

Random planar graphs with bounds on the maximum and minimum degrees

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Abstract Let $P_{n,d,D}$ denote the graph taken uniformly at random from the set of all labelled planar graphs on $\{1, 2, \dots, n\}$ with minimum degree at least $d(n)$ and maximum degree at most $D(n)$. We use counting arguments to investigate the probability that $P_{n,d,D}$ will contain given components and subgraphs, showing exactly when this is bounded away from 0 and 1 as $n \rightarrow \infty$.

Keywords planar graphs · random graphs · bounded degrees · labelled graphs

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1 Introduction

Random planar graphs have recently been the subject of much activity, and many properties of the standard random planar graph P_n (taken uniformly at random from the set of all labelled planar graphs on $\{1, 2, \dots, n\}$) are now known. For example, in [6] it was shown that P_n will asymptotically almost surely (that is, with probability tending to 1 as n tends to infinity) contain at least linearly many copies of any given planar graph. By combining the counting methods of [6] with some rather precise results of [5], obtained using generating functions, the exact limiting probability for the event that P_n will contain any given component is also known.

More recently, attention has turned to the graph $P_{n,m}$ taken uniformly at random from the set of all labelled planar graphs on $\{1, 2, \dots, n\}$ with exactly $m(n)$ edges, and the probability that $P_{n,m}$ will contain given components/subgraphs has been investigated in [1] and [4].

Clearly, $P_{n,m}$ can be thought of as a random planar graph with a restriction on the average degree. In this paper, we shall instead study a random planar graph with restrictions on the maximum and minimum degrees, again investigating the probability that such a graph will contain given components/subgraphs. As with P_n and $P_{n,m}$, we shall work solely with labelled graphs.

Given functions $d(n)$ and $D(n)$, let $\mathcal{P}(n, d, D)$ denote the set of all labelled planar graphs on $\{1, 2, \dots, n\}$ with maximum degree at most $D(n)$ and minimum degree at least $d(n)$ (i.e. with all degrees between $d(n)$ and $D(n)$, inclusive) and let $P_{n,d,D}$ denote a graph taken uniformly at random from this set (thus, $P_{n,0,n-1} = P_n$, the standard random planar graph). Note that all graphs with maximum degree at most 2 are planar, and so the interest lies with the case when $D(n) \geq 3$ for all n .

The structure of this paper shall be based on that of [6], where the standard random planar graph was studied. Hence, we will start in Section 2 by establishing a lower bound for the probability that $P_{n,d,D}$ will be connected (and, hence, an upper bound for the probability that it will contain any given component). In Section 3 we shall use this to show that there exists a non-zero finite ‘growth constant’ for $|\mathcal{P}(n, d, D)|$, and in Section 4 we will use this second fact to show that $P_{n,d,D}$ is likely to have many special ‘appearance’-type copies of certain graphs. In Section 5, we shall then use this last result to deduce a lower bound for the probability that $P_{n,d,D}$ will contain given components. Finally, in Section 6, we will prove that $P_{n,d,D}$ has linearly many copies of most, but not all, planar H satisfying $\Delta(H) \leq \liminf D(n)$ (see Table 1).

$\delta(H) = D(n) \text{ for all } n \Rightarrow \left\{ \begin{array}{l} \liminf \mathbf{P} > 0 \text{ (Theorem 13)} \\ \& \limsup \mathbf{P} < 1 \text{ (Theorem 2)} \end{array} \right.$
$\left. \begin{array}{l} \delta(H) < D(n) \text{ for all } n \\ \& \exists n : d(n) = D(n) = 4 \end{array} \right\} \Rightarrow \mathbf{P} \rightarrow 1 \text{ (Theorem 17)}$
$\left. \begin{array}{l} \delta(H) < D(n) \text{ for all } n \\ \& d(n) = D(n) = 4 \text{ for all } n \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \mathbf{P} \rightarrow 1 \text{ if } \exists \text{ 4-regular planar } G \supset H \text{ (Theorem 17)} \\ \mathbf{P} = 0 \text{ otherwise (trivial observation)} \end{array} \right.$

Table 1 A description of $\mathbf{P} := \mathbf{P}[P_{n,d,D} \text{ will have a copy of } H]$ for connected planar H with $\Delta(H) \leq \liminf D(n)$.

2 Connectivity

We will start by examining the probability that our random graph is connected. Not only is this topic interesting in its own right, but the results given here will also be important ingredients in later sections.

Recall that we must have $D(n) \geq 3$ for planarity to have any impact. The main result of this section (Theorem 2) will be to show that, given this, the probability that $P_{n,d,D}$ will be connected is bounded away from 0.

The proof will be based on that of Theorem 2.1 of [6], but will be slightly more complicated, as the bound on the maximum degree means that $\mathcal{P}(n, d, D)$ is not edge-addable (i.e. the class $\mathcal{P}(n, d, D)$ is not closed under the operation of inserting an edge between two components). Hence, we shall first prove a very helpful result on short cycles:

Lemma 1 *Let $k < \frac{1}{15}$, and let S be a planar graph with at most $k|S|$ vertices of degree ≤ 2 . Then S must contain at least $(\frac{1-15k}{28})|S|$ cycles of size ≤ 6 . In particular, if S has at most $\frac{|S|}{43}$ vertices of degree ≤ 2 then S must contain at least $\frac{|S|}{43}$ cycles of size ≤ 6 .*

Proof Fix a planar embedding of S . We shall first show that this embedding must have at least $(\frac{1-15k}{14})|S|$ faces of size ≤ 6 (where, as usual, the ‘size’ of a face denotes the number of edges in the associated facial boundary, with an edge counted twice if it appears twice in the boundary), and we will later deduce the lemma from this fact.

We shall argue by contradiction. Let f_i denote the number of faces of size i and suppose that $\sum_{i \leq 6} f_i < (\frac{1-15k}{14})|S|$. We have

$$\begin{aligned} 2e(S) &= \sum_i i f_i \\ &\geq 7 \sum_{i \geq 7} f_i \\ &> 7 \left(\sum_i f_i - \left(\frac{1-15k}{14} \right) |S| \right), \text{ by our supposition} \\ &= 7 \left(e(S) - |S| + \kappa(S) + 1 - \left(\frac{1-15k}{14} \right) |S| \right), \text{ by Euler's formula} \\ &> 7 \left(e(S) - \left(\frac{15(1-k)}{14} \right) |S| \right). \end{aligned}$$

Thus, $(\frac{15(1-k)}{2})|S| > 5e(S)$. But $e(S) \geq \frac{3(1-k)|S|}{2}$, since S contains at least $(1-k)|S|$ vertices of degree ≥ 3 , and so $5e(S) \geq (\frac{15(1-k)|S|}{2})$. Thus, we obtain our desired contradiction, and so it must be that we have at least $(\frac{1-15k}{14})|S|$ faces of size ≤ 6 .

Let us now consider these faces of size ≤ 6 . Note that the boundary of a face of size ≤ 6 must contain a cycle of size ≤ 6 as a subgraph unless it is acyclic, in which case it must be the entire graph S . But if S were acyclic, then at least half of the vertices would have degree ≤ 2 (since we would have $e(S) \leq |S| - 1$), and this would contradict the conditions of this lemma. Thus, for each of our faces of size ≤ 6 , the boundary must contain a cycle of size ≤ 6 as a subgraph.

Each edge of S can only be in at most two faces of the embedding, and so each cycle can only be in at most two faces. Thus, S must contain at least $(\frac{1-15k}{28})|S|$ distinct cycles of size ≤ 6 . \square

We shall now use Lemma 1 to obtain our aforementioned main result:

Theorem 2 *There exists a constant $c > 0$ such that*

$$\mathbf{P}[P_{r,d,D} \text{ will be connected}] > c$$

for all constants $r, d, D \in \mathbf{N} \cup \{0\}$ satisfying $D \geq 3$ and $\mathcal{P}(r, d, D) \neq \emptyset$.

Sketch of Proof We shall show that there are many ways to construct a graph in $\mathcal{P}(r, d, D)$ with $k - 1$ components from a graph in $\mathcal{P}(r, d, D)$ with k components, by combining two components. Our stated lower bound for the proportion of graphs with exactly one component will then follow by ‘cascading’ this result downwards.

If $D > 6$, we shall see that we may obtain sufficiently many ways to combine components simply by inserting edges between them that don’t interfere with this upper bound on the maximum degree.

If $D \leq 6$, we will sometimes also delete an edge from a small cycle in order to maintain $\Delta \leq D$. We shall use Lemma 1 to show that we have lots of choices for this small cycle, and then the fact that it is small (combined with the knowledge that $D < 7$) will help us to bound the amount of double-counting.

Full Proof Choose any $r, d, D \in \mathbf{N} \cup \{0\}$ with $D \geq 3$ and $\mathcal{P}(r, d, D) \neq \emptyset$. We shall show that there exists a strictly positive constant c , independent of r, d and D , such that $\mathbf{P}[P_{r,d,D} \text{ will be connected}] > c$.

Let $\mathcal{P}^t(r, d, D)$ denote the set of graphs in $\mathcal{P}(r, d, D)$ with exactly t components. For $k > 1$, we shall construct graphs in $\mathcal{P}^{k-1}(r, d, D)$ from graphs in $\mathcal{P}^k(r, d, D)$.

