# On graphs having maximal independent sets of exactly $t$ distinct cardinalities 

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#### Abstract

For a given positive integer $t$ we consider graphs having maximal independent sets of precisely $t$ distinct cardinalities and restrict our attention to those that have no vertices of degree one. In the situation when $t$ is four or larger and the length of the shortest cycle is at least $6 t-6$, we completely characterize such graphs.


Keywords: maximal independent set, girth, cycle
AMS subject classification: 05C69, 05C38

## 1 Introduction

A well-covered graph (Plummer [6]) is one in which every maximal independent set of vertices is of one cardinality and is hence a maximum independent set. Finbow, Hartnell and Whitehead [5] defined the class $\mathcal{M}_{t}$ to consist of those graphs which have exactly $t$ different sizes of maximal independent sets. Finbow, Hartnell and Nowakowski [4] proved that the well-covered graphs (the $\mathcal{M}_{1}$ collection) of girth (the length of a shortest cycle) 6 or more, with the exceptions of $K_{1}$ and $C_{7}$, have the property that every vertex has degree one or has exactly one vertex of degree one in its neighborhood. Thus, $C_{7}$ is the unique graph in $\mathcal{M}_{1}$ with girth at least 6

[^0]that has minimum degree at least two. The graphs in $\mathcal{M}_{2}$ of girth 8 or more have also been characterized (5). There are precisely five graphs in $\mathcal{M}_{2}$ of girth at least 8 and minimum degree 2 or more, namely the cycles $C_{8}, C_{9}, C_{10}, C_{11}$ and $C_{13}$. This implies there are no $\mathcal{M}_{1}$ graphs of girth at least 8 with minimum degree 2 or more and no $\mathcal{M}_{2}$ graphs of girth 14 or more and having minimum degree at least 2. For related work on the class $\mathcal{M}_{t}$ see [1] and [2].

In this paper we investigate the graphs in $\mathcal{M}_{t}$ that have minimum degree at least 2 and higher girth and establish that the characterization of these in $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ is part of a general pattern. In particular, for $t \geq 3$ we show that among graphs with minimum degree at least $2, \mathcal{M}_{t}$ does not contain a graph of girth at least $6 t+2$ and that $C_{6 t-4}, C_{6 t-3}, C_{6 t-2}, C_{6 t-1}$ and $C_{6 t+1}$ are the only exceptions for girth at least $6 t-4$. Furthermore, if $t \geq 4$, then these cycles along with $C_{6 t-6}$ are the only graphs in $\mathcal{M}_{t}$ that have minimum degree at least 2 and girth at least $6 t-6$.

Let $G$ be a finite simple graph. A vertex of degree 1 is called a leaf and any vertex that is adjacent to a leaf is called a support vertex. If $C$ is a cycle in a graph $G$ and $u$ and $v$ belong to $C$, we let $u C v$ denote the shorter of the two $u, v$-paths that are part of $C$. For $A \subseteq V(G)$ and $u$ a vertex in $G, d(u, A)$ will denote the length of a shortest path in $G$ from $u$ to a vertex of $A$. We will use $\mathcal{M}(G)$ to denote the collection of all maximal independent sets of $G$ and we define the independence spectrum (spectrum for short) of $G$ to be the set $\mathcal{S}(G)=\{|I|: I \in \mathcal{M}(G)\}$. The class $\mathcal{M}_{t}$ consists of those graphs $G$ for which $|\mathcal{S}(G)|=t$. The spectrum is not necessarily a set of consecutive positive integers (e.g., $\mathcal{S}\left(K_{2,4,5}\right)=\{2,4,5\}$ ), but for paths and cycles it is. We denote the set of positive integers between $p$ and $q$ inclusive by $[p, q]$. The following proposition is easy to establish.

