# On graphs having maximal independent sets of exactly t distinct cardinalities

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#### Abstract

For a given positive integer t we consider graphs having maximal independent sets of precisely t distinct cardinalities and restrict our attention to those that have no vertices of degree one. In the situation when t is four or larger and the length of the shortest cycle is at least 6t-6, we completely characterize such graphs.

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#### 1 Introduction

A well-covered graph (Plummer [6]) is one in which every maximal independent set of vertices is of one cardinality and is hence a maximum independent set. Finbow, Hartnell and Whitehead [5] defined the class  $\mathcal{M}_t$  to consist of those graphs which have exactly t different sizes of maximal independent sets. Finbow, Hartnell and Nowakowski [4] proved that the well-covered graphs (the  $\mathcal{M}_1$  collection) of girth (the length of a shortest cycle) 6 or more, with the exceptions of  $K_1$  and  $C_7$ , have the property that every vertex has degree one or has exactly one vertex of degree one in its neighborhood. Thus,  $C_7$  is the unique graph in  $\mathcal{M}_1$  with girth at least 6

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that has minimum degree at least two. The graphs in  $\mathcal{M}_2$  of girth 8 or more have also been characterized ([5]). There are precisely five graphs in  $\mathcal{M}_2$  of girth at least 8 and minimum degree 2 or more, namely the cycles  $C_8$ ,  $C_9$ ,  $C_{10}$ ,  $C_{11}$  and  $C_{13}$ . This implies there are no  $\mathcal{M}_1$  graphs of girth at least 8 with minimum degree 2 or more and no  $\mathcal{M}_2$  graphs of girth 14 or more and having minimum degree at least 2. For related work on the class  $\mathcal{M}_t$  see [1] and [2].

In this paper we investigate the graphs in  $\mathcal{M}_t$  that have minimum degree at least 2 and higher girth and establish that the characterization of these in  $\mathcal{M}_1$  and  $\mathcal{M}_2$  is part of a general pattern. In particular, for  $t \geq 3$  we show that among graphs with minimum degree at least 2,  $\mathcal{M}_t$  does not contain a graph of girth at least 6t + 2 and that  $C_{6t-4}$ ,  $C_{6t-3}$ ,  $C_{6t-2}$ ,  $C_{6t-1}$  and  $C_{6t+1}$  are the only exceptions for girth at least 6t - 4. Furthermore, if  $t \geq 4$ , then these cycles along with  $C_{6t-6}$  are the only graphs in  $\mathcal{M}_t$  that have minimum degree at least 2 and girth at least 6t - 6.

Let G be a finite simple graph. A vertex of degree 1 is called a *leaf* and any vertex that is adjacent to a leaf is called a *support vertex*. If G is a cycle in a graph G and G and G and G and G belong to G, we let G denote the shorter of the two G, G and G are part of G. For G is a vertex in G, G will denote the length of a shortest path in G from G to a vertex of G. We will use G be denote the collection of all maximal independent sets of G and we define the *independence spectrum* (*spectrum* for short) of G to be the set G and we define the *independence spectrum* (*spectrum* for short) of G to be the set G and we define the *independence spectrum* (*spectrum* for short) of G to be the set G and we define the *independence spectrum* (*spectrum* for short) of G to be the set G for which G is a spectrum is not necessarily a set of consecutive positive integers (e.g., G and G inclusive by G in G inclusive by G inclusive G incl

**Proposition 1** For each positive integer n at least 3,

$$\mathcal{S}(C_n) = [\lceil n/3 \rceil, \lceil n/2 \rceil]$$
 and  $\mathcal{S}(P_n) = [\lceil n/3 \rceil, \lceil n/2 \rceil]$ .

Hence,  $C_n \in \mathcal{M}_t$  and  $P_n \in \mathcal{M}_s$  where  $t = \lfloor n/2 \rfloor - \lceil n/3 \rceil + 1$  and  $s = \lceil n/2 \rceil - \lceil n/3 \rceil + 1$ .

The following lemma from [5] will be used throughout—often without mention.

**Lemma 2** [5] If the graph G belongs to  $\mathcal{M}_t$  and I is an independent set of G, then for every component C of G-N[I] there exists  $k \leq t$  such that  $C \in \mathcal{M}_k$ . In addition,  $G-N[I] \in \mathcal{M}_r$  for some  $r \leq t$ .

