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ORIGINAL PAPER

Tank-Ring Factors in Supereulerian Claw-Free Graphs

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Abstract A graph G has a tank-ring factor F if F is a connected spanning subgraph with all vertices of degree 2 or 4 that consists of one cycle C and disjoint triangles attaching to exactly one vertex of C such that every component of $G - C$ contains exactly two vertices. In this paper, we show the following results. (1) Every supereulerian claw-free graph G with 1-hourglass property contains a tank-ring factor. (2) Every supereulerian claw-free graph with 2-hourglass property is Hamiltonian.

Keywords Connected even factor · Cycle · Claw-free graph · Tank-ring factor

1 Introduction

We will consider the class of undirected finite graphs without loops or multiple edges, and use [1] for terminology and notation not defined here. Let G be a graph. We denote by $\Delta(G)$ the maximum degree of G. For a vertex v of G, the neighborhood of v is the

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Fig. 1 Hourglass

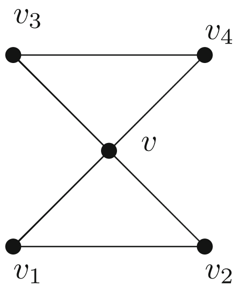
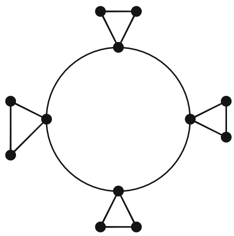


Fig. 2 Tank-ring Factor



set of all vertices that are adjacent to v, and will be denoted by $N(v)$. For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G - H$ and $G[S]$ the induced subgraphs of G by $V(G) - V(H)$ and S, respectively. We denote by $N_H(S)$ the set of all vertices of H adjacent to some vertex of S, and let $N(S) = \bigcup_{x \in S} N(x)$ and $d_H(S) = |N_H(S)|$. For a cycle C with a fixed orientation, and two vertices x and y on C, we define the segment $C(x, y)$ to be the set of vertices on C from x to y (excluding x and y), and x^+ and x^- denote the successor and the predecessor of x according to the orientation of C, respectively. A cycle of length k is called a k-cycle. A Hamiltonian cycle in a graph is a cycle that passes through all vertices of the graph. A graph is called *claw-free* if it does not contain a copy of $K_{1,3}$ as an induced subgraph. An *hourglass* is the unique graph with degree sequence 4,2,2,2,2 (i.e., two triangles meeting in exactly one vertex) (see Fig. 1). The vertex of degree 4 in an hourglass is called *the center of the hourglass*, and in labelling an induced hourglass we always use its center as first vertex of an induced hourglass. A connected factor of a graph is its connected spanning subgraph. A connected *even* [2,2s]-*factor* of a graph G is a connected factor with all vertices of degree $i(i = 2, 4, \dots, 2s)$, where $s \geq 1$ is an integer. In particular, a connected even factor with all vertices of degree 2 or 4 is called a *connected* [2,4]-*factor*. A graph G has a *tank-ring factor* F if F is a connected spanning subgraph with all vertices of degree 2 or 4 that consists of one cycle C and disjoint triangles attaching to exactly one vertex of C such that every component of $G - C$ contains exactly two vertices (see Fig. 2). Thus a Hamiltonian cycle is a connected even [2,2]-factor, and is a tank-ring factor without triangles. A connected even factor F can be decomposed into edge disjoint cycles $C_1, C_2, \dots, C_k$, which are said *a cycle decomposition of* F. A *trail* is a sequence $u_0e_1u_1e_2 \dots e_ru_r$ with alternative vertices and edges and with no repeated edges and $e_i = u_{i-1}u_i (1 \leq i \leq r)$. A graph G is *supereulerian* if G has a spanning closed trail (not necessary to contain every edge). Matthews and Sumner [8] made the following conjecture.

Conjecture 1 [8]. *Every* 4*-connected claw-free graph is Hamiltonian.*

Broersma et al. [2] proved the following results.