Proposition 1 For each positive integer $n$ at least 3,

$$
\mathcal{S}\left(C_{n}\right)=[\lceil n / 3\rceil,\lfloor n / 2\rfloor] \quad \text { and } \quad \mathcal{S}\left(P_{n}\right)=[\lceil n / 3\rceil,\lceil n / 2\rceil] .
$$

Hence, $C_{n} \in \mathcal{M}_{t}$ and $P_{n} \in \mathcal{M}_{s}$ where $t=\lfloor n / 2\rfloor-\lceil n / 3\rceil+1$ and $s=\lceil n / 2\rceil-\lceil n / 3\rceil+1$.
The following lemma from [5] will be used throughout-often without mention.
Lemma 2 [5 If the graph $G$ belongs to $\mathcal{M}_{t}$ and $I$ is an independent set of $G$, then for every component $C$ of $G-N[I]$ there exists $k \leq t$ such that $C \in \mathcal{M}_{k}$. In addition, $G-N[I] \in \mathcal{M}_{r}$ for some $r \leq t$.

Lemma 2 will most often be used in the following way. We will find an independent set $I$ in a graph $G$ and demonstrate that $G-N[I]$ has a component that is in the class $\mathcal{M}_{s}$ for some $s>t$ and conclude that $G \notin \mathcal{M}_{t}$. The following lemma will be used in that context with Lemma 2 .

Lemma 3 If a cycle $C$ is in $\mathcal{M}_{t}$ and a new vertex is added as a leaf adjacent to a single vertex of $C$, then the resulting graph belongs to $\mathcal{M}_{t+1}$.

Proof. Assume $\mathcal{S}(C)=[k, k+t-1]$. Let $H$ be the graph formed by adding a leaf $x$ adjacent to $y$. Let $u$ and $v$ be the neighbors of $y$ on $C$. Note that $\{I \in$ $\mathcal{M}(H): y \in I\}=\{J \in \mathcal{M}(C): y \in J\}$, and because of the symmetry of the cycle, $\mathcal{S}(C)=\{|J|: J \in \mathcal{M}(C), y \in J\}$. Also, $\{I \in \mathcal{M}(H): u \in I\}=\{J \cup\{x\}$ : $J \in \mathcal{M}(C), u \in J\}$. This shows that $[k, k+t] \subseteq \mathcal{S}(H)$. If $H$ has a maximal independent set $A$ of size less than $k$, then $x \in A$ and neither $u$ nor $v$ is in $A$, for otherwise $A \cap C$ is a maximal independent set in $C$ of cardinality less than $k$. But now $A^{\prime}=(A-\{x\}) \cup\{y\} \in \mathcal{M}(C)$ and $\left|A^{\prime}\right|<k$, a contradiction. Therefore, $\mathcal{S}(H)=[k, k+t]$. We conclude that $H \in \mathcal{M}_{t+1}$.

In the class of graphs with leaves there is no connection between girth and the size of the spectrum. This can be seen by the following general construction. Let $t \geq 2$ and $g \geq 3$ be integers. Let $H$ be the graph formed by adding a single leaf adjacent to each vertex of a cycle of order $g$. For a single vertex $x$ on the cycle attach a path $v_{1}, v_{2}, \ldots, v_{2 t-3}$ to $H$ by making $x$ and $v_{1}$ adjacent. Then add two leaves adjacent to $v_{i}$ if $i$ is odd, and add one leaf adjacent to $v_{j}$ if $j$ is even. The resulting graph of order $2 g+5 t-7$ has girth $g$ and belongs to the class $\mathcal{M}_{t}$. (The spectrum of this graph is $[g+2 t-3, g+3 t-4]$.) For this reason we will henceforth consider only graphs having minimum degree at least 2. For ease of reference we denote the class of graphs that are in $\mathcal{M}_{t}$ and have no leaves (i.e., minimum degree at least 2) by $\mathcal{M}_{t}^{2}$. Note that $\mathcal{M}_{t}^{2} \subseteq \mathcal{M}_{t}$. In the course of several of our proofs we will show that some given graph is not in $\mathcal{M}_{t}^{2}$ by demonstrating it does not belong to $\mathcal{M}_{t}$.

