Critical independent sets and König–Egerváry graphs

Vadim E. Levit
Ariel University Center of Samaria, Israel
levitv@ariel.ac.il

Eugen Mandrescu
Holon Institute of Technology, Israel
eugen_m@hit.ac.il

Abstract

A set S of vertices is *independent* in a graph G, and we write $S \in \mathrm{Ind}(G)$, if no two vertices from S are adjacent, and $\alpha(G)$ is the cardinality of an independent set of maximum size, while $\mathrm{core}(G)$ denotes the intersection of all maximum independent sets [18].

G is called a *König–Egerváry graph* if its order equals $\alpha(G) + \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching. The number $def(G) = |V(G)| - 2\mu(G)$ is the *deficiency* of G [22].

The number $d(G) = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$ is the *critical difference* of G. An independent set A is *critical* if $|A| - |N(A)| = d(G)$, where $N(S)$ is the neighborhood of S, and $\alpha_c(G)$ denotes the maximum size of a critical independent set [27].

In [15] it was shown that G is König–Egerváry graph if and only if there exists a maximum independent set that is also critical, i.e., $\alpha_c(G) = \alpha(G)$.

In this paper we prove that:

(i) $d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G) = def(G)$ for every König–Egerváry graph G;

(ii) G is König–Egerváry graph if and only if every maximum independent set of G is critical.

Keywords: independent set, maximum matching, critical difference, critical independent set, deficiency, core.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F, and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex

$v \in V$ is the set $N(v) = \{w : w \in V \ \ and \ vw \in E\}$, while $N(A) = \cup\{N(v) : v \in A\}$ and $N[A] = A \cup N(A)$ for $A \subset V$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the set of all the independent sets of G. An independent set of maximum size will be referred to as a *maximum independent set* of G, and the *independence number* of G is $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Let us denote the set $\{S : S$ *is a maximum independent set of* $G\}$ by $\Omega(G)$, and let $\mathrm{core}(G) = \cap\{S : S \in \Omega(G)\}$ [18]. A set $A \subseteq V(G)$ is a *local maximum independent set* of G if $A \in \Omega(G[N[A]])$ [17].

Theorem 1.1 *[23] Every local maximum independent set of a graph is a subset of a maximum independent set.*

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G.

It is well-known that $\lfloor |V|/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq |V|$ hold for any graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then G is called a *König-Egerváry graph*. We attribute this definition to Deming [7], and Sterboul [26]. These graphs were studied by Korach [12], Lovasz [21], Lovasz and Plummer [22], Bourjolly and Pulleyblank [3], Pulleyblank [25], and generalized by Bourjolly, Hammer and Simeone [2], Paschos and Demange [24]. Several properties of König-Egerváry graphs are presented in [16], [19], [20].

According to a well-known result of König [11], and Egerváry [9], any bipartite graph is a König-Egerváry graph. This class includes non-bipartite graphs as well (see, for instance, the graphs H_1 and H_2 in Figure 1).

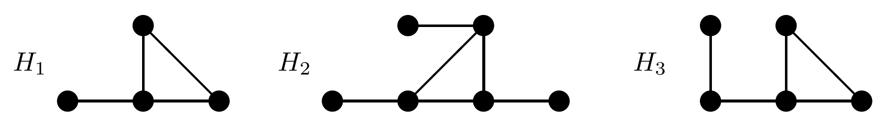

Figure 1: Only H_3 is not a König–Egerváry graph, as $\alpha(H_3) + \mu(H_3) = 4 < 5 = |V(H_3)|$.

It is easy to see that if G is a König-Egerváry graph, then $\alpha(G) \geq \mu(G)$, and that a graph G having a perfect matching is a König-Egerváry graph if and only if $\alpha(G) = \mu(G)$.

The number $d(G) = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$ is called the *critical difference* of G. An independent set A is *critical* if $|A| - |N(A)| = d(G)$, and the *critical independence number* $\alpha_c(G)$ is the cardinality of a maximum critical independent set [27]. Clearly, $\alpha_c(G) \leq \alpha(G)$. The problem of finding a critical independent set is polynomially solvable [1], [27].

Proposition 1.2 *[14] If S is a critical independent set, then there is a matching from $N(S)$ into S.*

If S is an independent set of a graph G and $H = G - S$, then we write $G = S * H$. Evidently, any graph admits such representations. For instance, if $E(H) = \emptyset$, then $G = S * H$ is bipartite; if H is complete, then $G = S * H$ is a *split graph* [10].