Let the graph $G \in \mathcal{P}^k(r, d, D)$ and let us denote the k components of G by S_1, S_2, \dots, S_k , where $|S_i| = n_i$ for all i . Without loss of generality, we may assume that S_1, S_2, \dots, S_k are ordered so that S_i contains at least $\frac{n_i}{43}$ vertices of degree $< D$ iff $i \leq l$, for some fixed $l \in \{0, 1, \dots, k\}$. Note that we must have $l = k$ if $D > 6$, since (by planarity) $e(S_i) < 3n_i$ and so we can only have at most $\frac{6n_i}{7}$ vertices of degree ≥ 7 .

For $1 \leq i < j \leq k$, let us construct a new graph $G_{i,j} \in \mathcal{P}^{k-1}(r, d, D)$ as follows:

Case (a): if $j \leq l$ (note that this is always the case if $D > 6$)

Insert an edge between a vertex in S_i of degree $< D$ (we have at least $\frac{n_i}{43}$ choices for this) and a vertex in S_j of degree $< D$ (we have at least $\frac{n_j}{43}$ choices for this). See Figure 1.

The constructed graph $G_{i,j}$ (see Figure 1) is planar and has exactly $k - 1$ components. It is also clear that we have $d \leq \delta(G_{i,j}) \leq \Delta(G_{i,j}) \leq D$, since we have not deleted any edges from the original graph and have only inserted an edge between two vertices with degree $< D$. Thus, $G_{i,j} \in \mathcal{P}^{k-1}(r, d, D)$.

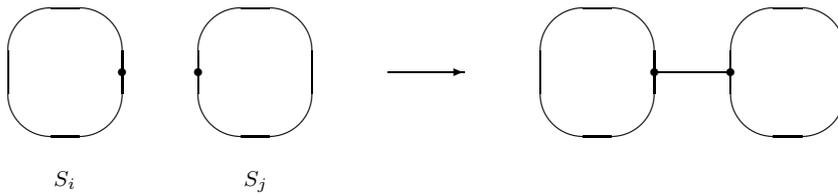


Fig. 1 Constructing the graph $G_{i,j}$ in case (a).

Case (b): if $j > l$ and $n_i > 1$ (in which case $D \leq 6$)

If $j > l$, then S_j contains less than $\frac{n_j}{43}$ vertices of degree $< D$. Thus, by Lemma 1, S_j must contain at least $\frac{n_j}{43}$ cycles of size ≤ 6 . Delete an edge uv in one of these cycles (we have at least $\frac{3}{D+D^2+D^3+D^4} \frac{n_j}{43} \geq \frac{3}{6+6^2+6^3+6^4} \frac{n_j}{43}$ choices for this edge, since each cycle must contain at least 3 edges and each edge is in at most $(D-1)^{m-2} < D^{m-2}$ cycles of size m), insert an edge between u and a vertex $w \in S_i$ (we have n_i choices for w), delete an edge between w and $x \in \Gamma(w)$ (we have at least one choice for x , since $n_i > 1$), and insert an edge between x and v (planarity is preserved, since we may draw S_j so that the face containing u and v is on the outside, and similarly we may draw S_i so that the face containing w and x is on the outside). See Figure 2.

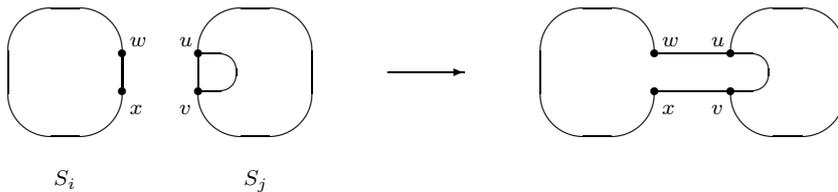


Fig. 2 Constructing the graph $G_{i,j}$ in case (b).

Since the deleted edge uv was in a cycle, it was not a cut-edge, and so the vertex set $V(S_j)$ is still connected. The deleted edge wx may have been a cut-edge in S_i , but since we have also inserted edges from w to $u \in V(S_j)$ and from x to $v \in V(S_j)$ it must be that the vertex set $V(S_i) \cup V(S_j)$ is now connected. Thus, the constructed planar graph $G_{i,j}$ has exactly $k-1$ components. By construction, the degrees of the vertices have not changed, and so we have $d \leq \delta(G_{i,j}) \leq \Delta(G_{i,j}) \leq D$. Thus, $G_{i,j} \in \mathcal{P}^{k-1}(r, d, D)$.

Case (c): if $j > l$ and $n_i = 1$ (in which case $D \leq 6$)

Delete any edge uv in S_j (we have at least n_j choices for this, since S_j cannot be a forest if $j > l$) and insert edges uw and vw , where w is the unique vertex in S_i (see Figure 3).

The constructed graph $G_{i,j}$ is planar and has exactly $k-1$ components. Note that the degrees have not changed, except that we now have $\deg(w) = 2$. But since $D \geq 3$, we still have $d \leq \delta(G_{i,j}) \leq \Delta(G_{i,j}) \leq D$. Thus, we have

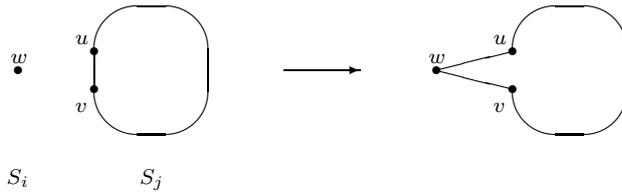


Fig. 3 Constructing the graph $G_{i,j}$ in case (c).

$G_{i,j} \in \mathcal{P}^{k-1}(r, d, D)$.

Let $z = \frac{3}{43(6+6^2+6^3+6^4)} = \min \left\{ \left(\frac{1}{43}\right)^2, \frac{3}{43(6+6^2+6^3+6^4)} \right\}$. Then in all cases we have at least zn_in_j choices when constructing the new graph. Thus, from our initial graph G , we have at least $\sum_{i<j} zn_in_j = z \sum_{i<j} n_in_j$ ways to construct a graph in $\mathcal{P}^{k-1}(r, d, D)$. Note that if $x \leq y$ then $xy > (x-1)(y+1)$, so $\sum_{i<j} n_in_j$ is at least what it would be if one component in G had order $r - (k-1)$ and the other $k-1$ components were isolated vertices. Thus, $z \sum_{i<j} n_in_j \geq z \left(\frac{1}{2}(k-1)(k-2) + (k-1)(r-k+1) \right) = (k-1) \left(r - \frac{k}{2} \right) z$. Hence, for $k > 1$, we have at least $(k-1) \left(r - \frac{k}{2} \right) z |\mathcal{P}^k(r, d, D)| \geq (k-1) \frac{r}{2} z |\mathcal{P}^k(r, d, D)|$ ways to construct (not necessarily distinct) graphs in $\mathcal{P}^{k-1}(r, d, D)$.

Let us now consider the amount of double-counting:

Given one of our constructed graphs, there are at most 3 possibilities for how the graph was obtained (case (a), (b) or (c)).

If case (a) was used (which must be so if $D > 6$), then we can re-obtain the original graph simply by deleting the inserted edge, for which there are at most $r - (k-1) < r$ possibilities, since it must now be a cut-edge. Thus, if case (a) was used, we have less than r possibilities for the original graph.

If case (b) was used, then we can re-obtain the original graph by locating the vertices u, v, w and x , deleting the two inserted edges (uw and vx) and re-inserting the two deleted edges (uv and wx). Note that we have at most r possibilities for which vertex is u . We know that u and v were originally on a cycle of size ≤ 6 , and so v is still at distance at most 5 from u . Since the graph has maximum degree at most D , we therefore have at most $D^2 + D^3 + D^4 + D^5$ possibilities for v . Once we have located u and v , we then have at most D possibilities for w and at most D possibilities for x , since w and x are now neighbours of u and v , respectively. Thus, if case (b) was used, we have at most $D^2(D^2 + D^3 + D^4 + D^5)r \leq 36(6^2 + 6^3 + 6^4 + 6^5)r$ possibilities for the original graph.

If case (c) was used, then we can re-obtain the original graph by locating the vertices u, v and w , deleting the two inserted edges (uw and vw) and re-inserting the deleted edge (uv). We have at most r possibilities for which vertex is w , and given w we then know which edges to delete and insert, as v and w are the only vertices adjacent to u . Thus, if case (c) was used, we have at most r possibilities for the original graph.

Therefore, there are less than r possibilities for the original graph if $D > 6$, since case (a) must have been used, and less than $r + 36(6^2 + 6^3 + 6^4 + 6^5)r + r = 2r(1 + 18(6^2 + 6^3 + 6^4 + 6^5))$ possibilities for the original graph if $D \leq 6$, since any of case (a), case (b) or case (c) may have been used.

Let $\alpha = \frac{z}{4(1+18(6^2+6^3+6^4+6^5))} = \min \left\{ \frac{z}{2}, \frac{z}{4(1+18(6^2+6^3+6^4+6^5))} \right\}$. Then we have shown that we can construct at least $\alpha(k-1)|\mathcal{P}^k(r, d, D)|$ *distinct* graphs in $|\mathcal{P}^{k-1}(r, d, D)|$, and so $|\mathcal{P}^{k-1}(r, d, D)| \geq \alpha(k-1)|\mathcal{P}^k(r, d, D)|$ for all $k > 1$.