Lemma 2 will most often be used in the following way. We will find an independent set I in a graph G and demonstrate that G - N[I] has a component that is in the class  $\mathcal{M}_s$  for some s > t and conclude that  $G \notin \mathcal{M}_t$ . The following lemma will be used in that context with Lemma 2.

**Lemma 3** If a cycle C is in  $\mathcal{M}_t$  and a new vertex is added as a leaf adjacent to a single vertex of C, then the resulting graph belongs to  $\mathcal{M}_{t+1}$ .

**Proof.** Assume S(C) = [k, k+t-1]. Let H be the graph formed by adding a leaf x adjacent to y. Let u and v be the neighbors of y on C. Note that  $\{I \in \mathcal{M}(H) : y \in I\} = \{J \in \mathcal{M}(C) : y \in J\}$ , and because of the symmetry of the cycle,  $S(C) = \{|J| : J \in \mathcal{M}(C), y \in J\}$ . Also,  $\{I \in \mathcal{M}(H) : u \in I\} = \{J \cup \{x\} : J \in \mathcal{M}(C), u \in J\}$ . This shows that  $[k, k+t] \subseteq S(H)$ . If H has a maximal independent set A of size less than k, then  $k \in A$  and neither  $k \in A$  and neither  $k \in A$  and neither  $k \in A$  and  $k \in A$  but now  $k \in A \cap A$  is a maximal independent set in  $K \in A$  and neither  $K \in A$  and neit

In the class of graphs with leaves there is no connection between girth and the size of the spectrum. This can be seen by the following general construction. Let  $t \geq 2$  and  $g \geq 3$  be integers. Let H be the graph formed by adding a single leaf adjacent to each vertex of a cycle of order g. For a single vertex x on the cycle attach a path  $v_1, v_2, \ldots, v_{2t-3}$  to H by making x and  $v_1$  adjacent. Then add two leaves adjacent to  $v_i$  if i is odd, and add one leaf adjacent to  $v_j$  if j is even. The resulting graph of order 2g + 5t - 7 has girth g and belongs to the class  $\mathcal{M}_t$ . (The spectrum of this graph is [g + 2t - 3, g + 3t - 4].) For this reason we will henceforth consider only graphs having minimum degree at least 2. For ease of reference we denote the class of graphs that are in  $\mathcal{M}_t$  and have no leaves (i.e., minimum degree at least 2) by  $\mathcal{M}_t^2$ . Note that  $\mathcal{M}_t^2 \subseteq \mathcal{M}_t$ . In the course of several of our proofs we will show that some given graph is not in  $\mathcal{M}_t^2$  by demonstrating it does not belong to  $\mathcal{M}_t$ .

The remainder of this paper is devoted to verifying the entries in the following table.

	girth								
	6t - 6	6t - 5	6t - 4	6t - 3	6t - 2	6t - 1	6t	6t + 1	$\geq 6t + 2$
t = 1				Δ	Δ	Δ	Ø	$C_7$	Ø
t=2	Δ	Δ	$C_8$	$C_9$	$C_{10}$	$C_{11}$	Ø	$C_{13}$	Ø
t = 3	$C_{12}$	Δ	$C_{14}$	$C_{15}$	$C_{16}$	$C_{17}$	Ø	$C_{19}$	Ø
t = 4	$C_{18}$	Ø	$C_{20}$	$C_{21}$	$C_{22}$	$C_{23}$	Ø	$C_{25}$	Ø
$t \ge 5$	$C_{6t-6}$	Ø	$C_{6t-4}$	$C_{6t-3}$	$C_{6t-2}$	$C_{6t-1}$	Ø	$C_{6t+1}$	Ø

Table 1: Graphs of given girth in  $\mathcal{M}_t^2$ 

The entry for a given girth (written as a function of t) and a given value of t should be interpreted as follows. If a specific graph is given, then this is the unique graph of that girth that belongs to  $\mathcal{M}_t^2$ . For example,  $C_{15}$  is the only graph of girth 15 in  $\mathcal{M}_3^2$ . If  $\emptyset$  appears, then there are no graphs of that girth in  $\mathcal{M}_t^2$ . When the

entry is  $\Delta$ , then it is known that  $\mathcal{M}_t^2$  contains at least one graph of that girth (and it is not just a cycle). Some of these type of entries have been verified in previous papers. For example, see [4] and [5] for  $\mathcal{M}_1^2$  and  $\mathcal{M}_2^2$ , respectively.