Theorem 2 [2] *Every* 4*-connected claw-free graph has a connected* [2,4]*-factor.*

Every 4-edge-connected graph is supereulerian [4] (also see [3]). Theorem 2 has been generalized as follows in [6]

Theorem 3 [6]. *Every supereulerian claw-free graph contains a connected even* [2,4]*-factor.*

A graph G is said to have 1*-hourglass property* (or *hourglass property*) if one pair of nonadjacent vertices of every induced hourglass H in G has a common neighbor outside H. A graph G is said to have 2*-hourglass property* if there are two pairs $\{v_1, v_3\}$ and $\{v_2, v_4\}$ of nonadjacent vertices of every induced hourglass H in G such that each pair has a common neighbor outside H and v_1v_2 and v_3v_4 are two disjoint edges in H. Obviously, 2-hourglass property implies 1-hourglass property. Recently, Kaiser et al. [5] showed the following result using hourglass property, which is a special case of Conjecture 1.

Theorem 4 [5]. *Every* 4*-connected claw-free graph with* 1*-hourglass property is Hamiltonian.*

An example in [5] shows that the 4-connectivity condition in Theorem 6 is required. Thus, a natural problem is that if the condition of 4-connectivity is reduced, then what conclusions can we obtain? In this paper, we will explore the problem and obtain the following results.

Theorem 5 *Every supereulerian claw-free graph with* 1*-hourglass property contains a tank-ring factor.*

Corollary 6 *Every* 4*-edge-connected claw-free graph with* 1*-hourglass property contains a tank-ring factor.*

Corollary is best possible in the sense that the edge-connectivity of a graph G cannot be reduced to 3. To see this, we copy an example from [5]. Let k be a negative integer and $G(k)$ the graph obtained from the Petersen graph PTS_{10} by adding at least k edges to every vertex and subdividing every original edge of PTS_{10}. Then the line graph $L(G(k))$ is 3-edge-connected claw-free graph and has no induced hourglass (and hence has 1-hourglass property). But $G(k)$ contain no tank-ring factors.

A graph G is said to have *weak* 2*-hourglass property* if there are two pairs of nonadjacent vertices of every induced hourglass H in G such that each pair has a common neighbor outside H. Obviously, 2-hourglass property implies weak 2-hourglass property. The graph in Figure 3 is a supereulerian graph with weak 2-hourglass property but not Hamiltonian, which shows that 2-hourglass property in Theorem 7 is required. If we replace 1-hourglass property in Theorem 5 by 2-hourglass property, we obtain the following result.

Theorem 7 *Every supereulerian claw-free graph with* 2*-hourglass property is Hamiltonian.*

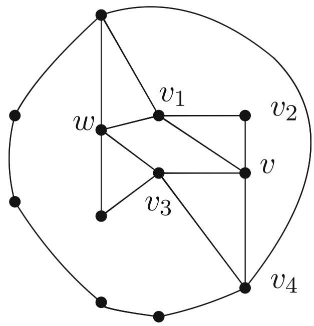


Fig. 3 A non-hamiltonian claw-free graph with weak 2-hourglass property. $\{v_1, v_3\}$, $\{v_1, v_4\}$

Corollary 8 *Every* 4*-edge-connected claw-free graph with* 2*-hourglass property is hamiltonian.*

To show the sharpness of Corollary 8, we copy an example from [5]. Let k be an negative integer and $G(k)$ the graph obtained from the Petersen graph PTS_{10} by adding at least k edges to every vertex and subdividing every original edge of PTS_{10}. Then the line graph $L(G(k))$ of $G(k)$ is 3-edge-connected claw-free graph and has no induced hourglass (and hence has 2-hourglass property). But $L(G(k))$ is not hamiltonian.

2 Proofs of Main Results

In this section, our aim is to prove our results. In order to prove our other theorems, we first show the following lemmas. Recall that if a connected even factor F can be decomposed into edge disjoint cycles $C_1, C_2, \ldots, C_k$, then we say that $C_1, C_2, \ldots, C_k$ is *a cycle decomposition of* F. The proofs of Lemmas 2.1–2.2 were given in [7]. However, the paper [7] was unpublished, so we provide these simple proofs for confirming the correction here.