Let us define p_k to be $\frac{|\mathcal{P}^{k+1}(r, d, D)|}{|\mathcal{P}^k(r, d, D)|}$ and let $p = p_0 = \frac{|\mathcal{P}^1(r, d, D)|}{|\mathcal{P}^0(r, d, D)|} = \mathbf{P}[P_{r, d, D} \text{ will be connected}]$. From the previous paragraph, we know $|\mathcal{P}^{k+1}(r, d, D)| \leq \frac{|\mathcal{P}^k(r, d, D)|}{\alpha k}$ for all $k > 0$, and so $p_k \leq \frac{p}{\alpha^k k!}$ for all $k \geq 0$. We must have $\sum_{k \geq 0} p_k = 1$, so $\sum_{k \geq 0} \frac{p}{\alpha^k k!} \geq 1$ and hence $p \geq \left(\sum_{k \geq 0} \frac{(\frac{1}{\alpha})^k}{k!} \right)^{-1} = e^{-\frac{1}{\alpha}}$. \square

3 Growth Constants

We shall now look at the topic of ‘growth constants’, which will play a vital role in the proofs of Section 4.

It is known from [6] that there exists a finite constant $\gamma_l > 0$ such that $\left(\frac{|\mathcal{P}(n, 0, n-1)|}{n!} \right)^{1/n} \rightarrow \gamma_l$ as $n \rightarrow \infty$. In this section, we shall use our connectivity bound from Theorem 2 to also obtain (in Theorems 4 and 5) growth constants for $\mathcal{P}(n, d, D)$ for the case when $d(n)$ is a constant and $D(n)$ is any monotonically non-decreasing function (it will turn out that the result for this restricted case is all that will be required for later sections).

We shall follow the proof of Theorem 3.3 of [6], which will require us to first state the following useful lemma:

Proposition 3 (see, for example, Lemma 11.6 of [7]) *Let $f : \mathbf{N} \rightarrow \mathbf{R}^+$ be a function such that $f(n) > 0$ for all large n and $f(i+j) \geq f(i) \cdot f(j)$ for all $i, j \in \mathbf{N}$. Then $(f(n))^{1/n} \rightarrow \sup_n ((f(n))^{1/n})$ as $n \rightarrow \infty$.*

We may now use Proposition 3 to obtain our growth constant result:

Theorem 4 *Let $d \in \{0, 1, \dots, 5\}$ be a constant and let $D(n)$ be a monotonically non-decreasing integer-valued function that for all large n satisfies $D(n) \geq \max\{d, 3\}$ and $(d, D(n)) \notin \{(3, 3), (5, 5)\}$. Then there exists a finite constant $\gamma_{d, D} > 0$ such that*

$$\left(\frac{|\mathcal{P}(n, d, D)|}{n!} \right)^{\frac{1}{n}} \rightarrow \gamma_{d, D} \text{ as } n \rightarrow \infty.$$

Proof We shall follow the method of proof of Theorem 3.3 of [6]. Let c be the constant given by Theorem 2 and let $g(n, d, D) = \frac{c^2 |\mathcal{P}(n, d, D)|}{2 \cdot n!}$ for all $n \in \mathbf{N}$.

We shall show that $g(n, d, D)$ satisfies the conditions of Proposition 3, which we will then use to deduce our result.

To show $g(n, d, D) > 0$ for all large n , it clearly suffices to prove that $\mathcal{P}(n, d, D)$ is non-empty for all large n . This is true, but fairly tedious to demonstrate and so we will omit the details (see [2] for a full discussion).

Let us now show that g satisfies the supermultiplicative condition:

Let $i, j \in \mathbf{N}$ and let us denote by $\mathcal{P}_c(i, d, D)$ and $\mathcal{P}_c(j, d, D)$ the set of connected graphs in $\mathcal{P}(i, d, D)$ and $\mathcal{P}(j, d, D)$, respectively. Then, by Theorem 2, we know that there exists a constant $c > 0$ such that we have $|\mathcal{P}_c(i, d, D)| \geq c|\mathcal{P}(i, d, D)|$ and $|\mathcal{P}_c(j, d, D)| \geq c|\mathcal{P}(j, d, D)|$. We may form a graph in $\mathcal{P}(i+j, d, D)$ by choosing i of the $i+j$ vertices $\left(\binom{i+j}{i} \text{ choices}\right)$, placing a connected planar graph G_1 with $|G_1| = i$ and $d \leq \delta(G_1) \leq \Delta(G_1) \leq D(i)$ on the chosen vertices ($|\mathcal{P}_c(i, d, D)| \geq c|\mathcal{P}(i, d, D)|$ choices), and then placing a connected planar graph G_2 with $|G_2| = j$ and $d \leq \delta(G_2) \leq \Delta(G_2) \leq D(j)$ on the remaining j vertices ($|\mathcal{P}_c(j, d, D)| \geq c|\mathcal{P}(j, d, D)|$ choices). If $i = j$, then we need to divide by two to avoid double-counting. Note that the constructed graph will have maximum degree at most $\max\{D(i), D(j)\}$ and so will indeed be in $\mathcal{P}(i+j, d, D)$, since D is a monotonically non-decreasing function. Thus,

$$|\mathcal{P}(i+j, d, D)| \geq \frac{c^2}{2} \binom{i+j}{i} |\mathcal{P}(i, d, D)| \cdot |\mathcal{P}(j, d, D)| \text{ for all } i, j$$

and, therefore,

$$\begin{aligned} g(i+j, d, D) &= \frac{c^2 |\mathcal{P}(i+j, d, D)|}{2(i+j)!} \\ &\geq \frac{c^4 \binom{i+j}{i} |\mathcal{P}(i, d, D)| |\mathcal{P}(j, d, D)|}{4(i+j)!} \\ &= \frac{c^2 |\mathcal{P}(i, d, D)|}{2 \cdot i!} \frac{c^2 |\mathcal{P}(j, d, D)|}{2 \cdot j!} \\ &= g(i, d, D) \cdot g(j, d, D). \end{aligned}$$

Let $\gamma_{d,D} = \sup_n ((g(n, d, D))^{1/n})$. By Proposition 3, it now only remains to show that $\gamma_{d,D} < \infty$. But clearly $\mathcal{P}(n, d, D) \subset \mathcal{P}(n, 0, n-1)$, the set of all labelled planar graphs on $\{1, 2, \dots, n\}$, and it is known that $\left(\frac{|\mathcal{P}(n, 0, n-1)|}{n!}\right)^{\frac{1}{n}}$ converges to a finite constant as $n \rightarrow \infty$ (see [6]), so we are done. \square

By the same proof, we may also obtain an analogous result to Theorem 4 for the case when $D(n) = d \in \{3, 5\}$ for all n :

Theorem 5 *Let $D \in \{3, 5\}$ be a constant. Then there exists a finite constant $\gamma_{D,D} > 0$ such that*

$$\left(\frac{|\mathcal{P}(2n, D, D)|}{(2n)!}\right)^{\frac{1}{2n}} \rightarrow \gamma_{D,D} \text{ as } n \rightarrow \infty.$$

4 Appearances

We shall now look at special subgraphs in $P_{n,d,D}$ called ‘appearances’, with the aim of turning some of these into components in Section 5.

We will produce separate results for the cases when we have $d(n) < D(n)$ for all n (Theorem 8) and when $d(n) = D(n)$ for all n (Theorem 10), because for the latter case it will be more awkward to later convert these subgraphs into components without violating our bound on the minimum degree.

We will deal with the $d(n) < D(n)$ case first. The main work will be done in Lemma 7, where we shall follow the lines of a proof of [6] (using the growth constants of Section 3) to obtain an appearance result for the case when $d(n)$ is a constant and $D(n)$ is a monotonically non-decreasing function. We shall then extend this into a more general form in Theorem 8, before finally noting that the $d(n) = D(n)$ case follows from a similar proof.

We start with a definition of appearances:

Definition 6 *Let H be a graph on the vertex set $\{1, 2, \dots, h\}$, and let G be a graph on the vertex set $\{1, 2, \dots, n\}$, where $n > h$. Let $W \subset V(G)$ with $|W| = h$, and let the ‘root’ r_W denote the least element in W . We say that H appears at W in G if (a) the increasing bijection from $1, 2, \dots, h$ to W gives an isomorphism between H and the induced subgraph $G[W]$ of G ; and (b) there is exactly one edge in G between W and the rest of G , and this edge $e_W = r_W v_W$ is incident with the root r_W .*

We call e_W the associated cut-edge of the appearance, and we say that $TE_W := E(G[W]) \cup \{e_W\}$ is the total edge set of the appearance (see Figure 4).

When working in $\mathcal{P}(n, d, D)$, we say that the appearance is cut-able if we have $\min\{\deg_G r_W, \deg_G v_W\} > d$, and we let $f_H(G)$ denote the number of cut-able appearances of H in G (that is, the number of sets $W \subset V(G)$ such that there is a cut-able appearance of H at W).