The remainder of this paper is devoted to verifying the entries in the following table.

|  | girth |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $6 t-6$ | $6 t-5$ | $6 t-4$ | $6 t-3$ | $6 t-2$ | $6 t-1$ | $6 t$ | $6 t+1$ | $\geq 6 t+2$ |  |
| $t=1$ |  |  |  | $\Delta$ | $\Delta$ | $\Delta$ | $\emptyset$ | $C_{7}$ | $\emptyset$ |  |
| $t=2$ | $\Delta$ | $\Delta$ | $C_{8}$ | $C_{9}$ | $C_{10}$ | $C_{11}$ | $\emptyset$ | $C_{13}$ | $\emptyset$ |  |
| $t=3$ | $C_{12}$ | $\Delta$ | $C_{14}$ | $C_{15}$ | $C_{16}$ | $C_{17}$ | $\emptyset$ | $C_{19}$ | $\emptyset$ |  |
| $t=4$ | $C_{18}$ | $\emptyset$ | $C_{20}$ | $C_{21}$ | $C_{22}$ | $C_{23}$ | $\emptyset$ | $C_{25}$ | $\emptyset$ |  |
| $t \geq 5$ | $C_{6 t-6}$ | $\emptyset$ | $C_{6 t-4}$ | $C_{6 t-3}$ | $C_{6 t-2}$ | $C_{6 t-1}$ | $\emptyset$ | $C_{6 t+1}$ | $\emptyset$ |  |

Table 1: Graphs of given girth in $\mathcal{M}_{t}^{2}$

The entry for a given girth (written as a function of $t$ ) and a given value of $t$ should be interpreted as follows. If a specific graph is given, then this is the unique graph of that girth that belongs to $\mathcal{M}_{t}^{2}$. For example, $C_{15}$ is the only graph of girth 15 in $\mathcal{M}_{3}^{2}$. If $\emptyset$ appears, then there are no graphs of that girth in $\mathcal{M}_{t}^{2}$. When the
entry is $\Delta$, then it is known that $\mathcal{M}_{t}^{2}$ contains at least one graph of that girth (and it is not just a cycle). Some of these type of entries have been verified in previous papers. For example, see [4] and [5] for $\mathcal{M}_{1}^{2}$ and $\mathcal{M}_{2}^{2}$, respectively.

## 2 Establishing Table Entries

We begin by showing that for a given positive integer $t$ the only graphs in $\mathcal{M}_{t}$ with large enough girth must have leaves. The next result was proved for well-covered graphs $(t=1)$ in [3]. Proposition 1 shows it is sharp in terms of girth.

Theorem 4 Let $t$ be a positive integer. If $g(G) \geq 6 t+2$ and $\delta(G) \geq 2$, then $G \in \mathcal{M}_{r}(G)$ for some $r>t$.

Proof. Assume $t \geq 2$. Let $G$ have girth at least $6 t+2$ and minimum degree at least two. We will show that $G$ has maximal independent sets of at least $t+1$ different sizes. Choose a cycle $C=v_{1}, v_{2}, \ldots, v_{s}$ of minimum length in $G$.

Assume first that $s \geq 6 t+4$ and let $P$ denote the path $v_{3}, v_{4}, \ldots, v_{6 t+1}$. Since $\delta(G) \geq 2$ and $g(G)=s$, each vertex $u \notin C$ that is adjacent to a vertex of $P$ has another neighbor $u^{\prime}$ that does not belong to $P$ and is not adjacent to any vertex of $P$. Choose one such neighbor $u^{\prime}$ for each $u$ and let $J$ denote the set of these neighbors. By the girth restriction it follows that the set $I=J \cup\left\{v_{1}, v_{6 t+3}\right\}$ is independent. (If $s=6 t+2$, then proceed as above except let $I=J \cup\left\{v_{1}\right\}$.) However, $P$ is a component of $G-N[I]$ and by Proposition 11, $P \in \mathcal{M}_{t+1}$. Similar to the proof of Lemma 2 this implies that $G$ has maximal independent sets of at least $t+1$ different sizes.