Lemma 2.1 *Let F be a connected* [2,4]*-factor with minimal number of vertices of degree* 4 *in a claw-free graph G. Then every vertex of degree 4 in F is the center of an induced hourglass in G.*

Proof If F has no vertex of degree 4, then we are done. Let S be a set of vertices of degree 4 in F. Let v be any vertex of degree 4 in F, and let $N_F(v) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality, assume that v_1, v, v_2 and v_3, v, v_4 are contained in two distinct cycles with a common vertex v, respectively, of a cycle decomposition of F. If $v_1v_3 \in E(G)$, then $F - vv_1 - vv_3 + v_1v_3$ is a connected [2,4]-factor with fewer vertices of degree 4 than F, a contradiction. Hence $v_1v_3 \notin E(G)$. By symmetry, v_1v_4, v_2v_3, v_2v_4 are not edges of G. Since G is claw-free, this fact implies that v_1v_2 and v_3v_4 are edges of G. Thus v is the center of the induced hourglass $G[v, v_1, v_2, v_3, v_4]$. Thus Lemma 2.1 is true. □

Lemma 2.2 [7] *Let F be a connected* [2,4]*-factor with minimal number of vertices of degree* 4 *in a claw-free graph G and $C_1, C_2, \ldots, C_p$ be a cycle decomposition of F. Then any pair of cycles C_i and C_j in F meets at most one vertex.*

Proof Assume that $P = \{C_1, C_2, \ldots, C_p\}$. If $|V(C_i) \cap V(C_j)| \geq 2 (i \neq j)$, then at least one of $\{C_i, C_j\}$ has order at least 4 (say C_i). Let $u \in V(C_i) \cap V(C_j)$. Then u is the center of the induced hourglass $G[N_F(u) \cup \{u\}]$ by Lemma 2.2 Let $N_F(u) = \{u_1, u_2, u_3, u_4\}$ and $u_1, u_2 \in V(C_i)$ and $u_3, u_4 \in V(C_j)$, and let $u_1u_2 \in E(G)$. If $u_1u_2 \notin E(F)$, then $F' = F - u_1u - u_2u + u_1u_2$ is a [2,4]-factor with fewer vertices of degree 4 than F. Since $|V(C_i) \cap V(C_j)| \geq 2$ and C_i contains at least 4 vertices, F' is connected. This contradiction shows $u_1u_2 \in E(F)$. Thus uu_1u_2u is a triangle in F and $d_F(u_1) = d_F(u_2) = 4$, and so C_i contains at least 4 vertices. Removing the edges uu_1, u_1u_2, u_2u from F, we obtain a new $[2, 4]$-factor F'' containing fewer vertices of degree 4 than F. Since $|V(C_i) \cap V(C_j)| \geq 2$, F'' is connected. This contradiction shows $|V(C_i) \cap V(C_j)| \leq 1$. This completes the proof of Lemma 2.2. $\square$

A cycle C of a graph G is said to have 2-*component property* if every component of $G - C$ contains at most two vertices. In the following, we will prove a stronger result than Theorem 5. That is,

Theorem 2.3 *Let G be a connected claw-free graph with* 1*-hourglass property and contain a connected* [2,4]*-factor. Then G contains a tank-ring factor.*

Proof Let F be a connected $[2, 4]$-factor in a connected claw-free graph G. Furthermore, assume that F contains least number of vertices of degree 4 in F among connected $[2, 4]$-factors of G. Let S be the set of vertices of degree 4 in F and $m = |S|$. If $m = 0$, then we are done. Thus $m \geq 1$. By Lemma 2.1, every vertex of S is the center of an induced hourglass of G. Let $C_1, C_2, \ldots, C_p$ be a cycle decomposition of F and let $P = \{C_1, C_2, \ldots, C_p\}$. Then, by Lemma 2.2, $|V(C_i) \cap V(C_j)| \leq 1$ for $i, j = 1, 2, \ldots, p (i \neq j)$. Obviously, if $V(C_i) \cap V(C_j) \neq \emptyset$, then $V(C_i) \cap V(C_j) \subseteq S$.

Since $m \geq 1$, $p = |P| \geq 2$. Without loss of generality assume that C_1 is a longest cycle among $\{C_1, C_2, \ldots, C_p\}$ and C_1 contains as many vertices as possible. We say that two distinct cycles C_i and C_j are adjacent if $V(C_i) \cap V(C_j) \neq \emptyset$. Let

$$A_1 = \{C_k \in P : C_k \text{ is adjacent to } C_1\}.$$

Then we have the following fact.

Claim 1 *Let F be a connected* $[2, 4]$*-factor with minimal value of* $|S|$*, and subject to this, C_1 a longest cycle. If $C_i \in A_1$, then* $|V(C_i)| = 3$.