We are now ready to give our first result on appearances in $P_{n,d,D}$. We shall start by assuming that $d(n)$ is constant and $D(n)$ is monotonically non-decreasing, as in Section 3, but we will later (in Theorem 8) get rid of these conditions. The statement of the result may seem complicated, but basically it just asserts that for any ‘sensible’ choice of H , there will probably be lots of cut-able appearances of H in $P_{n,d,D}$. Clearly, ‘sensible’ entails that we must have $\delta(H) \geq d$, $\Delta(H) \leq D(n)$ and $\deg_H(1) + 1 \leq D(n)$, and as always we will require that $D(n) \geq 3$ (note that it follows from these conditions that we must also have $d < D(n)$). The proof is based on that of Theorem 4.1 of [6].

Lemma 7 *Let H be a (fixed) connected planar graph on $\{1, 2, \dots, h\}$. Then there exists a constant $\alpha(h) > 0$ such that, given any constant $d \leq \delta(H)$*

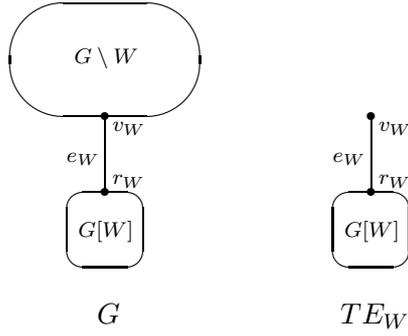


Fig. 4 An appearance at W in G and its total edge set.

and any monotonically non-decreasing integer-valued function $D(n)$ satisfying $\liminf_{n \rightarrow \infty} D(n) \geq \max\{\Delta(H), \deg_H(1) + 1, 3\}$, we have

$$\mathbf{P}[f_H(P_{n,d,D}) \leq \alpha n] < e^{-\alpha n} \text{ for all sufficiently large } n.$$

Sketch of Proof We choose a specific α and suppose that the result is false for $n = k$, where k is suitably large. Using Theorem 4, it then follows that there are many graphs $G \in \mathcal{P}(k, d, D)$ with $f_H(G) \leq \alpha k$.

From each such G , we construct graphs in $\mathcal{P}((1 + \delta)k, d, D)$, for a fixed $\delta > 0$. If G has lots of vertices with degree $< D(k)$, then we do this simply by attaching appearances of H to some of these vertices. If G has few vertices with degree $< D(k)$, then we attach appearances of H to small cycles in G and also delete appropriate edges. By Lemma 1, we have lots of choices for these small cycles and, since G has few vertices with degree $< D(k)$, we may assume that we don't interfere with any vertices of minimum degree.

The fact that the original graphs satisfied $f_H \leq \alpha k$, together with the knowledge that any deleted edges were in small cycles, is then used to show that there is not much double-counting, and so we find that we have constructed so many graphs in $\mathcal{P}((1 + \delta)k, d, D)$ that we contradict Theorem 4.

Full Proof Let $p \in \left(0, \frac{1}{7(6^2+6^3+6^4+6^5)}\right)$, let $\beta = \frac{344e^2(h+7)(6^2+6^3+6^4+6^5)h!(\gamma_{d,D})^h}{p}$, and let $\alpha \in \left(0, \frac{p}{344e^2(h+7)(6^2+6^3+6^4+6^5)h!(\gamma_l)^h}\right)$, where we recall that $\gamma_l \approx 27.2268$ denotes the growth constant for $\mathcal{P}(n, 0, n-1)$. Clearly $\gamma_{d,D} \leq \gamma_l$, so $\alpha\beta < 1$ and hence there exists an $\epsilon \in \left(0, \frac{1}{3}\right)$ such that $(\alpha\beta)^\alpha = 1 - 3\epsilon$.

By Theorem 4, there exists an N such that

$$(1 - \epsilon)^n n! (\gamma_{d,D})^n \leq |\mathcal{P}(n, d, D)| \leq (1 + \epsilon)^n n! (\gamma_{d,D})^n \text{ for all } n \geq N. \quad (1)$$

Suppose (aiming for a contradiction) that we can find a value $k > N$ such that $\mathbf{P}[f_H(P_{k,d,D}) \leq \alpha k] \geq e^{-\alpha k}$, and let \mathcal{G} denote the set of graphs in $\mathcal{P}(k, d, D)$ such that $G \in \mathcal{G}$ iff $f_H(G) \leq \alpha k$. Then we must have $|\mathcal{G}| \geq e^{-\alpha k} |\mathcal{P}(k, d, D)| \geq e^{-\alpha k} (1 - \epsilon)^k k! (\gamma_{d,D})^k$.

Let $\delta = \frac{\lceil \alpha k \rceil h}{k}$. We may assume that k is sufficiently large that $\lceil \alpha k \rceil \leq 2\alpha k$. Thus, $\delta \leq 2\alpha h < 1$ (by our definition of α). This fact will be useful later.

We shall construct graphs in $\mathcal{P}((1 + \delta)k, d, D)$:

Choose δk special vertices (we have $\binom{(1+\delta)k}{\delta k}$ choices for these) and partition them into $\lceil \alpha k \rceil$ unordered blocks of size h (we have $\binom{\delta k}{h, \dots, h} \frac{1}{\lceil \alpha k \rceil!}$ choices for this). On each of the blocks, put a copy of H such that the increasing bijection from $\{1, 2, \dots, h\}$ to the block is an isomorphism between H and this copy. Note that we may assume that k is large enough that $D(k) \geq \liminf_{n \rightarrow \infty} D(n)$, and so the root, r_B , of a block (i.e. the lowest numbered vertex in it) satisfies $\deg(r_B) < D(k)$, by the conditions of the theorem. On the remaining k vertices, we place a planar graph G with $d \leq \delta(G) \leq \Delta(G) \leq D(k)$ and $f_H(G) \leq \alpha k$ (we have at least $|\mathcal{G}|$ choices for this).

We shall continue our construction in one of two ways, depending on the number of vertices of degree $D(k)$ in G :

Case (a): If G has at least $\frac{pk}{43}$ vertices of degree $< D(k)$ (note that this is certainly the case if $D(k) \geq 7$).

For each block B , we choose a *different* non-special vertex $v_B \in V(G)$ with $\deg(v_B) < D(k)$ (we have at least $\binom{pk/43}{\lceil \alpha k \rceil} \lceil \alpha k \rceil!$ choices for this, since certainly $\alpha < \frac{p}{86}$ and we may assume that k is large enough that $\lceil \alpha k \rceil \leq 2\alpha k$), and we insert the edge $r_B v_B$ from the root of the block to this vertex, creating a cut-able appearance of H at B (see Figure 5). Note that we have not deleted

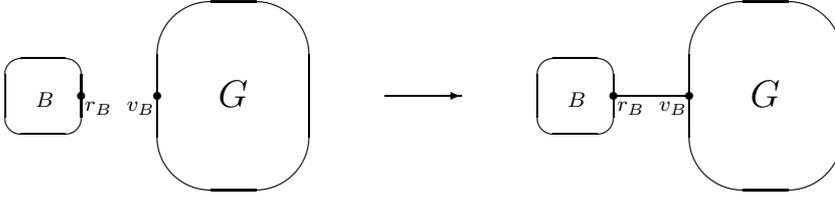


Fig. 5 Creating a cut-able appearance of H at B in case (a).

any edges, so we shall still have minimum degree at least d , and we have only inserted edges between vertices of degree $< D(k)$, so we still have maximum degree at most $D(k)$, which is at most $D((1 + \delta)k)$ by monotonicity of D . Thus, our new graph is indeed in $\mathcal{P}((1 + \delta)k, d, D)$.

Hence, for each graph G with at least $\frac{pk}{43}$ vertices of degree $< D(k)$, we find that we can construct at least $\binom{(1+\delta)k}{\delta k} \binom{\delta k}{h, \dots, h} \cdot \frac{1}{\lceil \alpha k \rceil!} \cdot \binom{pk/43}{\lceil \alpha k \rceil} \lceil \alpha k \rceil!$ different graphs in $\mathcal{P}((1 + \delta)k, d, D)$.

Case (b): If G has less than $\frac{pk}{43}$ vertices of degree $< D(k)$ (in which case $D(k) < 7$).

Before describing the case (b) continuation of our construction, it shall first be useful to investigate the number of short cycles in G :

If G has less than $\frac{pk}{43} < \frac{k}{43}$ vertices of degree $< D(k)$, then (by Lemma 1) G contains at least $\frac{k}{43}$ cycles of size at most 6. A vertex can only be in at most $(D(k))^2 + (D(k))^3 + (D(k))^4 + (D(k))^5 \leq 6^2 + 6^3 + 6^4 + 6^5$ cycles of size at most 6, so G must have at most $\frac{pk(6^2+6^3+6^4+6^5)}{43}$ cycles of size at most 6 that contain a vertex of degree $< D(k)$. In particular, G must have at least $\frac{(1-(6^2+6^3+6^4+6^5)p)k}{43}$ cycles of size at most 6 that don't contain a vertex of degree d , since $d \leq \delta(H) < \deg_H(1) + 1 \leq D(k)$. Since a vertex can only be in at most $6^2 + 6^3 + 6^4 + 6^5$ cycles of size at most 6, each cycle of size at most 6 can only have a vertex in common with at most $6(6^2 + 6^3 + 6^4 + 6^5)$ other cycles of size at most 6. Thus, G must have a set of at least $\frac{\left(\frac{1-(6^2+6^3+6^4+6^5)p}{6(6^2+6^3+6^4+6^5)}\right)k}{43} > \frac{pk}{43}$ vertex-disjoint cycles of size at most 6 that don't contain a vertex of degree d (using the fact that $p < \frac{1}{7(6^2+6^3+6^4+6^5)}$). We shall call these cycles 'special'.