# 2 Establishing Table Entries

We begin by showing that for a given positive integer t the only graphs in  $\mathcal{M}_t$  with large enough girth must have leaves. The next result was proved for well-covered graphs (t=1) in [3]. Proposition 1 shows it is sharp in terms of girth.

**Theorem 4** Let t be a positive integer. If  $g(G) \ge 6t + 2$  and  $\delta(G) \ge 2$ , then  $G \in \mathcal{M}_r(G)$  for some r > t.

**Proof.** Assume  $t \geq 2$ . Let G have girth at least 6t + 2 and minimum degree at least two. We will show that G has maximal independent sets of at least t + 1 different sizes. Choose a cycle  $C = v_1, v_2, \ldots, v_s$  of minimum length in G.

Assume first that  $s \geq 6t + 4$  and let P denote the path  $v_3, v_4, \ldots, v_{6t+1}$ . Since  $\delta(G) \geq 2$  and g(G) = s, each vertex  $u \notin C$  that is adjacent to a vertex of P has another neighbor u' that does not belong to P and is not adjacent to any vertex of P. Choose one such neighbor u' for each u and let J denote the set of these neighbors. By the girth restriction it follows that the set  $I = J \cup \{v_1, v_{6t+3}\}$  is independent. (If s = 6t + 2, then proceed as above except let  $I = J \cup \{v_1\}$ .) However, P is a component of G - N[I] and by Proposition 1,  $P \in \mathcal{M}_{t+1}$ . Similar to the proof of Lemma 2 this implies that G has maximal independent sets of at least t+1 different sizes.

If s = 6t+3, let P be the path  $v_3, v_4, \ldots, v_{6t+2}$ . The set J is chosen as before, and now  $G - N[J \cup \{v_1\}]$  has the path P of order 6t as a component. By Proposition 1 it once again follows that G has at least t+1 distinct sizes of maximal independent sets.

For any positive integer t it follows from Proposition 1 that  $C_{6t+1} \in \mathcal{M}_t$ . In [4] it was shown that  $C_7$  is the only well-covered graph of girth 7 and minimum degree 2 or more. The following theorem shows the similar result is true for larger values of t.

**Theorem 5** Let  $t \geq 2$  be an integer. The cycle  $C_{6t+1}$  is the only graph of girth 6t + 1 in  $\mathcal{M}_t^2$ , and  $\mathcal{M}_t^2$  contains no graphs of girth 6t.

**Proof.** By Proposition 1 the cycle of order 6t + 1 belongs to  $\mathcal{M}_t^2$ . Suppose G is a graph not isomorphic to  $C_{6t+1}$  such that g(G) = 6t + 1 and  $\delta(G) \geq 2$ . Then G

has an induced cycle C of order 6t+1, and C has a vertex w of degree at least 3. Since g(G)=6t+1 and  $\delta(G)\geq 2$  we can find an induced path w,a,b,c, such that none of a,b or c belongs to C. Let  $X=\{u\in V(G):d(u,C)=2\}-N(a)$  and let  $Y=\{u\in V(G):d(u,a)=2,d(u,w)=3\}$ . For any two vertices on C there is a path using part of C of length at most 3t joining them. Since  $g(G)\geq 13$  it follows that Y is independent. Suppose two vertices  $x_1,x_2\in X$  are adjacent. Let  $x_1,v_1,w_1$  and  $x_2,v_2,w_2$  be paths in G with  $w_1$  and  $w_2$  on the cycle C. Then the cycle  $x_1,v_1,w_1Cw_2,v_2,x_2,x_1$  has length at most 3t+5. But then  $3t+5\geq 6t+1$ , which implies that t=1, a contradiction. Finally, if a vertex in X is adjacent to a vertex in Y, then a similar argument shows that G has a cycle of length at most 3t+6 which also leads to a contradiction.

Therefore,  $X \cup Y$  is an independent set. One of the components of the graph  $G - N[X \cup Y]$  is the cycle C with a single leaf a attached at the support vertex w. By Lemma 3 this component is in  $\mathcal{M}_{t+1}$ . An application of Lemma 2 then shows that  $G \notin \mathcal{M}_t^2$ .