Proof Otherwise, let v be a vertex such that $v \in V(C_i)$ and $v \in V(C_1)$. Then $v \in S$, and then v is the center of the induced hourglass $H := G[N_F(v) \cup \{v\}]$ by Lemma 2.1. Let v_i be the same as in Lemma 2.1 for $i = 1, 2, 3, 4$, and $v_1, v_2 \in V(C_i)$ and $v_3, v_4 \in V(C_1)$. Assume $i = 2$ and $C_1 = (v_3 \ldots v_4vv_3)$ and $C_2 = (vv_2 \ldots v_1v)$, then $v_3^- = v$, $v_4^+ = v$ (on C_1) and $v_2^- = v$, $v_1^+ = v$ (on C_2). Since v is the center of the hourglass H, there is one vertex w outside of H such that w is adjacent to v_3 or v_4 and v_1 or v_2. By symmetry, without loss of generality assume that $wv_2, wv_4 \in E(G)$.

Suppose that $w \in V(C_j)$ for $j \geq 3$. If $w^+v_4 \in E(G)$, then $F - ww^+ - vv_4 - vv_2 + v_2w + v_4w^+$ is a connected $[2, 4]$-factor with fewer vertices of degree 4 than F, a contradiction. Thus $w^+v_2 \in E(G)$. Removing the edges ww^+ (on C_3), vv_4 (on C_1),

vv_2 (on C_2) and adding the edges v_2w^+, v_4w into F, we obtain a new connected [2, 4]-factor containing fewer vertices of degree 4 than F, a contradiction. Thus $w \notin V(C_j)$ for $j \geq 3$. We further have the following fact.

Claim 1.1 *$w \notin V(C_2)$, that is, $w \in V(C_1)$.*

Proof Assume that $w \in V(C_2)$. If $|V(C_2(v_2, w))| \geq 1$, then removing v_2v, vv_4 from F and adding wv_2, wv_4 into F we obtain two new cycles C_1' and C_2' and C_1' contains more vertices than C_1, a contradiction. Note that $V(C_1') \cap V(C_2') = \{w\}$. Thus $V(C_2(v_2, w)) = \emptyset$. That is, $w = v_2^+$. Removing vv_1, vv_4, $v_2^+v_2$ from F and adding v_1v_2, $v_2^+v_4$ into F we obtain a new connected [2, 4]-factor which contains fewer vertices of degree 4 than F, a contradiction. Thus Claim 1.1 is true.

Now we complete the proof of Claim 1.

By Claim 1.1, $w \in V(C_1)$. If $w^+w^- \in E(G)$, then replacing w^+ww^- by w^+w^- and removing the edges vv_2 from C_2 and vv_4 from C_1 and adding the path v_2wv_4, we obtain a new cycle C' and a new connected [2, 4]-factor $F' = C' \cup_{j=3}^{j=p} C_j$ containing fewer vertices of degree 4 than F, a contradiction. Thus $w^+w^- \notin E(G)$. Since $G[w, w^+, w^-, v_2] \neq K_{1,3}$, $w^+v_2 \in E(G)$ or $w^-v_2 \in E(G)$ (say $w^+v_2 \in E(G)$). If $vv_2^+ \in E(G)$, then replacing $vv_2v_2^+$ by vv_2^+ on C_2 we obtain a cycle C_2', and replacing ww^+ by wv_2w^+ on C_1 we obtain a cycle C_1'. Note that $C_1' \cup C_2' \cup_{j=3}^{j=p} C_j$ is a connected [2, 4]-factor with the same number of vertices of degree 4 as F and C_1' contains more vertices than C_1, a contradiction. Thus $vv_2^+ \notin E(G)$.

We have $vw \notin E(G)$ since otherwise, removing the edges v_2v, vv_4, vv_3 from F and adding v_2w^+, vw, v_4v_3 into F, we obtain a new connected [2, 4]-factor F' which contains fewer vertices of degree 4 than F, a contradiction. Thus $vw \notin E(G)$. Since $G[v_2, v_2^+, v, w] \neq K_{1,3}$, $v_2^+w \in E(G)$. Removing edges $v_2v_2^+$, ww^+, v_2v, vv_1 from F and adding the edges v_2^+w, v_2w^+, v_2v_1 into F, we obtain a new connected [2, 4]-factor F' which contains fewer vertices of degree 4 than F, a contradiction. Thus Claim 1 is proved.