Recall that we have $\lceil \alpha k \rceil$ blocks isomorphic to H . For each block B , choose a *different* one of our 'special' cycles (we have at least $\binom{pk/43}{\lceil \alpha k \rceil} \lceil \alpha k \rceil!$ choices for this), delete an edge $u_B v_B$ in the cycle and insert an edge $r_B v_B$ from the root of the block to a vertex v_B that was incident to the deleted edge, creating an appearance of H at B (see Figure 6). Note that the deleted edge was between

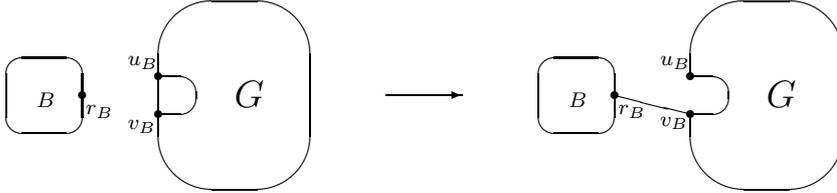


Fig. 6 Creating a cut-able appearance of H at B in case (b).

two vertices of degree $> d$, so we still have minimum degree at least d (and v_B will still have degree $> d$, so the appearance will be cut-able). Recall that the root of each block has degree $< D(k)$, so we still have maximum degree at most $D(k)$, which is at most $D((1 + \delta)k)$ by monotonicity of D . Thus, our constructed graph is indeed in $\mathcal{P}((1 + \delta)k, d, D)$.

Thus, for each graph G with less than $\frac{pk}{43}$ vertices of degree $< D(k)$, we find that we can construct at least $\binom{(1+\delta)k}{\delta k} \binom{\delta k}{h\dots h} \cdot \frac{1}{\lceil \alpha k \rceil!} \cdot \binom{pk/43}{\lceil \alpha k \rceil} \lceil \alpha k \rceil!$ different graphs in $\mathcal{P}((1 + \delta)k, d, D)$.

We have shown that, regardless of whether case (a) or case (b) is used, for each G we can construct at least

$$\binom{(1+\delta)k}{\delta k} \binom{\delta k}{h\dots h} \cdot \frac{1}{\lceil \alpha k \rceil!} \cdot \binom{pk/43}{\lceil \alpha k \rceil} \lceil \alpha k \rceil!$$

$$\begin{aligned}
&= \frac{((1+\delta)k)!}{k!} \frac{1}{(h!)^{\lceil \alpha k \rceil}} \binom{pk/43}{\lceil \alpha k \rceil} \\
&\geq \frac{((1+\delta)k)!}{k!} \frac{1}{(h!)^{\lceil \alpha k \rceil}} \frac{(pk/43 - \lceil \alpha k \rceil + 1)^{\lceil \alpha k \rceil}}{\lceil \alpha k \rceil!} \\
&\geq \frac{((1+\delta)k)!}{k!} \frac{1}{(h!)^{\lceil \alpha k \rceil}} \left(\frac{pk}{86}\right)^{\lceil \alpha k \rceil} \frac{1}{\lceil \alpha k \rceil!} \\
&\quad \left(\text{since certainly } \alpha < \frac{p}{86} \text{ and so } \frac{pk}{43} - \alpha k \geq \frac{pk}{86} \right) \\
&\geq \frac{((1+\delta)k)!}{k!} \left(\frac{pk}{86h!\lceil \alpha k \rceil}\right)^{\lceil \alpha k \rceil} \\
&\geq \frac{((1+\delta)k)!}{k!} \left(\frac{p}{172h!\alpha}\right)^{\lceil \alpha k \rceil} \\
&\quad (\text{since we may assume } k \text{ is large enough that } \lceil \alpha k \rceil \leq 2\alpha k)
\end{aligned}$$

different graphs in $\mathcal{P}((1+\delta)k, d, D)$. Thus, recalling that we have at least $e^{-\alpha k}(1-\epsilon)^k k! (\gamma_{d,D})^k$ choices for G , we find that we can in total construct at least $e^{-\alpha k}(1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^k \left(\frac{p}{172h!\alpha}\right)^{\lceil \alpha k \rceil}$ (not necessarily distinct) graphs in $\mathcal{P}((1+\delta)k, d, D)$.

We are now at the half way point of our proof, and it remains to investigate the amount of double-counting, i.e. how many times each of our constructed graphs will have been built. Given one of our constructed graphs, G' , there are at most two possibilities for how the graph was obtained (case (a) or case(b)), and we shall now examine these two cases separately:

If case (a) was used, then we can re-obtain the original graph, G , simply by deleting the $\lceil \alpha k \rceil$ cut-able appearances that were deliberately added. Thus, in order to bound the amount of double-counting under case (a), we only need to investigate $f_H(G')$:

Suppose W is a cut-able appearance of H in G' . We shall consider how many possibilities there are for W :

(i) If we don't have $TE_W \subset E(G)$, then the total edge set of W must intersect the total edge set of one of our deliberately created appearances. Note that the total edge set of an appearance of H can only intersect at most $|H|$ other total edge sets of appearances of H (since there are at most $|H|$ cut-edges in the total edge set and each of these can have at most one 'orientation' that provides an appearance of H), so we have at most $(h+1)\lceil \alpha k \rceil$ possibilities for W (including the possibility that W is one of our deliberately created appearances).

If $TE_W \subset E(G)$, then W must have been an appearance of H in G :

(ii) If W was a *cut-able* appearance of H in G , then there are at most $\lceil \alpha k \rceil$ possibilities for W , by definition of \mathcal{G} .

(iii) If W was an appearance of H in G that was not cut-able, then the unique vertex $v \in V(G') \setminus W$ incident to the root of W must have had $\deg(v) =$

d originally and must have been chosen as v_B by some block B . Hence, we have at most $\lceil \alpha k \rceil$ possibilities for v and thus at most $d \lceil \alpha k \rceil$ possibilities for W .

Thus, if case (a) was used, then $f_H(G') \leq (h + d + 2) \lceil \alpha k \rceil$, giving us at most $\binom{(h+d+2)\lceil \alpha k \rceil}{\lceil \alpha k \rceil} \leq ((h + d + 2)e)^{\lceil \alpha k \rceil} \leq ((h + 7)e)^{\lceil \alpha k \rceil}$ possibilities for G .

If case (b) was used, we can re-obtain the original graph, G , by deleting the $\lceil \alpha k \rceil$ appearances that were deliberately added and re-inserting the $\lceil \alpha k \rceil$ deleted edges. Note that once we have identified the appearances that were deliberately added, we have at most $((D(k))^2 + (D(k))^3 + (D(k))^4 + (D(k))^5)^{\lceil \alpha k \rceil} \leq (6^2 + 6^3 + 6^4 + 6^5)^{\lceil \alpha k \rceil}$ possibilities for the edges that were deleted, since for each appearance we will automatically know one endpoint, v , of the corresponding deleted edge and we know that the other endpoint, u , will now be at most distance 5 from v , since uv was originally part of a cycle of size ≤ 6 . Hence, as with case (a), it now remains to examine how many possibilities there are for the $\lceil \alpha k \rceil$ appearances that were deliberately added.

Suppose W is a cut-able appearance of H in G' .

(i) If we don't have $TE_W \subset E(G)$, then we have at most $(h + 1) \lceil \alpha k \rceil$ possibilities for W , as with case (a).

(ii) If $TE_W \subset E(G)$ and W was an appearance of H in G , then note that this appearance must have already been cut-able, since it is clear that we have $\deg_{G'} \leq \deg_G$ for all vertices that were in $V(G)$. Hence, there are at most $\lceil \alpha k \rceil$ possibilities for W , by definition of \mathcal{G} .

(iii) If $TE_W \subset E(G)$ and W was *not* an appearance of H in G , then there must have originally been either another edge between W and $V(G) \setminus W$ other than e_W , or another edge between vertices in W . This deleted edge must be of the form $u_B v_B$ for some block B , and so W must contain either u_B or v_B (or both). However, if $v_B \in W$ then $r_B v_B$ would belong to the total edge set of W , which would contradict our assumption that $TE_W \subset E(G)$. Thus, $u_B \in W$ and $v_B \notin W$ (see Figure 7). Recall that the deleted edge $u_B v_B$ was originally

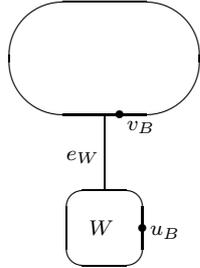


Fig. 7 The appearance W in case (iii).

part of a cycle of size ≤ 6 and that no other edges from the same cycle were deleted. Thus, there is still a $u_B - v_B$ path in G' consisting of the other edges

in the cycle. But, since $u_B \in W$, $v_B \notin W$ and e_W is the unique edge between W and $G' \setminus W$, it must be that e_W belongs to this path, i.e. e_W must have been one of the other (at most 5) edges in the cycle. Thus, we have at most $5\lceil\alpha k\rceil$ possibilities for e_W , and hence for W (since e_W must be ‘oriented’ so that $v_B \notin W$).