Proposition 1.3 *[19] G is a König-Egerváry graph if and only if $G = H_1 * H_2$, where $V(H_1) \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |V(H_2)|$.*

Let M be a maximum matching of a graph G. To adopt Edmonds's terminology [8], we recall the following terms for G relative to M. An *alternating path* from a vertex x to a vertex y is a x,y-path whose edges are alternating in and not in M. A vertex x is *exposed* relative to M if x is not the endpoint of a heavy edge. An odd cycle C with $V(C) = \{x_0, x_1, ..., x_{2k}\}$ and $E(C) = \{x_i x_{i+1} : 0 \leq i \leq 2k-1\} \cup \{x_{2k}, x_0\}$, such that $x_1x_2, x_3x_4, ..., x_{2k-1}x_{2k} \in M$ is a *blossom* relative to M. The vertex x_0 is the *base* of the blossom. The *stem* is an even length alternating path joining the base of a blossom and an exposed vertex for M. The base is the only common vertex to the blossom and the stem. A *flower* is a blossom and its stem. A *posy* consists of two (not necessarily disjoint) blossoms joined by an odd length alternating path whose first and last edges belong to M. The endpoints of the path are exactly the bases of the two blossoms. The following result of Sterboul, characterizes König-Egerváry graphs in terms of forbidden configurations.

Theorem 1.4 *[26] For a graph G, the following properties are equivalent:*
(i) *G is a König-Egerváry graph;*
(ii) *there exist no flower and no posy relative to some maximum matching M;*
(iii) *there exist no flower and no posy relative to any maximum matching M.*

In [21] is given a characterization of König-Egerváry graphs having a perfect matching, in terms of certain forbidden subgraphs with respect to a specific perfect matching of the graph. In [13] is given the following characterization of König-Egerváry graphs in terms of excluded structures.

Theorem 1.5 *[13] Let M be a maximum matching in a graph G. Then G is a König-Egerváry graph if and only if G does not contain one of the forbidden configurations, depicted in Figure 2, with respect to M.*

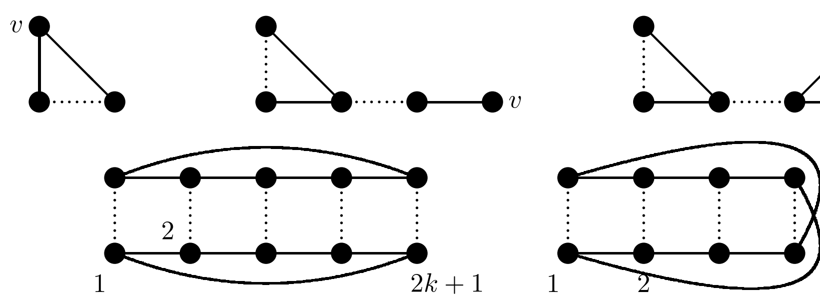

Figure 2: Forbidden configurations. The vertex v is not adjacent to the matching edges (namely, dashed edges).

In [15] it was shown that G is a König-Egerváry graph if and only if $\alpha_c(G) = \alpha(G)$, thus giving a positive answer to the Graffiti.pc 329 conjecture [6].

The *deficiency* of G, denoted by $def(G)$, is defined as the number of exposed vertices relative to a maximum matching [22]. In other words, $def(G) = |V(G)| - 2\mu(G)$.

In this paper we prove that the critical difference for a König-Egerváry graph G is given by

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G),$$

and using this finding, we show that G is a König-Egerváry graph if and only if each of its maximum independent sets is critical.

2 Results

Proposition 2.1 *Any critical independent set is a local maximum independent set.*

Proof. Suppose, on the contrary, that there is a critical independent set S such that $S \notin \Psi(G)$, i.e., there exists some independent set $A \subseteq N[S]$, larger than S. It follows that $|A \cap N(S)| > |S - S \cap A|$, and this contradicts the fact that, according to Proposition 1.2, there is a matching from $A \cap N(S)$ to S, in fact, from $A \cap N(S)$ to $S - S \cap A$. ■

The converse of Proposition 2.1 is not true; e.g., the set $\{d, h\}$ is a local maximum independent set of the graph G_1 from Figure 3, but it is not critical.

Using Theorem 1.1, we easily deduce the following result.

Corollary 2.2 *[5] Every critical independent set is included in some maximum independent set.*

Theorem 2.3 *If G is a König-Egerváry graph, then*

(i) *[19] $G - N[core(G)]$ has a perfect matching and it is also a König-Egerváry graph.*

(ii) *[19]* $N(\text{core}(G)) = \cap \{V(G) - S : S \in \Omega(G)\}$.