Claim 2 $A_1 = P - \{C_1\}$.

Proof Assume that there is a cycle C_j in $P - \{C_1\}$ such that $C_j \notin A_1$. Then we can find two distinct cycles (say C_2 and C_p) such that $C_p \in A_1$, $C_2 \notin A_1$ and $C_2 \cap C_p \neq \emptyset$. Let $V(C_p) = \{u, v, z\}$ and $v \in V(C_p) \cap V(C_1)$ and $u \in V(C_p) \cap V(C_2)$. Then $u, v \in S$. Let $N_F(u) = \{u_1, u_2, v, z\}$ and $N_F(v) = \{v_1, v_2, u, z\}$. Then, by Lemma 2.1, $G[N_F(u) \cup \{u\}]$ and $G[N_F(v) \cup \{v\}]$ are hourglasses and $v, z \notin N(\{u_1, u_2\})$ and $u, z \notin N(\{v_1, v_2\})$. From the assumption of Theorem 2.3, there is a vertex w outside of G[$N_F(v)$] such that w is adjacent to v_1 or v_2 and z or u since $G[N_F(v) \cup \{v\}]$ is an hourglass. By symmetry, assume that $v_2w \in E(G)$. Let $C_1 = (vv_1 \ldots v_2v)$ and $C_2 = (uu_1 \ldots u_2u)$. Then $v_1^- = v$ and $v_2^+ = v$ (on C_1) and $u_1^- = v$ and $u_2^+ = v$ (on C_2). We further have the following fact.

Claim 2.1 *If $uw \in E(G)$, then $w \notin V(C_1)$.*

Proof Otherwise, we have $w \neq v_2^-$ since otherwise removing the edges v_1v, v_2w (on C_1), vu (on C_p) and adding the edges v_1v_2, uw into F we obtain a new connected

$[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$, a contradiction. Similarly, $w \neq v_1^+$.

Note that $w^+w^- \notin E(G)$ since otherwise replacing the path w^-ww^+ by the edge w^-w^+ on C_1 and removing vu (on C_p), vv_2 (on C_1) and adding the edges v_2w and wu into F, we obtain a connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$. Since $G[w, w^-, w^+, u] \neq K_{1,3}$, $w^-u \in E(G)$ or $w^+u \in E(G)$ (say $w^-u \in E(G)$). Since $G[u, z, u_1, w] \neq K_{1,3}$ and $zu_1 \notin E(G)$, $u_1w \in E(G)$ or $zw \in E(G)$.

If $u_1w \in E(G)$, then removing the edges ww^-, v_1v, vv_2 (on C_1) and uu_1 (on C_2) and adding the edges u_1w, uw^-, v_1v_2 into F, we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since v is not vertex of degree 4 in F'. Thus $u_1w \notin E(G)$. It follows that $zw \in E(G)$. Removing the edges v_1v, vv_2, ww^- (on C_1), zu (on C_p) and adding the edges zw, uw^-, v_1v_2 into F we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$. This contradiction shows that Claim 2.1 is true. □

Claim 2.2 *If $uw \in E(G)$, then $w \notin V(C_2)$.*

Proof Since we did not use the maximality of C_1 in the proof of Claim 2.1, by symmetry, Claim 2.2 is true.

Claim 2.3 $uw \notin E(G)$.

Proof Otherwise, by Claims 2.1 and 2.2, $w \notin V(C_1 \cup C_2)$, and so there is a cycle (say C_3) such that $w \in V(C_3)$. Since $G[w, w^-, v_2, u] \neq K_{1,3}$ and $uv_2 \notin E(G)$, $w^-v_2 \in E(G)$ or $w^-u \in E(G)$. If $w^-v_2 \in E(G)$, then removing the edges vv_2 (on C_1), vu (on C_p), w^-w on C_3) and adding the edges w^-u and wv_2, we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$, a contradiction. Thus $w^-u \in E(G)$. Similarly, we can get a contradiction. Thus Claim 2.3 is true.

Now we complete the proof of Claim 2.