Thus, if case (b) was used, then $f_H(G') \leq (h+7)\lceil\alpha k\rceil$, and so we have at most $\binom{(h+7)\lceil\alpha k\rceil}{\lceil\alpha k\rceil} (6^2 + 6^3 + 6^4 + 6^5)^{\lceil\alpha k\rceil} \leq ((h+7)(6^2 + 6^3 + 6^4 + 6^5)e)^{\lceil\alpha k\rceil}$ possibilities for G .

We have shown that each graph in $\mathcal{P}((1+\delta)k, d, D)$ is constructed at most $((h+7)e)^{\lceil\alpha k\rceil} + ((h+7)(6^2 + 6^3 + 6^4 + 6^5)e)^{\lceil\alpha k\rceil} \leq 2((h+7)(6^2 + 6^3 + 6^4 + 6^5)e)^{\lceil\alpha k\rceil} \leq y^{\lceil\alpha k\rceil}$ times, where y denotes $2e(h+d+2)(6^2 + 6^3 + 6^4 + 6^5)$. Thus, the number of *distinct* graphs that we have constructed in $\mathcal{P}((1+\delta)k, d, D)$ must be at least

$$\begin{aligned}
& e^{-\alpha k} (1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^k \left(\frac{p}{172h!\alpha y} \right)^{\lceil\alpha k\rceil} \\
& \geq e^{-\alpha k} (1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^{(1+\delta)k} (\gamma_{d,D})^{-\lceil\alpha k\rceil h} \left(\frac{p}{172h!\alpha y} \right)^{\lceil\alpha k\rceil}, \\
& \quad \text{since } \delta k = \lceil\alpha k\rceil h \\
& \geq (1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^{(1+\delta)k} \left(\frac{172eh!\alpha y (\gamma_{d,D})^h}{p} \right)^{-\lceil\alpha k\rceil}, \\
& \quad \text{since } e^{-\alpha k} \geq e^{-\lceil\alpha k\rceil} \\
& \geq (1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^{(1+\delta)k} (\alpha\beta)^{-\lceil\alpha k\rceil} \\
& \geq (1-\epsilon)^k ((1+\delta)k)! (\gamma_{d,D})^{(1+\delta)k} (\alpha\beta)^{-\alpha k}, \quad \text{since } \alpha\beta < 1 \\
& = \left(\frac{1-\epsilon}{1-3\epsilon} \right)^k ((1+\delta)k)! (\gamma_{d,D})^{(1+\delta)k}, \quad \text{since } (\alpha\beta)^\alpha = 1-3\epsilon \\
& \geq \left(\frac{1-\epsilon}{1-3\epsilon} \right)^k \frac{|\mathcal{P}((1+\delta)k, d, D)|}{(1+\epsilon)^{(1+\delta)k}}, \quad \text{by (1)} \\
& > \left(\frac{1-\epsilon}{(1-3\epsilon)(1+\epsilon)^2} \right)^k |\mathcal{P}((1+\delta)k, d, D)|, \quad \text{since } \delta < 1 \text{ (page 11)} \\
& > |\mathcal{P}((1+\delta)k, d, D)|, \quad \text{since } (1-3\epsilon)(1+\epsilon)^2 = 1-\epsilon-5\epsilon^2-3\epsilon^3.
\end{aligned}$$

Hence, we have our desired contradiction. \square

As mentioned, we shall now see that we can actually drop the conditions that $d(n)$ is a constant and $D(n)$ is monotonically non-decreasing:

Theorem 8 *Let H be a (fixed) connected planar graph on $\{1, 2, \dots, h\}$. Then there exists a constant $\alpha(h) > 0$ such that, given any integer-valued functions*

$d(n)$ and $D(n)$ satisfying $\limsup_{n \rightarrow \infty} d(n) \leq \delta(H)$ and $\liminf_{n \rightarrow \infty} D(n) \geq \max\{\Delta(H), \deg_H(1) + 1, 3\}$, we have

$$\mathbf{P}[f_H(P_{n,d,D}) \leq \alpha n] < e^{-\alpha n} \text{ for all sufficiently large } n.$$

Proof Suppose we can find a graph H and functions $d(n)$ and $D(n)$ that satisfy the conditions of this lemma, but not the conclusion, and let $\alpha = \alpha(h)$ be as given by Lemma 7. Then there exist arbitrarily large ‘bad’ n for which $\mathbf{P}[f_H(P_{n,d,D}) \leq \alpha n] \geq e^{-\alpha n}$.

Let n_1 be one of these bad n and let us try to find a bad $n_2 > n_1$ with $D(n_2) \geq D(n_1)$. Let us then try to find a bad $n_3 > n_2$ with $D(n_3) \geq D(n_2)$, and so on. We will either (a) obtain an infinite sequence $n_1, n_2, n_3 \dots$ with $n_1 < n_2 < n_3 < \dots$ and $D(n_1) \leq D(n_2) \leq D(n_3) \leq \dots$, or (b) we will find a value n_k such that all bad $n > n_k$ have $D(n) \leq D(n_k)$.

Note that we must have $d(n) \in \{0, 1, 2, 3, 4, 5\}$ for all n . Hence, in case (a) there must exist a constant d such that infinitely many of our n_i satisfy $d(n_i) = d$ (we shall call these n_i ‘special’). Let the function D^* be defined by setting $D^*(n) = D(n_1)$ for all $n \leq n_1$ and $D^*(n) = D(n_j)$ for all $n \in \{n_{j-1} + 1, n_{j-1} + 2, \dots, n_j\}$ for all $j > 1$. Then D^* is a monotonically non-decreasing integer-valued function satisfying $\liminf_{n \rightarrow \infty} D^*(n) \geq \liminf_{n \rightarrow \infty} D(n) \geq \max\{\Delta(H), \deg_H(1) + 1, 3\}$. Hence, since $d \leq \limsup_{n \rightarrow \infty} d(n) \leq \delta(H)$, by Lemma 7 it must be that we have $\mathbf{P}[f_H(P_{n,d,D^*}) \leq \alpha n] < e^{-\alpha n}$ for all sufficiently large n . But recall that our infinitely many ‘special’ n_i satisfy $(d, D^*(n_i)) = (d(n_i), D(n_i))$, and so $\mathbf{P}[f_H(P_{n_i,d,D^*}) \leq \alpha n_i] = \mathbf{P}[f_H(P_{n_i,d,D}) \leq \alpha n_i] \geq e^{-\alpha n_i}$ for these n_i . Thus, we obtain a contradiction.

In case (b), note that we have $d(n_i) \in \{0, 1, 2, 3, 4, 5\}$ for all i and that we also have $D(n_i) \in \{3, 4, \dots, D(n_k)\}$ for all $i \geq k$. Hence, there must exist constants d and D such that infinitely many of our n_i satisfy $(d(n_i), D(n_i)) = (d, D)$. But by Lemma 7 we have $\mathbf{P}[f_H(P_{n,d,D}) \leq \alpha n] < e^{-\alpha n}$ for all large n , and so we again obtain a contradiction. \square

In the remainder of this section, we will look at the case when $d(n) = D(n)$. This time, we shall find it more convenient to introduce the concept of ‘2-appearances’:

Definition 9 *Let J be a connected graph on the vertices $\{1, 2, \dots, |J|\}$. Given a graph G , we say that J 2-appears at $W \subset V(G)$ if (a) the increasing bijection from $\{1, 2, \dots, |J|\}$ to W gives an isomorphism between J and the induced subgraph $G[W]$ of G ; and (b) there are exactly two edges, $e_1 = r_1 v_1$ and $e_2 = r_2 v_2$, in G between $W \supset \{r_1, r_2\}$ and $V(G) \setminus W \supset \{v_1, v_2\}$, these edges are non-adjacent (i.e. $r_1 \neq r_2$ and $v_1 \neq v_2$), and v_1 and v_2 are also non-adjacent (see Figure 8).*

We will now give our result on 2-appearances in $P_{n,d,D}$. As the method is very similar to that of Lemma 7, we omit the full details (see Lemma 87 of [2] for a complete proof):

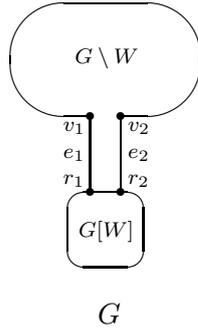


Fig. 8 A 2-appearance at W in G .