(iii) *[20]* $\alpha(G) + |\cap \{V(G) - S : S \in \Omega(G)\}| = \mu(G) + |\cap \{S : S \in \Omega(G)\}|$.

Let us notice that for non-König-Egerváry graphs every relation between $\alpha(G) - \mu(G)$ and $|\text{core}(G)| - |N(\text{core}(G))|$ is possible.

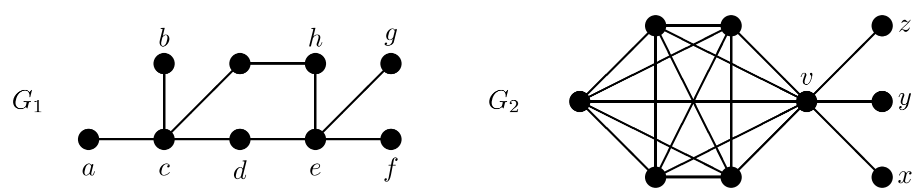

Figure 3: $\alpha(G_1) = 6$, $\mu(G_1) = 3$, $\text{core}(G_1) = \{a, b, d, g, f\}$ and $N(\text{core}(G_1)) = \{c, e\}$, while $\alpha(G_2) = 4$, $\mu(G_2) = 3$, $\text{core}(G_2) = \{x, y, z\}$, and $N(\text{core}(G_2)) = \{v\}$.

The non-König-Egerváry graphs from Figure 3 satisfy:

$$\alpha(G_1) - \mu(G_1) = 3 = |\text{core}(G_1)| - |N(\text{core}(G_1))|$$

and

$$\alpha(G_2) - \mu(G_2) = 1 < 2 = |\text{core}(G_2)| - |N(\text{core}(G_2))| .$$

The opposite direction of the above inequality may be found in $G_3 = K_{2n} - e, n \geq 3$:

$$\alpha(G_3) - \mu(G_3) = 2 - n > 4 - 2n = 2 - (2n - 2) = |\text{core}(G_3)| - |N(\text{core}(G_3))| .$$

Theorem 2.4 *If G is König-Egerváry graph, then*

$$d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G) = def(G).$$

Proof. Firstly, let us prove that $\alpha(G)-\mu(G) \geq |S|-|N(S)|$ holds for every $S \in \mathrm{Ind}(G)$, i.e., $d(G) \leq \alpha(G) - \mu(G)$. If $\alpha(G) = \mu(G)$, then G has a perfect matching and

$$|S| - |N(S)| \leq 0 = \alpha(G) - \mu(G)$$

holds for every $S \in \mathrm{Ind}(G)$.

Suppose that $\alpha(G) > \mu(G)$. Let $S_0 \in \Omega(G)$ and M be a maximum matching, i.e., $|M| = |V(G) - S_0| = \mu(G)$. Assume that $S \in \mathrm{Ind}(G)$ satisfies $|S| - |N(S)| > 0$. Then one can write $S = S_1 \cup S_2 \cup S_3$, where $S_3 \subseteq V(G) - S_0, S_1 \cup S_2 \subset S_0, S_1 \cap S_2 = \emptyset$, and S_2 contains all $v \in S$ matched by M with some vertex of $V(G) - S_0$. Since M is a maximum matching, we get $|S_2| - |N(S_2)| \leq 0$ and $|S_3| - |N(S_3)| \leq 0$. Consequently, we obtain

$$\alpha(G) - \mu(G) = |S_0| - |V(G) - S_0| \geq |S_1| \geq |S| - |N(S)|,$$

as required (see Figure 4 for various examples of S).

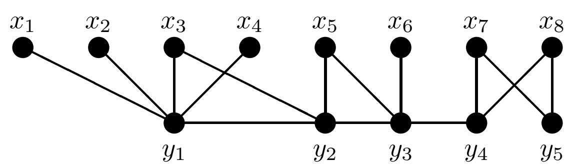

Figure 4: $S_0 = \{x_i : 1 \leq i \leq 8\}, M = \{y_1x_4, y_2x_5, y_3x_6, y_4x_7, y_5x_8\}$, $S = S_1 \cup S_2 \cup S_3$, where $S_2 = \{x_5\}, S_3 = \{y_4, y_5\}$, while S_1 belongs to $\{\{x_1, x_2\}, \{x_1x_3\}, \{x_3\}\}$.