By Claim 2.3, we have $zw \in E(G)$. A similar argument to the proof of Claim 2.3 shows $w \in V(C_1)$ or $w \in V(C_2)$. If $w \in V(C_2)$, then $w^+ \neq u_2$ since otherwise removing uu_1 and zu and adding zu_2^+ we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = 2$. If $zw^+ \in E(G)$, then removing the edges ww^+ (on C_2), vv_2 (on C_1), vu, zu (on C_p) and adding the edges zw^+ and v_2w, we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = d_{F'}(v) = 2$. Thus $zw^+ \notin E(G)$. It follows that $v_2w^+ \in E(G)$. Removing vv_2 (on C_1), vu, uz (on C_p) and ww^+ (on C_2) and adding the edges zw and w^+v_2, we obtain a new connected $[2, 4]$-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = d_{F'}(v) = 2$. This contradiction shows $w \notin V(C_2)$. Thus $w \in V(C_1)$. By a similar argument to Claim 1.1, we have $w^+w^- \notin E(G)$. Since $G[w, w^+, w^-, z] \neq K_{1,3}$, $w^+z \in E(G)$ or $w^-z \in E(G)$ (say $w^+z \in E(G)$).

Since $G[N_F(u) \cup \{u\}]$ is an hourglass, there is a common neighbor w' outside of $G[N_F(u) \cup \{u\}]$ such that w' is adjacent to u_1 or u_2 and z or v. A similar argument to Claim 2.3 shows that $w'v \notin E(G)$. Thus $w'z \in E(G)$. By symmetry, assume that

$u_1w' \in E(G)$. By a similar argument to the above, we have that $w' \in V(C_2)$ and $zw'^{+} \in E(G)$ or $w'^{-}z \in E(G)$ (say $w'^{+}z \in E(G)$).

If $ww' \in E(G)$, then removing the edges ww^{+} (on C_1), $w'w'^{+}$ (on C_2), zu, zv, vu (on C_p) and adding the edges zw^{+}, ww' and $w'^{+}z$, we obtain a new connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = d_{F'}(v) = 2$. Thus $ww' \notin E(G)$. Since $G[z, w, w', v] \neq K_{1,3}$, $vw \in E(G)$. Removing the edges ww^{+}, v_2v, vv_1 (on C_1), zv (on C_p) and adding the edges zw^{+}, wv and v_1v_2 into F, we obtain a new connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$. Thus Claim 2 is true.

Let

$$S_1 = V(G) - V(C_1).$$

Then we have the following Claim.

Claim 3 *For any vertex v in S, $G[\{v\} \cup N_F(v)]$ has no common neighbor w in $S_1 - N_F(v)$.*

Proof Otherwise, let $N_F(v) = \{v_1, v_2, v_3, v_4\}$ and $v_1, v_2 \in S_1$ and $v_3, v_4 \in V(C_1)$. Then $v_1v_2 \in E(G)$ and $v_3v_4 \in E(G)$. By symmetry, assume that $v_1w, v_3w \in E(G)$. Let $G[\{u\} \cup N_F(u)]$ be another hourglass and $N_F(u) = \{u_1, u_2, u_3, u_4\}$ (where $u_1, u_2 \in S_1$ and $u_3, u_4 \in V(C_1)$). Then $u_1u_2, u_3u_4 \in E(G)$. Without loss of generality assume that $w = u_1$ and $C_1 = (v_3 \ldots u_3uu_4 \ldots v_4vv_3)$. Then $uv_3 \notin E(G)$ since otherwise removing the edges u_3u, uu_4, v_3v (on C_1), v_1v, u_1u and adding v_3u, u_3u_4, v_1u_1, we obtain a new connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = d_{F'}(v) = 2$. Thus $uv_3 \notin E(G)$. Since $G[u_1, v_1, v_3, u] \neq K_{1,3}$, $uv_1 \in E(G)$. Similarly, we can obtain a contradiction. Thus Claim 3 is true.

Two vertices x and y are *consecutive* on the cycle C_1 if xy is the edge on C_1 (i.e., $x^{+} = y$ or $x^{-} = y$). We have the following fact.