Theorem 10 *Let $D \geq 3$ be a constant, let H be a (fixed) D -regular connected planar graph on $\{1, 2, \dots, h\}$, and let $f \in E(H)$ be a non cut-edge. Then there exist constants $\alpha(h) > 0$ and $N(h)$ such that*

$$\mathbf{P}[P_{n,D,D} \text{ will not have a set of at least } \alpha n \text{ 2-appearances of } H \setminus f] < e^{-\alpha n} \begin{cases} \text{for all } n \geq N \text{ if } D = 4 \\ \text{for all even } n \geq N \text{ if } D \in \{3, 5\}. \end{cases}$$

Sketch of Proof We choose a specific α and suppose that the result is false for $n = k$, where k is suitably large. Using Theorem 4/Theorem 5 (depending on the parity of D), it then follows that there are many graphs $G \in \mathcal{P}(k, D, D)$ with at most αk 2-appearances of $H \setminus f$.

From each such G , we construct graphs in $\mathcal{P}((1 + \delta)k, D, D)$, for a fixed $\delta > 0$, by replacing some edges in G with 2-appearances of $H \setminus f$.

The fact that the original graphs had few 2-appearances of $H \setminus f$ can then be used to show that there is not much double-counting, and so we find that we have built so many graphs in $\mathcal{P}((1 + \delta)k, D, D)$ that we contradict Theorem 4/Theorem 5.

5 Components

We shall now use our appearance results from the previous section to investigate the probability of $P_{n,d,D}$ having given components.

We already know from Section 2 that (assuming $D(n) \geq 3$ for all n , as always) $\liminf \mathbf{P}[P_{n,d,D} \text{ will be connected}] > 0$, so certainly it must be that $\limsup \mathbf{P}[P_{n,d,D} \text{ will have a component isomorphic to } H] < 1$ for all H . In this section, we will now see (in Theorem 13) that for all feasible H we also have $\liminf \mathbf{P}[P_{n,d,D} \text{ will have a component isomorphic to } H] > 0$.

As we are going to be using Theorems 8 and 10 from Section 4, we will start by dealing with the $d(n) < D(n)$ and $d(n) = D(n)$ cases separately (in

Lemmas 11 and 12, respectively), but we shall then combine these results in Theorem 13.

We start with the case when $d(n) < D(n)$ for all n :

Lemma 11 *Let $d(n)$ and $D(n)$ be any integer-valued functions that for all n satisfy $D(n) \geq 3$ and $d(n) < D(n)$. Then, given any (fixed) connected planar graph H with $\limsup_{n \rightarrow \infty} d(n) \leq \delta(H) \leq \Delta(H) \leq \liminf_{n \rightarrow \infty} D(n)$, we have*

$$\liminf_{n \rightarrow \infty} \mathbf{P}[P_{n,d,D} \text{ will have a component isomorphic to } H] > 0.$$

Proof Let $\mathcal{N}^<$ denote the set of values of n for which $\delta(H) < D(n)$. We shall start by proving the result for $\mathcal{N}^<$.

Without loss of generality (by symmetry), we may assume that we have $V(H) = \{1, 2, \dots, |H|\}$ and that $\deg_H(1) = \delta(H)$. Thus, by Theorem 8, we know there exist $\alpha > 0$ and N_1 such that $\mathbf{P}[f_H(P_{n,d,D}) \leq \alpha n] < e^{-\alpha n}$ for $\{n \in \mathcal{N}^< : n \geq N_1\}$. Hence, given any $\epsilon > 0$, there certainly exists an N_2 such that $\mathbf{P}[f_H(P_{n,d,D}) \geq \alpha n] > \epsilon$ for $\{n \in \mathcal{N}^< : n \geq N_2\}$. Let us suppose that we can find an $n \in \mathcal{N}^<$ satisfying $n \geq N_2$ for which $\mathbf{P}[P_{n,d,D} \text{ will have a component isomorphic to } H] < \frac{\epsilon}{2}$ (if not, then we are done) and let $\mathcal{G}(n, d, D)$ denote the set of graphs in $\mathcal{P}(n, d, D)$ which have both (i) at least αn cut-able appearances of H and (ii) no components isomorphic to H .

We may construct a graph in $\mathcal{P}(n, d, D)$ with a component isomorphic to H simply by taking a graph in $\mathcal{G}(n, d, D)$ (at least $\frac{\epsilon}{2} |\mathcal{P}(n, d, D)|$ choices) and deleting the associated cut-edge from a cut-able appearance of H (at least αn choices). Thus, we have at least $\frac{\epsilon \alpha n}{2} |\mathcal{P}(n, d, D)|$ ways to construct (not necessarily distinct) graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H (see Figure 9).

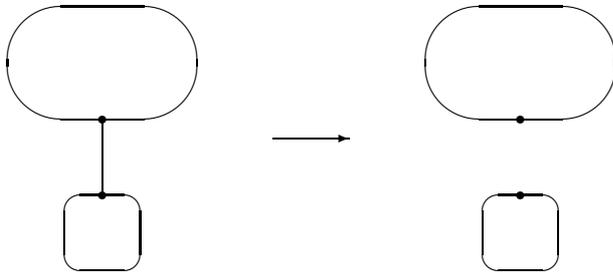


Fig. 9 Using an appearance to construct a component isomorphic to H .

Given one of our constructed graphs, there will be at most two possibilities for which is our deliberately created component (since there were no components isomorphic to H in the original graph and we can have only created at most two when we deleted the cut-edge) and then at most n possibilities for the vertex in the rest of the graph that this component was attached to originally. Hence, the amount of double-counting is at most $2n$, and so we find

that we can construct at least $\frac{\epsilon\alpha}{4}|\mathcal{P}(n, d, D)|$ *distinct* graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H , which is what we wanted to prove.

Now let $\mathcal{N}^=$ denote the set of values of n for which $\delta(H) = D(n)$. It only remains to prove the result for $\mathcal{N}^=$. If $\mathcal{N}^= \neq \emptyset$, then we must have $\delta(H) \geq 3$, since $D(n) \geq 3$. Hence, there exists a non cut-edge $f \in E(H)$. Let $H' = H \setminus f$ and note that $\delta(H') = D(n) - 1 \geq d(n)$ for all $n \in \mathcal{N}^=$. Thus, we may use Theorem 8 with H' and then follow the same proof as with $\mathcal{N}^<$ to find constants $\epsilon' > 0$ and N_3 such that we have $\mathbf{P}[P_{n,d,D}$ will have a component isomorphic to $H'] > \epsilon'$ for $\{n \in \mathcal{N}^= : n \geq N_3\}$. Let us suppose that we can find an $n \in \mathcal{N}^=$ satisfying $n \geq N_3$ for which $\mathbf{P}[P_{n,d,D}$ will have a component isomorphic to $H] < \frac{\epsilon'}{2}$ (if not, then we are done) and let $\mathcal{H}(n, d, D)$ denote the set of graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H' but without any components isomorphic to H .

Given a graph in $\mathcal{H}(n, d, D)$, we may construct a graph in $\mathcal{P}(n, d, D)$ with a component isomorphic to H simply by choosing a component isomorphic to H' and adding an appropriate edge. The amount of double-counting will be at most $3|H|$, since we will know exactly where the modified component is, and so we find that the number of *distinct* graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H must be at least $\frac{|\mathcal{H}(n,d,D)|}{3|H|} \geq \frac{\epsilon'|\mathcal{P}(n,d,D)|}{6|H|}$, and so we are done. \square

We shall now see an analogous result for when $d(n) = D(n)$ for all n :

Lemma 12 *Let $D \geq 3$ be a constant and let H be a (fixed) D -regular connected planar graph. Then there exist constants $\epsilon(H) > 0$ and $N(H)$ such that*

$$\begin{aligned} \mathbf{P}[P_{n,D,D} \text{ will have a component isomorphic to } H] \\ > \epsilon \begin{cases} \text{for all } n \geq N \text{ if } D = 4 \\ \text{for all even } n \geq N \text{ if } D \in \{3, 5\}. \end{cases} \end{aligned}$$

Proof In order to simplify parity matters, we shall just prove the result for $D = 4$, but the $D \in \{3, 5\}$ cases will follow in a completely analogous way.

Without loss of generality, we may assume that $V(H) = \{1, 2, \dots, |H|\}$. Let $f \in E(H)$ be an arbitrary non cut-edge. Then, by Theorem 10, we know that there exists $\alpha > 0$ and there exists N_1 such that for all $n \geq N_1$ we have $\mathbf{P}[P_{n,D,D}$ does not have $\geq \alpha n$ 2-appearances of $H \setminus f] < e^{-\alpha n}$. Thus, given any $\delta > 0$, there certainly exists an N_2 such that $\mathbf{P}[P_{n,d,D}$ has $\geq \alpha n$ 2-appearances of $H \setminus f] > \delta$ for all $n \geq N_2$. Let us now suppose that we can find an $n \geq N_2$ for which $\mathbf{P}[P_{n,d,D}$ will have a component isomorphic to $H] < \frac{\delta}{2}$ (if not, then we are done) and let $\mathcal{G}(n, d, D)$ denote the set of graphs in $\mathcal{P}(n, d, D)$ which have both (i) at least αn 2-appearances of H and (ii) no components isomorphic to H .