Since $\mathrm{core}(G)$ is an independent set of G,

$$\alpha(G) - \mu(G) \geq |\mathrm{core}(G)| - |N(\mathrm{core}(G))|.$$

Since G is a König-Egerváry graph

$$\alpha(G) + \mu(G) = |V(G)| = |\mathrm{core}(G)| + |N(\mathrm{core}(G))| + |V(G - N[\mathrm{core}(G)])|.$$

Assuming that

$$\alpha(G) - \mu(G) > |\mathrm{core}(G)| - |N(\mathrm{core}(G))|,$$

we obtain the following contradiction

$$2\alpha(G) > 2|\mathrm{core}(G)| + |V(G - N[\mathrm{core}(G)])| = 2|\mathrm{core}(G)| + 2\alpha(V(G - N[\mathrm{core}(G)])) = 2\alpha(G),$$

because $|V(G - N[\mathrm{core}(G)])| = 2\alpha(V(G - N[\mathrm{core}(G)]))$ by Theorem 2.3*(i)*.

Therefore, we get that $\alpha(G)-\mu(G) = |\mathrm{core}(G)|-|N(\mathrm{core}(G))|$. Actually, this equality immediately follows from Theorem 2.3*(ii)*,*(iii)*, but the current way of proof exploits different aspects of $\mathrm{Ind}(G)$.

Further, using the inequality $d(G) \leq \alpha(G) - \mu(G)$ and the equality

$$\alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))|,$$

we finally deduce that

$$\begin{aligned} |\mathrm{core}(G)| - |N(\mathrm{core}(G))| &\leq \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\} = d(G) \\ &\leq \alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))|, \end{aligned}$$

i.e.,

$$\alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = d(G).$$

Since G is a König-Egerváry graph,

$$\alpha(G) - \mu(G) = \alpha(G) + \mu(G) - 2\mu(G) = |V(G)| - 2\mu(G) = def(G),$$

and this completes the proof. ■

Corollary 2.5 *If G is a König-Egerváry graph, then $d(G) = 0$ if and only if G has a perfect matching.*

Remark 2.6 *There exist non-König-Egerváry graphs enjoying the equalities*

$$d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G),$$

see, for instance, the graph G from Figure 5.

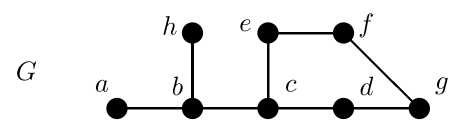

Figure 5: G has $\alpha(G) = 4, \mu(G) = 3$, $\mathrm{core}(G) = \{a, h\}$ and $N(\mathrm{core}(G)) = \{b\}$.

Theorem 2.7 *The following assertions are equivalent:*
(i) *G is a König-Egerváry graph;*
(ii) *there is $S \in \Omega(G)$, such that S is critical, i.e., $\alpha_c(G) = \alpha(G)$;*
(iii) *every $S \in \Omega(G)$ is critical.*

Proof. *(i) $\Longrightarrow$ (iii)* Let $S \in \Omega(G)$, $A = S-\mathrm{core}(G)$ and $B = V - S-\mathrm{core}(G)$. By Proposition 2.3, we get that $|A| = |B|$, since $G - N[\mathrm{core}(G)]$ has a perfect matching. Hence, we obtain that:

$$\begin{aligned} |S| - |N(S)| &= |A| + |\mathrm{core}(G)| - (|B| + |N(\mathrm{core}(G))| \\ &= |\mathrm{core}(G)| - |N(\mathrm{core}(G))|. \end{aligned}$$

In other words, according to Theorem 2.4*(ii)*, the equality $|S| - |N(S)| = d(G)$ is true for each $S \in \Omega(G)$.

(iii) $\Longrightarrow$ (ii) It is clear.

(ii) $\Longrightarrow$ (i) This was done in [15]. For the sake of completeness we add the proof.

There is a critical independent set S with $|S| = \alpha_c(G) = \alpha(G)$. By Proposition 1.2, there exists a matching M from $N(S)$ into S, and clearly, $|M| = |N(S)| = \mu(G)$. Hence, we finally obtain that $|V(G)| = |S| + |N(S)| = \alpha(G) + \mu(G)$, i.e., G is a König-Egerváry graph. ■

3 Conclusions

In this paper we give a new characterization of König-Egerváry graphs. On the one hand, it is similar in form to Sterboul's theorem [26]. On the other hand it extends Larson's finding [15]. We found that the critical difference of a König-Egerváry graph G is given by

$$d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G) = def(G).$$

It seems interesting to find other families of graphs satisfying these equalities.

References

[1] A. A. Ageev, *On finding critical independent and vertex sets*, SIAM J. Discrete Mathematics **7** (1994) 293–295.