Claim 4 *For any vertex v in S, either v_1 or v_2 has two consecutive neighbors on C_1, where $v_1, v_2 \in N_F(v) \cap S_1$ and $v_1v_2 \in E(G)$. If v_2 has no two consecutive neighbors on C_1 but has neighbor w_1 on C_1, then $w_1^{-}w_1^{+} \in E(G)$.*

Proof Let $N_F(v) = \{v_1, v_2, v_3, v_4\}$. Then $v_3, v_4 \in V(C_1)$ and $v_3v_4 \in E(G)$. By 1-hourglass property, there is a vertex w outside $G[N_F(v) \cup \{v\}]$ such that w is adjacent to v_1 or v_2 and v_3 or v_4. By symmetry, assume that $v_1w, v_3w \in E(G)$. By Claim 3, $w \in V(C_1)$. Let $C_1 = (v_3 \ldots w \ldots v_4vv_3)$. Then $v_3^{-} = v$. By a similar argument to Claim 1.1, we have $w^{+}w^{-} \notin E(G)$. Since $G[w, w^{+}, w^{-}, v_1] \neq K_{1,3}$, $w^{+}v_1 \in E(G)$ or $w^{-}v_1 \in E(G)$ (say $w^{+}v_1 \in E(G)$). Thus v_1 has two consecutive neighbors on C_1. If v_2 also has two consecutive neighbors x and y on C_1, then replacing the edge ww^{+} by the path wv_1w^{+} and the edge xy by the path xv_2y and removing the edges v_1v, v_2v, v_1v_2 from F, we obtain a new connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(v) = 2$. This contradiction shows that v_2 has no two consecutive neighbors on C_1. If v_2 has a neighbor x on C_1, then $x^{+}x^{-} \in E(G)$ since $x^{+}v_2, x^{+} - v_2 \notin E(G)$ and $G[x, x^{+}, x^{-}, v_2] \neq K_{1,3}$. Thus we have proved Claim 4.

Let

$$T = \{x \in S_1 : x \text{ has no consecutive neighbors on } C_1\}$$

Then $|T| \geq 1$ since otherwise the tank-ring factor is hamiltonian and so $S_1 = \emptyset$ and we are done. We further have the following fact.

Claim 5 *For any vertex z in T, z has at most one neighbor in T.*

Proof Suppose that $G[N_F(v) \cup \{v\}]$ is an hourglass and and $N_F(v) = \{z, z_1, v_1, v_2\}$, where $v, v_1, v_2 \in V(C_1)$, $z, z_1 \notin V(C_1)$ and $v_1v_2 \in E(G)$. If $d_T(z) \geq 2$, assume $zx, zy \in E(G)$ and $x, y \in T$. Obviously, $yz_1, z_1x \notin E(G)$ since otherwise, e.g., $xz_1 \in E(G)$, using the new cycle $vzxz_1v$ and Claim 4, G has a new connected [2, 4]-factor containing fewer vertices of degree 4, a contradiction. Similarly, $yv, xv \notin E(G)$. Since $G[z, z_1, x, y] \neq K_{1,3}$, $xy \in E(G)$ and then $G[z, Z_1, v, x, y]$ induces an hourglass. It follows that there is a vertex w such that w is adjacent to x or y and v or z_1. Assume that $xw, vw \in E(G)$ (the proofs of other cases are similar). Then we have $w \in V(C_1)$ since otherwise assume $N_F(u) \cup \{u\} = \{w, w_1, u_1, u_2\} \cup \{u\}$ induces an hourglass, where $ww_1, u_1u_2 \in E(G)$ and $u_1, u_2 \in V(C_1)$ and $w_1, w \notin V(C_1)$. Then $w_1x, w_1v \notin E(G)$ since otherwise, e.g., $w_1x \in E(G)$, using the cycle $w_1xzz_1vww_1$, we get a new connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(u) = 2$, a contradiction. Thus $w \in V(C_1)$. By Claim 4, $w^+w^- \in E(G)$. Using the cycle vz_1zxwv and the edge w^+w^- and Claim 4, we can get a contradiction. Thus Claim 5 is true.