Now let G be a graph of girth 6t, and as above find an induced cycle C of length 6t. This time let  $X = \{u \in V(G) : d(u,C) = 2\}$ . This set is independent unless there is a cycle of the form  $x_1, v_1, w_1 C w_2, v_2, x_2, x_1$  that has length at most 3t + 5. But this means  $3t + 5 \ge 6t$  contradicting our assumption that  $t \ge 2$ . Hence X is independent. The cycle C is one of the components of G - N[X]. Since  $C_{6t} \in \mathcal{M}_{t+1}$ , Lemma 2 implies that  $G \notin \mathcal{M}_t^2$ .

By following a line of reasoning similar to the first part of the proof of Theorem 5 one can prove the following result. The proof is omitted. As noted earlier, Theorem 6 also holds for t = 2. See [5].

**Theorem 6** Let  $t \geq 3$  be a positive integer. For each integer n such that  $6t - 4 \leq n \leq 6t - 1$ , the cycle  $C_n$  is the unique graph of girth n that belongs to  $\mathcal{M}_t^2$ .

We now establish the uniqueness (for  $t \geq 3$ ) of the table entry corresponding to those graphs with no leaves whose shortest cycle has length 6t-6 and which have maximal independent sets of exactly t distinct cardinalities.

**Theorem 7** For each integer  $t \geq 3$ , the cycle  $C_{6t-6}$  is the only graph of girth 6t-6 that belongs to  $\mathcal{M}_t^2$ .

**Proof.** The cycle of order 6t - 6 is in  $\mathcal{M}_t^2$  by Proposition 1. Suppose that G is a graph of girth 6t - 6 with no leaves. If G is not  $C_{6t-6}$ , then we can find an induced cycle C of length 6t - 6 in G with w, a, b, c, X and Y defined as in the proof of Theorem 5. The set Y is independent because  $g(G) \geq 12$ , and X is independent since  $t \geq 3$ . If some vertex of X is adjacent to a vertex of Y, then G contains a cycle

of length at most 3t - 3 + 6. It follows that  $3t + 3 \ge g(G) = 6t - 6$ , or equivalently t < 3.

If the set  $X \cup Y$  is independent, then  $G - N[X \cup Y]$  has a component isomorphic to a cycle of length 6t - 6 with a single leaf attached at w. By Lemma 3 this component is in  $\mathcal{M}_{t+1}$  and so it follows from Lemma 2 that  $G \notin \mathcal{M}_t$ .

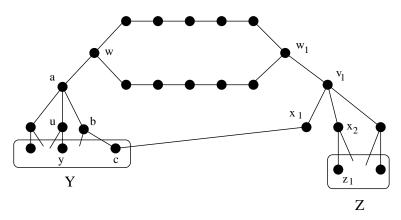


Figure 1: Part of G

Thus we may assume that t=3 and that  $X \cup Y$  is not independent. Without loss of generality we may assume that c from Y is adjacent to  $x_1$  such that  $x_1 \in X$  and  $x_1, v_1, w_1$  is a path where  $w_1$  is on the cycle C. See Figure 1. By using the fact that C has length 12 and g(G)=12 we infer that the length of  $wCw_1$  is 6. Let  $X'=X-N(v_1)$  and let  $Z=\{u: d(u,v_1)=2, d(u,w_1)=3, ux_1 \notin E(G)\}$ . It is clear that Z is independent.

As above, if a vertex of Z is adjacent to a vertex h of X', then if d(h,w) > 2 a cycle of length at most 11 is present and if d(h,w) = 2 then G contains a cycle of length 10, contradicting g(G) = 12. Suppose  $z_1 \in Y \cap Z$ , say  $z_1 = y$  as in Figure 1. Then  $z_1 \neq c$ , and  $a, b, c, x_1, v_1, x_2, z_1, u, a$  is a cycle, contradicting the girth assumption. Similarly, since G has no cycles of length 9, it follows that  $Z \cup Y$  is independent.

The set  $X' \cup Y \cup Z$  is independent, and one of the components of the graph  $G - N[X' \cup Y \cup Z]$  is the cycle C with a single leaf attached at vertices w and  $w_1$ . But this component has spectrum  $\{4, 5, 6, 7, 8\}$  from which it follows that  $G \notin \mathcal{M}_3$ .