If $s=6 t+3$, let $P$ be the path $v_{3}, v_{4}, \ldots, v_{6 t+2}$. The set $J$ is chosen as before, and now $G-N\left[J \cup\left\{v_{1}\right\}\right]$ has the path $P$ of order $6 t$ as a component. By Proposition $\mathbb{\square}$ it once again follows that $G$ has at least $t+1$ distinct sizes of maximal independent sets.

For any positive integer $t$ it follows from Proposition 1] that $C_{6 t+1} \in \mathcal{M}_{t}$. In [4] it was shown that $C_{7}$ is the only well-covered graph of girth 7 and minimum degree 2 or more. The following theorem shows the similar result is true for larger values of $t$.

Theorem 5 Let $t \geq 2$ be an integer. The cycle $C_{6 t+1}$ is the only graph of girth $6 t+1$ in $\mathcal{M}_{t}^{2}$, and $\mathcal{M}_{t}^{2}$ contains no graphs of girth $6 t$.

Proof. By Proposition 1 the cycle of order $6 t+1$ belongs to $\mathcal{M}_{t}^{2}$. Suppose $G$ is a graph not isomorphic to $C_{6 t+1}$ such that $g(G)=6 t+1$ and $\delta(G) \geq 2$. Then $G$
has an induced cycle $C$ of order $6 t+1$, and $C$ has a vertex $w$ of degree at least 3 . Since $g(G)=6 t+1$ and $\delta(G) \geq 2$ we can find an induced path $w, a, b, c$, such that none of $a, b$ or $c$ belongs to $C$. Let $X=\{u \in V(G): d(u, C)=2\}-N(a)$ and let $Y=\{u \in V(G): d(u, a)=2, d(u, w)=3\}$. For any two vertices on $C$ there is a path using part of $C$ of length at most $3 t$ joining them. Since $g(G) \geq 13$ it follows that $Y$ is independent. Suppose two vertices $x_{1}, x_{2} \in X$ are adjacent. Let $x_{1}, v_{1}, w_{1}$ and $x_{2}, v_{2}, w_{2}$ be paths in $G$ with $w_{1}$ and $w_{2}$ on the cycle $C$. Then the cycle $x_{1}, v_{1}, w_{1} C w_{2}, v_{2}, x_{2}, x_{1}$ has length at most $3 t+5$. But then $3 t+5 \geq 6 t+1$, which implies that $t=1$, a contradiction. Finally, if a vertex in $X$ is adjacent to a vertex in $Y$, then a similar argument shows that $G$ has a cycle of length at most $3 t+6$ which also leads to a contradiction.

Therefore, $X \cup Y$ is an independent set. One of the components of the graph $G-N[X \cup Y]$ is the cycle $C$ with a single leaf $a$ attached at the support vertex $w$. By Lemma 3 this component is in $\mathcal{M}_{t+1}$. An application of Lemma 2 then shows that $G \notin \mathcal{M}_{t}^{2}$.

Now let $G$ be a graph of girth $6 t$, and as above find an induced cycle $C$ of length $6 t$. This time let $X=\{u \in V(G): d(u, C)=2\}$. This set is independent unless there is a cycle of the form $x_{1}, v_{1}, w_{1} C w_{2}, v_{2}, x_{2}, x_{1}$ that has length at most $3 t+5$. But this means $3 t+5 \geq 6 t$ contradicting our assumption that $t \geq 2$. Hence $X$ is independent. The cycle $C$ is one of the components of $G-N[X]$. Since $C_{6 t} \in \mathcal{M}_{t+1}$, Lemma 2 implies that $G \notin \mathcal{M}_{t}^{2}$.

By following a line of reasoning similar to the first part of the proof of Theorem 5 one can prove the following result. The proof is omitted. As noted earlier, Theorem6 also holds for $t=2$. See [5].