Claim 6 *For any vertex z in T, z has only one neighbor in $S_1 - T$.*

Proof Obviously, z has at least one neighbor in $S_1 - T$. If z has two neighbors z_1, y in $S_1 - T$, then $y \notin T$ and y has two neighbors w, w^+ on C_1 by Claim 4. Assume $N_F(y') = \{y, y_1, y'_1, y'_2\}$, where $y, y_1 \notin V(C_1)$ and $y', y'_1, y'_2 \in V(C_1)$ and $yy_1, y'_1y'_2 \in E(G)$. The $G[y', y, y_1, y'_1, y'_2]$ induces an hourglass. Let $V(C_1) = (y'_1y'y'_2 \ldots ww^+ \ldots v_1vv_2 \ldots y'_1)$. Then we have from the proof of Claim 4 that $wy'_2 \in E(G)$. We have $zy' \notin E(G)$ since otherwise we use the cycle $y'y_1yzy'$ and Claim 4 to get a connected [2, 4]-factor containing fewer vertices of degree 4 than F. Obviously, $wy' \notin E(G)$. Since $G[y, w, y', z] \neq K_{1,3}$, $wz \in E(G)$. By Claim 4, $w^+w^- \in E(G)$ since $z \in T$. Removing the edges $y'y'_2$, w^-w, ww^+ (on C_1) and yy' and adding the edges y'_2w, wy, w^-w^+ into F we get a connected [2, 4]-factor F' containing fewer vertices of degree 4 than F since $d_{F'}(y') = 2$. Thus Claim 6 is true.

By Claims 5 and 6, for any vertex z in T, z has at most two neighbors in S_1. By Claim 4, we put all vertices of S_1 with consecutive neighbors in C_1 into C_1, we get a cycle C such that every component of $G - C$ has at most two vertices by Claims 5 and 6. Thus we complete the proof of Theorem 2.3. □

By Theorem 2.3, we easily obtain that Theorem 5 is true.

The concept of the claw-free closure was defined in [9]. A graph G is called a *closed claw-free graph* if for any vertex v of G, $G[N(v)]$ is either a clique or an union of two cliques. We know from [9] that the closure $cl(G)$ of a claw-free graph G is a line graph of some triangle-free graph and a closed claw-free graph.

Theorem 2.4 [9]. *Let G be a claw-free graph. Then*

(1) *The closure $cl(G)$ of G is the line graph of some triangle-free graph,*

(2) *G is Hamiltonian if and only if $cl(G)$ is Hamiltonian.*

Using a similar proof to that of Property 3 in [5], we get the following fact.

Lemma 2.5 *Let G be a claw-free graph with 2-hourglass property. Then its closure $cl(G)$ has 2-hourglass property, too.*

In the following, we will prove a stronger result than Theorem 7.

Theorem 2.6 *Let G be a connected claw-free graph with 2-hourglass property and a connected $[2, 4]$-factor F. Then G is Hamiltonian.*

Proof From Theorem 2.4 and Lemma 2.5, we only consider the closure $cl(G)$ of G. Without loss of generality assume that $cl(G) = G$. Let F, $P = \{C_1, \ldots, C_p\}$ and A_1 be the same as in the proof of Theorem 2.3. Then $A_1 = P - \{C_1\}$, and for any cycle $C_j \in A_1$, $|V(C_j)| = 3$. Let S denote the set of vertices of degree 4 in F. If $A_1 = \emptyset$ or $|S| = 0$, then we are done. Thus $|S| \geq 1$ and $|A_1| \geq 1$. By Lemma 2.1, every vertex of S is the center of some induced hourglass in G. Note that F is a connected [2,4]-factor of least number of vertices of degree 4 among all connected [2,4]-factors, and from the proof of Theorem 2.3, F is a tank-ring factor in G (see Claims 1–6 in Theorem 2.3). For a vertex $v \in S$, let vv_1v_2v be a triangle and $G[\{v, v_1, v_2, v_3, v_4\}] = H$ an induced hourglass in G, where $v, v_3, v_4 \in V(C_1)$ and $v_1, v_2 \notin V(C_1)$. Since G satisfies 2-hourglass property, there are two pairs (such as $\{v_1, v_3\}$ and $\{v_2, v_4\}$) of nonadjacent vertices in H such that v_1, v_3 have a common neighbor w_1 outside H and v_2, v_4 have a common neighbors w_2 outside H. By Claims 3 and 4 of Theorem 2.3, these common neighbors w_1, w_2 are on C_1 and so v_1, v_2 have consecutive neighbors on C_1. Inserting v_1, v_2 into C_1, we get a connected [2,4]-factor of fewer vertices of degree 4 than F, a contradiction. Thus we complete the proof of Theorem 2.6. □

From Theorem 3 and Theorem 2.6, we know that Theorem 7 is true.

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