We may construct a graph in $\mathcal{P}(n, d, D)$ with a component isomorphic to H simply by taking a graph $G \in \mathcal{G}(n, d, D)$ (at least $\frac{\delta}{2}|\mathcal{P}(n, d, D)|$ choices);

choosing a 2-appearance W of $H \setminus e$ (at least βn choices); deleting the two edges of the form $v_1 r_1$ and $v_2 r_2$ for $\{r_1, r_2\} \subset W$ and $\{v_1, v_2\} \subset V(G) \setminus W$; and then finally inserting the two edges $v_1 v_2$ and $r_1 r_2$ (note that v_1 and v_2 were not originally adjacent, by the definition of a 2-appearance, and that r_1 and r_2 were also not originally adjacent, since they must be the two vertices of degree $D - 1$ in $G[W]$, by D -regularity of G , and we know $G[W]$ is isomorphic to $H \setminus f$). Thus, we have at least $\frac{\delta\beta n}{2} |\mathcal{P}(n, d, D)|$ ways to construct (not necessarily distinct) graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H (see Figure 10).

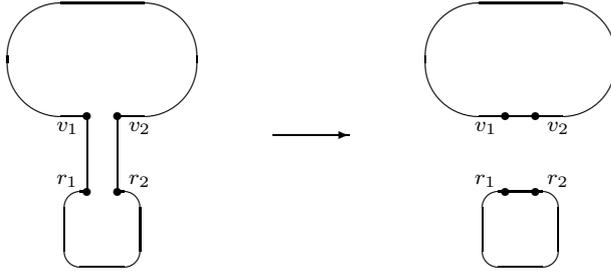


Fig. 10 Using a 2-appearances to construct a component isomorphic to H .

Given one of our constructed graphs, there will be at most two possibilities for which is our deliberately constructed component (since there were no components isomorphic to H in the original graph and we can have only created at most two). We then know which edge was inserted into it and have at most $3n$ (by planarity) possibilities for which edge was inserted in the rest of the graph. We also have two further possibilities for how the vertices in these two edges were connected originally. Hence, the amount of double-counting is at most $12n$, and so we find that we can construct at least $\frac{\delta\beta}{12} |\mathcal{P}(n, d, D)|$ *distinct* graphs in $\mathcal{P}(n, d, D)$ with a component isomorphic to H , which is what we wanted to prove. \square

We may now combine Lemmas 11 and 12 to obtain our full result:

Theorem 13 *Let $d(n)$ and $D(n)$ be any integer-valued functions, subject to $D(n) \geq 3$ for all n and $(d(n), D(n)) \notin \{(3, 3), (5, 5)\}$ for odd n . Then, given any (fixed) connected planar graph H satisfying $\limsup_{n \rightarrow \infty} d(n) \leq \delta(H) \leq \Delta(H) \leq \liminf_{n \rightarrow \infty} D(n)$, we have*

$$\liminf_{n \rightarrow \infty} \mathbf{P}[P_{n,d,D} \text{ will have a component isomorphic to } H] > 0.$$

6 Subgraphs

We will now use the results of the previous two sections to investigate the probability of $P_{n,d,D}$ having *subgraphs* isomorphic to given connected planar graphs. As always, we shall assume throughout that $D(n) \geq 3$ for all n .

Clearly, for those values of n for which $D(n) < \Delta(H)$, we must have $\mathbf{P}[P_{n,d,D} \text{ will have a copy of } H] = 0$. For sufficiently large n , it turns out that the only other time when we can have this is if $d(n) = D(n) = 4$ and H happens to be a graph that can *never* be a subgraph of a 4-regular planar graph.

Apart from the above exceptions, we shall see that the matter of whether $\mathbf{P}[P_{n,d,D} \text{ will have a copy of } H]$ is bounded away from 0 and/or 1 actually depends only on whether or not H is $D(n)$ -regular. For those values of n for which this is the case, any copy of H must be a component and so it already follows from Sections 2 and 5 that the probability must indeed be bounded away from both 0 and 1. If there aren't arbitrarily large values of n for which H is $D(n)$ -regular, though, we shall be able to use our appearance results of Section 4 to deduce (in Theorem 17) that $\mathbf{P}[P_{n,d,D} \text{ will have a copy of } H] \rightarrow 1$ (again, with the exception of the cases given in the previous paragraph).

We will start by working towards the $d(n) = D(n)$ case:

Lemma 14 *Let H be a (fixed) connected planar graph and let $D \in \{3, 4, 5\}$ be a constant. Suppose H is not D -regular, but that there exists a D -regular planar graph H^* that contains a copy of H . Then there exist constants $\beta(H) > 0$ and $N(H)$ such that*

$$\mathbf{P}[P_{n,D,D} \text{ will not have at least } \beta n \text{ copies of } H] < e^{-\beta n} \begin{cases} \text{for all } n \geq N \text{ if } D = 4 \\ \text{for all even } n \geq N \text{ if } D \in \{3, 5\}. \end{cases}$$

Proof Since H is not D -regular, it must be that H^* contains an edge $f = uv$ such that $H^* \setminus f$ also contains a copy of H . Without loss of generality, we may assume that f is not a cut-edge in H^* , since we could replace f_i with a copy of the appropriate graph from Figure 11 and in this way obtain a D -regular planar graph containing several non cut-edges that don't interfere with our copy of H_i . Thus, the result then follows from Theorem 10 with $H^* \setminus f$. \square

If $D \in \{3, 5\}$, then it can easily be shown that there does always exist a D -regular planar graph $H^* \supset H$ whenever $\Delta(H) \leq D$, since in these cases there exist planar graphs that are D -regular except for exactly one vertex with degree $D - 1$, and hence we can extend H into a D -regular planar graph simply by attaching an appropriate number of these graphs to any vertices of H that have degree less than D . This trick does not work for $D = 4$, however, since clearly a graph that is 4-regular except for exactly one vertex of degree 3 would have to have an odd sum of degrees! In fact, the following example

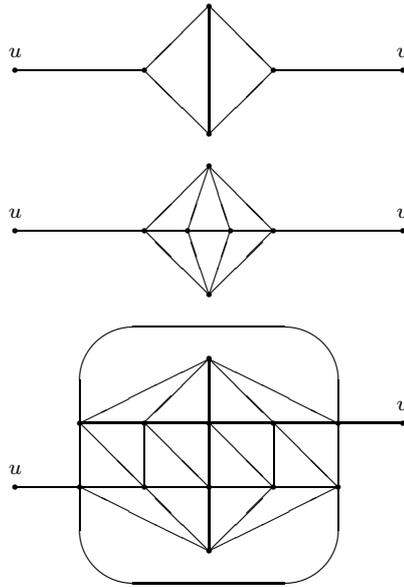


Fig. 11 Replacing the edge $f = uv$ (cases $D = 3$, $D = 4$ and $D = 5$).

shows that there do actually exist some planar graphs with maximum degree at most 4 that can't ever be contained in a 4-regular planar graph:

Example 15 *No 4-regular planar graph contains a copy of the graph K_5 minus an edge.*

Proof The graph $K_5 \setminus \{u, w\}$ is drawn with its *unique* planar embedding (see [8]) in Figure 12.

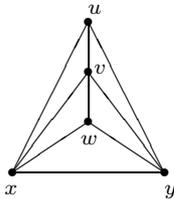


Fig. 12 The unique planar embedding of $K_5 \setminus \{u, w\}$.

Consider any planar graph $G \supset K_5 \setminus \{u, w\}$ with $\Delta(G) = 4$. Since we already have $\deg_H(v) = \deg_H(x) = \deg_H(y) = 4$, any new edge with at least one endpoint inside the triangle given by vxy must have both endpoints inside. Hence, the sum of degrees inside this triangle must remain odd, and so this region must still contain a vertex of odd degree. Thus, G is not 4-regular. \square

An $O(|H|^{2.5})$ time algorithm for determining whether or not a given graph H can ever be a subgraph of a 4-regular planar graph is given in [3].

It now only remains for us to deal with the case when $d(n) < D(n)$. But note that this can be deduced easily simply by applying Theorem 8 to a connected planar graph $H' \supset H$ with sufficiently high minimum degree. Hence, we have:

Lemma 16 *Let H be a (fixed) connected planar graph. Suppose $d(n)$ and $D(n)$ are integer-valued functions that for all large n satisfy (a) $d(n) < \min\{6, D(n)\}$, and (b) $D(n) \geq \max\{\Delta(H), \delta(H) + 1, 3\}$. Then there exists a constant $\beta(H) > 0$ such that*

$$\mathbf{P}[P_{n,d,D} \text{ will not have at least } \beta n \text{ copies of } H] = e^{-\Omega(n)}.$$

Combining all the results of this section, we obtain our full result:

Theorem 17 *Let H be a (fixed) connected planar graph. Suppose $d(n)$ and $D(n)$ are integer-valued functions that for all n satisfy (a) $d(n) \leq \min\{5, D(n)\}$, (b) $D(n) \geq \max\{\Delta(H), \delta(H) + 1, 3\}$, (c) $(d(n), D(n)) \notin \{(3, 3), (5, 5)\}$ for odd n , and also (d) $(d(n), D(n)) \neq (4, 4)$ if H happens to be a graph that can never be contained within a 4-regular planar graph. Then there exists a constant $\beta(H) > 0$ such that*

$$\mathbf{P}[P_{n,d,D} \text{ will not have at least } \beta n \text{ copies of } H] = e^{-\Omega(n)}.$$

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