Theorem 6 Let $t \geq 3$ be a positive integer. For each integer $n$ such that $6 t-4 \leq$ $n \leq 6 t-1$, the cycle $C_{n}$ is the unique graph of girth $n$ that belongs to $\mathcal{M}_{t}^{2}$.

We now establish the uniqueness (for $t \geq 3$ ) of the table entry corresponding to those graphs with no leaves whose shortest cycle has length $6 t-6$ and which have maximal independent sets of exactly $t$ distinct cardinalities.

Theorem 7 For each integer $t \geq 3$, the cycle $C_{6 t-6}$ is the only graph of girth $6 t-6$ that belongs to $\mathcal{M}_{t}^{2}$.

Proof. The cycle of order $6 t-6$ is in $\mathcal{M}_{t}^{2}$ by Proposition (1) Suppose that $G$ is a graph of girth $6 t-6$ with no leaves. If $G$ is not $C_{6 t-6}$, then we can find an induced cycle $C$ of length $6 t-6$ in $G$ with $w, a, b, c, X$ and $Y$ defined as in the proof of Theorem [5. The set $Y$ is independent because $g(G) \geq 12$, and $X$ is independent since $t \geq 3$. If some vertex of $X$ is adjacent to a vertex of $Y$, then $G$ contains a cycle
of length at most $3 t-3+6$. It follows that $3 t+3 \geq g(G)=6 t-6$, or equivalently $t \leq 3$.

If the set $X \cup Y$ is independent, then $G-N[X \cup Y]$ has a component isomorphic to a cycle of length $6 t-6$ with a single leaf attached at $w$. By Lemma 3 this component is in $\mathcal{M}_{t+1}$ and so it follows from Lemma 2 that $G \notin \mathcal{M}_{t}$.


Figure 1: Part of $G$
Thus we may assume that $t=3$ and that $X \cup Y$ is not independent. Without loss of generality we may assume that $c$ from $Y$ is adjacent to $x_{1}$ such that $x_{1} \in X$ and $x_{1}, v_{1}, w_{1}$ is a path where $w_{1}$ is on the cycle $C$. See Figure By using the fact that $C$ has length 12 and $g(G)=12$ we infer that the length of $w C w_{1}$ is 6 . Let $X^{\prime}=X-N\left(v_{1}\right)$ and let $Z=\left\{u: d\left(u, v_{1}\right)=2, d\left(u, w_{1}\right)=3, u x_{1} \notin E(G)\right\}$. It is clear that $Z$ is independent.

As above, if a vertex of $Z$ is adjacent to a vertex $h$ of $X^{\prime}$, then if $d(h, w)>2$ a cycle of length at most 11 is present and if $d(h, w)=2$ then $G$ contains a cycle of length 10 , contradicting $g(G)=12$. Suppose $z_{1} \in Y \cap Z$, say $z_{1}=y$ as in Figure 1. Then $z_{1} \neq c$, and $a, b, c, x_{1}, v_{1}, x_{2}, z_{1}, u, a$ is a cycle, contradicting the girth assumption. Similarly, since $G$ has no cycles of length 9 , it follows that $Z \cup Y$ is independent.

The set $X^{\prime} \cup Y \cup Z$ is independent, and one of the components of the graph $G-N\left[X^{\prime} \cup Y \cup Z\right]$ is the cycle $C$ with a single leaf attached at vertices $w$ and $w_{1}$. But this component has spectrum $\{4,5,6,7,8\}$ from which it follows that $G \notin \mathcal{M}_{3}$.

We now show that when $t \geq 4$ there is a "gap" at girth $6 t-5$ among the leafless graphs. That is, if $G$ has minimum degree at least 2 and the shortest cycle of $G$ has order $6 t-5$, then $G$ does not belong to $\mathcal{M}_{t}$.

Theorem 8 For each integer $t$ at least 4, the class $\mathcal{M}_{t}^{2}$ contains no graphs of girth $6 t-5$.