[2] J. - M. Bourjolly, P. L. Hammer, B. Simeone, *Node weighted graphs having König-Egervary property*, Math. Programming Study **22** (1984) 44-63.

[3] J. M. Bourjolly, W. R. Pulleyblank, *König-Egerváry graphs, 2-bicritical graphs and fractional matchings*, Discrete Applied Mathematics **24** (1989) 63–82.

[4] E. Boros, M. C. Golumbic, V. E. Levit, *On the number of vertices belonging to all maximum stable sets of a graph*, Discrete Applied Mathematics **124** (2002) 17–25.

[5] S. Butenko, S. Trukhanov, *Using Critical Sets to Solve the Maximum Independent Set Problem*, Operations Research Letters **35** (2007) 519-524.

[6] E. DeLaVina, *Written on the Wall II, Conjectures of Graffiti.pc*, http://cms.dt.uh.edu/faculty/delavinae/research/wowII/

[7] R. W. Deming, *Independence numbers of graphs - an extension of the König-Egerváry theorem*, Discrete Mathematics **27** (1979) 23–33.

[8] J. Edmonds, *Paths, trees and flowers*, Canadian Journal of Mathematics **17** (1965) 449-467.

[9] E. Egerváry, *On combinatorial properties of matrices*, Matematikai Lapok **38** (1931) 16–28.

[10] S. Földes, P. L. Hammer, *Split graphs*, Proceedings of 8th Southeastern Conference on Combinatorics, Graph Theory and Computing (F. Hoffman *et al.* eds), Louisiana State University, Baton Rouge, Louisiana, 311–315.

[11] D. König, *Graphen und Matrizen*, Matematikai Lapok **38** (1931) 116–119.

[12] E. Korach, *On dual integrality, min-max equalities and algorithms in combinatorial programming*, University of Waterloo, Department of Combinatorics and Optimization, Ph.D. Thesis, 1982.

[13] E. Korach, T. Nguyen, B. Peis, *Subgraph characterization of Red/Blue-Split Graph and König-Egerváry graphs*, Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM Press (2006) 842-850.

[14] C. E. Larson, *A note on critical independence reductions*, Bulletin of the Institute of Combinatorics and its Applications 5 (2007) 34-46.

[15] C. E. Larson, *A new characterization of König-Egerváry graphs*, The 2^{nd} Canadian Discrete and Algorithmic Mathematics Conference, May 25-28, 2009, CRM Montreal (Canada).

[16] V. E. Levit, E. Mandrescu, *Well-covered and König-Egerváry graphs*, Congressus Numerantium **130** (1998) 209–218.

[17] V. E. Levit, E. Mandrescu, *A new greedoid: the family of local maximum stable sets of a forest*, Discrete Applied Mathematics **124** (2002) 91-101.

[18] V. E. Levit, E. Mandrescu, *Combinatorial properties of the family of maximum stable sets of a graph*, Discrete Applied Mathematics **117** (2002) 149-161.

[19] V. E. Levit, E. Mandrescu, *On α^+-stable König-Egerváry graphs*, Discrete Mathematics **263** (2003) 179–190.

[20] V. E. Levit, E. Mandrescu, *On α-critical edges in König-Egervary graphs*, Discrete Mathematics **306** (2006) 1684-1693.

[21] L. Lovász, *Ear decomposition of matching covered graphs*, *Combinatorica* **3** (1983) 105-117.

[22] L. Lovász, M. D. Plummer, *Matching Theory*, Annals of Discrete Mathematics **29** (1986) North-Holland.

[23] G. L. Nemhauser and L. E. Trotter, Jr., *Vertex packings: structural properties and algorithms*, Mathematical Programming **8** (1975) 232-248.

[24] V. T. Paschos, M. Demange, *A generalization of König-Egerváry graphs and heuristics for the maximum independent set problem with improved approximation ratios*, European Journal of Operational Research **97** (1997) 580–592.

[25] W. R. Pulleyblank, *Matchings and Extensions*, in: *Handbook of Combinatorics, Volume 1* (eds. R. L. Graham, M. Grotschel and L. Lovász), MIT Press and North-Holland, Amsterdam (1995) 179-232.

[26] F. Sterboul, *A characterization of the graphs in which the transversal number equals the matching number*, Journal of Combinatorial Theory Series B **27** (1979) 228–229.

[27] C. Q. Zhang, *Finding critical independent sets and critical vertex subsets are polynomial problems*, SIAM J. Discrete Mathematics **3** (1990) 431-438.