We now show that when  $t \geq 4$  there is a "gap" at girth 6t - 5 among the leafless graphs. That is, if G has minimum degree at least 2 and the shortest cycle of G has order 6t - 5, then G does not belong to  $\mathcal{M}_t$ .

**Theorem 8** For each integer t at least 4, the class  $\mathcal{M}_t^2$  contains no graphs of girth 6t-5.

**Proof.** First observe that  $C_{6t-5} \in \mathcal{M}_{t-1}$ . Our approach will be similar as that pursued in earlier proofs, except that we will be attempting to isolate a cycle of length 6t-5 with a path of order 5 attached as in Figure 2. It is easy to check, using either  $\{a, c, e\}$  or  $\{a, d\}$  together with all possible maximal independent sets of a path of order 6t-6, that this component has spectrum [2t, 3t] and hence belongs to  $\mathcal{M}_{t+1}$ . This in turn implies via Lemma 2 that  $G \notin \mathcal{M}_t^2$ .

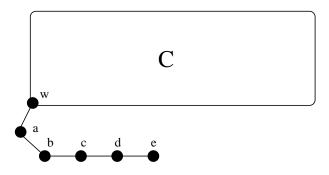


Figure 2: The cycle C with attachments

Suppose that G has girth 6t-5 and has minimum degree at least 2. Let C be an induced cycle of length 6t-5 in G. There must exist a vertex w on C having degree at least 3. For any two vertices on C there is a path on C joining them whose length is at most 3t-3. Because of the girth and minimum degree assumptions on G we can find a path w, a, b, c, d, e as in Figure 2. Let  $A = \{a, b, c, d, e\}$ . Let  $X = \{u : d(u, C) = 2\} - N(a)$  and let  $Y = \{u : u \notin C, d(u, A) = 2, d(u, w) \ge 2\}$ .

As in previous proofs it is straightforward to show that X is independent. Since  $g(G) = 6t - 5 \ge 19$  no pair of vertices in Y can be adjacent. Suppose first that  $X \cup Y$  is independent. The graph in Figure 2 is a component of  $G - N[X \cup Y]$ . As remarked at the outset, this shows that  $G \notin \mathcal{M}_t^2$ . We note that for  $t \ge 5$ , the girth restriction ensures that  $X \cup Y$  is independent.

Now consider t=4. Thus C is of length 19. Let  $s_1$  and  $s_2$  be the adjacent vertices on C that are at distance 9 from w. If both  $s_1$  and  $s_2$  are of degree two, then  $X \cup Y$  is independent or else a cycle of length 18 would exist in G. Assume then without loss of generality that  $s_1$  has a neighbor r that is not on C. Let  $U=N(r)-\{s_1\}$ . For each  $u_i \in U$  choose a vertex  $v_i \in N(u_i)-\{r\}$ , and set  $V=\{v_i: u_i \in U\}$ . Similarly, let  $B=N(a)-\{w\}$ . For each  $b_i \in B$  choose a vertex  $c_i \in N(b_i)-\{a\}$ , and set  $D=\{c_i: b_i \in B\}$ . Since g(G)=19 the set  $V \cup D \cup (X-U)$  is independent, and one of the components of  $G-N[V \cup D \cup (X-U)]$  is a cycle of order 19 with a single leaf a adjacent to w and a single leaf r adjacent to  $s_1$ . This component

belongs to  $\mathcal{M}_5$  which proves that  $G \notin \mathcal{M}_4^2$  and establishes the theorem.

## 3 Concluding Remarks

We have shown that for a positive integer  $t \geq 4$  and for each possible value of girth at least 6t-6, the class  $\mathcal{M}_t^2$  either contains exactly one graph of that girth (the cycle) or contains no graphs of that girth. It is interesting to note that as t grows there is an ever increasing gap—in terms of girth—between the unique graph of girth 6t-6 in  $\mathcal{M}_t^2$  and ones of smaller girth. For instance, we can show that  $\mathcal{M}_{31}^2$  contains no graphs of girth r for  $131 \leq r \leq 179$ . Hence the cycles  $C_{180}, C_{182}, C_{183}, C_{184}, C_{185}$  and  $C_{187}$  are the only leafless members of  $\mathcal{M}_{31}$  that have girth at least 131. Thus the six cycles are quite special in  $\mathcal{M}_t^2$ .

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