Proof. First observe that $C_{6 t-5} \in \mathcal{M}_{t-1}$. Our approach will be similar as that pursued in earlier proofs, except that we will be attempting to isolate a cycle of length $6 t-5$ with a path of order 5 attached as in Figure 2. It is easy to check, using either $\{a, c, e\}$ or $\{a, d\}$ together with all possible maximal independent sets of a path of order $6 t-6$, that this component has spectrum $[2 t, 3 t]$ and hence belongs to $\mathcal{M}_{t+1}$. This in turn implies via Lemma 2 that $G \notin \mathcal{M}_{t}^{2}$.


Figure 2: The cycle $C$ with attachments
Suppose that $G$ has girth $6 t-5$ and has minimum degree at least 2 . Let $C$ be an induced cycle of length $6 t-5$ in $G$. There must exist a vertex $w$ on $C$ having degree at least 3 . For any two vertices on $C$ there is a path on $C$ joining them whose length is at most $3 t-3$. Because of the girth and minimum degree assumptions on $G$ we can find a path $w, a, b, c, d, e$ as in Figure 2, Let $A=\{a, b, c, d, e\}$. Let $X=\{u: d(u, C)=2\}-N(a)$ and let $Y=\{u: u \notin C, d(u, A)=2, d(u, w) \geq 2\}$.

As in previous proofs it is straightforward to show that $X$ is independent. Since $g(G)=6 t-5 \geq 19$ no pair of vertices in $Y$ can be adjacent. Suppose first that $X \cup Y$ is independent. The graph in Figure 2 is a component of $G-N[X \cup Y]$. As remarked at the outset, this shows that $G \notin \mathcal{M}_{t}^{2}$. We note that for $t \geq 5$, the girth restriction ensures that $X \cup Y$ is independent.

Now consider $t=4$. Thus $C$ is of length 19. Let $s_{1}$ and $s_{2}$ be the adjacent vertices on $C$ that are at distance 9 from $w$. If both $s_{1}$ and $s_{2}$ are of degree two, then $X \cup Y$ is independent or else a cycle of length 18 would exist in $G$. Assume then without loss of generality that $s_{1}$ has a neighbor $r$ that is not on $C$. Let $U=N(r)-\left\{s_{1}\right\}$. For each $u_{i} \in U$ choose a vertex $v_{i} \in N\left(u_{i}\right)-\{r\}$, and set $V=\left\{v_{i}: u_{i} \in U\right\}$. Similarly, let $B=N(a)-\{w\}$. For each $b_{i} \in B$ choose a vertex $c_{i} \in N\left(b_{i}\right)-\{a\}$, and set $D=\left\{c_{i}: b_{i} \in B\right\}$. Since $g(G)=19$ the set $V \cup D \cup(X-U)$ is independent, and one of the components of $G-N[V \cup D \cup(X-U)]$ is a cycle of order 19 with a single leaf $a$ adjacent to $w$ and a single leaf $r$ adjacent to $s_{1}$. This component
belongs to $\mathcal{M}_{5}$ which proves that $G \notin \mathcal{M}_{4}^{2}$ and establishes the theorem.

## 3 Concluding Remarks

We have shown that for a positive integer $t \geq 4$ and for each possible value of girth at least $6 t-6$, the class $\mathcal{M}_{t}^{2}$ either contains exactly one graph of that girth (the cycle) or contains no graphs of that girth. It is interesting to note that as $t$ grows there is an ever increasing gap-in terms of girth-between the unique graph of girth $6 t-6$ in $\mathcal{M}_{t}^{2}$ and ones of smaller girth. For instance, we can show that $\mathcal{M}_{31}^{2}$ contains no graphs of girth $r$ for $131 \leq r \leq 179$. Hence the cycles $C_{180}, C_{182}, C_{183}, C_{184}, C_{185}$ and $C_{187}$ are the only leafless members of $\mathcal{M}_{31}$ that have girth at least 131. Thus the six cycles are quite special in $\mathcal{M}_{t}^{2}$.

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    ${ }^{\dagger}$ Research supported in part by the Wylie Enrichment Fund of Furman University.

