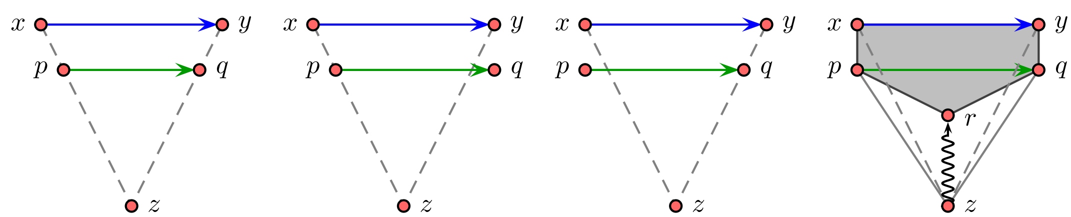
Proof. Say $\overrightarrow{xy}$ is an arc of A_i, where $i \in [\ell - 2]$. Let $S := \{x, y, p, q\}$. Since p and q are in the interior of $\mathrm{conv}(A_i)$, both x and y are corners of S. Both p and q are corners of

S, as otherwise $\overrightarrow{xy}$ is not empty. Thus S is in strictly convex position. S is empty by the definition of A_{i+1}. Thus S is a 4-hole.

Suppose that $\overrightarrow{pq}$ is not empty; that is, $\Delta(p,q,z) \cap A_{i+2} \neq \varnothing$. Let r be a point in $\Delta(p,q,z) \cap A_{i+2}$ closest to $\overline{pq}$. Thus $\Delta(p,q,r) \cap P = \varnothing$. Since $\{x,y,p,q\}$ is a 4-hole, $\{x,y,p,q,r\}$ is a 5-hole, as illustrated in Figure 6(b). This contradiction proves that $\overrightarrow{pq}$ is empty. □

As illustrated in Figure 7(a)–(c), we say the follower $\overrightarrow{pq}$ of $\overrightarrow{xy}$ is:

- *double-aligned* if $p \in \overline{xz}$ and $q \in \overline{yz}$,
- *left-aligned* if $p \in \overline{xz}$ and $q \notin \overline{yz}$,
- *right-aligned* if $p \notin \overline{xz}$ and $q \in \overline{yz}$.



(a) double-aligned (b) left-aligned (c) right-aligned (d) neither

FIGURE 7.

Claim 13. *If $\overrightarrow{pq}$ is the follower of an empty arc $\overrightarrow{xy}$, then $\overrightarrow{pq}$ is either double-aligned or left-aligned or right-aligned.*

Proof. Suppose that $\overrightarrow{pq}$ is neither double-aligned nor left-aligned nor right-aligned, as illustrated in Figure 7(d). Since $\overrightarrow{xy}$ is empty, $p \notin \Delta[x,y,z]$ and $q \notin \Delta[x,y,z]$. Let $D := (P \cap \Delta[p,q,z]) - \{p,q\}$. Thus $z \in D$ and $D \neq \varnothing$. Let r be a point in D closest to $\overline{pq}$. Thus $\Delta(r,p,q)$ is empty. By Claim 12, $\{x,y,p,q\}$ is a 4-hole. Thus $\{x,y,p,q,r\}$ is a 5-hole, which is the desired contradiction. □

Suppose that no arc of A_1 is empty. That is, $\Delta(x,y,z) \cap A_2 \neq \varnothing$ for each arc $\overrightarrow{xy}$ of A_1. Observe that $\Delta(x,y,z) \cap \Delta(p,q,z) = \varnothing$ for distinct arcs $\overrightarrow{xy}$ and $\overrightarrow{pq}$ of A_1 (since these triangles are open). Thus $|A_2| \geq |A_1|$, which contradicts the minimality of A_1.

Now assume that some arc $\overrightarrow{x_1y_1}$ of A_1 is empty. For $i = 2,3,\dots,\ell-1$, let $\overrightarrow{x_iy_i}$ be the follower of $\overrightarrow{x_{i-1}y_{i-1}}$. By Claim 12 (at each iteration), $\overrightarrow{x_iy_i}$ is empty. For some $i \in [2,\ell-2]$, the arc $\overrightarrow{x_iy_i}$ is not double-aligned, as otherwise $\{x_1,x_2,\dots,x_{\ell-2},z\}$ are collinear and $\{y_1,y_2,\dots,y_{\ell-2},z\}$ are collinear, which implies that $\{x_1,x_2,\dots,x_{\ell-1},z\}$ are collinear or $\{y_1,y_2,\dots,y_{\ell-1},z\}$ are collinear by Claim 13. Let i be the minimum integer in $[2,\ell-2]$ such that $\overrightarrow{x_iy_i}$ is not double-aligned. Without loss of generality, $\overrightarrow{x_iy_i}$

is left-aligned. On the other hand, $\overrightarrow{x_j y_j}$ is not left-aligned for all $j \in [i+1, \ell-1]$, as otherwise $\{x_1, x_2, \dots, x_{\ell-1}, z\}$ are collinear. Let j be the minimum integer in $[i+1, \ell-1]$ such that $\overrightarrow{x_j y_j}$ is not left-aligned. Thus $\overrightarrow{x_{j-1} y_{j-1}}$ is left-aligned and $\overrightarrow{x_j y_j}$ is not left-aligned. It follows that $\{x_{j-2}, y_{j-2}, y_{j-1}, y_j, x_{j-1}\}$ is a 5-hole, as illustrated in Figure 8. This contradiction proves that P contains ℓ collinear points or a 5-hole. □

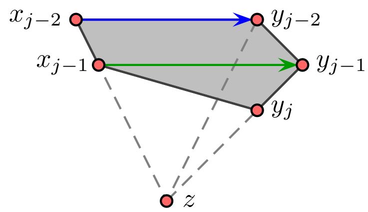


Figure 8.

We expect that the lower bound on $|P|$ in Theorem 3 is far from optimal. All known point sets with at most ℓ collinear points and no 5-hole have $O(\ell^2)$ points, the $\ell \times \ell$ grid for example. See [11, 23] for other examples.

Open Problem. For which values of ℓ is there an integer n such that every set of at least n points in the plane contains ℓ collinear points or a 6-hole?

This is true for $\ell = 3$ by the empty hexagon theorem. If this question is true for a particular value of ℓ then Conjecture 4 is true for $k = 6$ and the same value of ℓ. For $k \geq 7$ different methods are needed since there are point sets in general position with no 7-hole.

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