

LARGE CHROMATIC NUMBER AND RAMSEY GRAPHS

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ABSTRACT. Let $Q(n, \chi)$ denote the minimum clique size an n -vertex graph can have if its chromatic number is χ . Using Ramsey graphs we give an exact, albeit implicit, formula for the case $\chi \geq (n+3)/2$.

1. PRELIMINARIES

The clique number, the chromatic number, and the independence number of a graph $G = (V, \mathcal{E})$ are denoted by $\omega(G)$, $\chi(G)$, and $\alpha(G)$, respectively. Intuitively, large chromatic number must imply large cliques. Define

$$Q(n, c) := \min\{\omega(G) : |V(G)| = n \text{ and } \chi(G) = c\}.$$

It is obvious that $Q(n, n) = n$, (the only graph to investigate is K_n), it is not difficult to show that $Q(n, n-1) = n-1$ ($n \geq 2$) (the complement of the graph should be a star) and that $Q(n, n-2) \leq n-3$ for $n \geq 5$ (remove a five-cycle C_5 from K_n). Biró [2] determined $Q(n, n-k)$ for $k \leq 6$, whenever n is sufficiently large, $n > n_0(k)$.

$$\begin{array}{cccccccc} k & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ Q(n, n-k) & = & n & n-1 & n-3 & n-4 & n-6 & n-7 & n-9 \end{array}$$

Based on these values he was tempted to conjecture that if n is large enough, then $Q(n, n-k) = n-2k + \lceil k/2 \rceil$. He also showed $Q(n, n-k) \geq n-2k+3$ for $k \geq 5$ and n is large enough. Jahanbekam and West [6] observed that $Q(n, n-k)$ is at most the conjectured value whenever $n \geq 5k/2$ and they also asked if this threshold on n is both sufficient and necessary for equality. Their construction is the complement of $\lfloor k/2 \rfloor$ vertex disjoint C_5 's and a path P_3 if k is odd. The aim of this note is to give an exact formula for $Q(n, n-k)$ for $n \geq 2k+3$. Our results established the above conjecture for $k \leq 12$ and $k = 14$ but disproved it for any other value of k .

2. RAMSEY GRAPHS

The Ramsey number $R(3, \ell)$ is the minimum integer R such that every graph on $n \geq R$ vertices has either three independent vertices or a clique of size ℓ . It is well-known (Kim [7] and Ajtai, Komlós and Szemerédi [1]) that there are constants $\gamma_1, \gamma_2 > 0$ such that

$$(1) \quad \gamma_1 \frac{\ell^2}{\log \ell} < R(3, \ell) < \gamma_2 \frac{\ell^2}{\log \ell}$$

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hold for every $\ell \geq 3$. The first few values are also known (see, e.g., the survey [10])

$$(2) \quad \begin{array}{cccccccccccc} \ell & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ R(3, \ell) & = & 1 & 3 & 6 & 9 & 14 & 18 & 23 & 28 & 36 & 40 - 43 & 46 - 51 \end{array}$$

To state our result we need an inverse of this function. Let

$$\omega(x) := \min\{\omega(G) : |V(G)| = x \text{ and } \alpha(G) \leq 2\}.$$

We have $\omega(x) = \omega$ for $R(3, \omega) \leq x < R(3, \omega + 1)$. For $k \geq 1$ define

$$q(k) := \min \sum_{i=1}^s (\omega(2k_i + 1) - 1)$$

where the minimum is taken over all positive integers k_1, \dots, k_s with $k_1 + \dots + k_s = k$, $s \geq 1$. Also define $q(0) := 0$. From the tableaux (2) one can easily calculate the first few values of q

$$(3) \quad \begin{array}{cccccccc} k & = & 0 & 1-2 & 3 & 4 & 5-6 & 7-8 & 9-10 & 11-13 \\ \omega(2k+1) - 1 & = & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 6 \\ q(k) & = & 0 & 1 & 2 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{array}{cccc} k & = & 14-17 & 18-19 & 20 & 21-22 \\ \omega(2k+1) - 1 & = & 7 & 8 & ? & 9 \\ q(k) & = & 7 & 8 & ? & 9 \end{array}$$

It also follows from (1) that there exist $\gamma'_1, \gamma'_2 > 0$ such that for $k \geq 3$ we have

$$\gamma'_1 \sqrt{k \log k} < q(k) < \gamma'_2 \sqrt{k \log k}.$$

Theorem 2.1. *For $n \geq 2k + 3$*

$$Q(n, n - k) = n - 2k + q(k).$$

3. THE CHROMATIC GAP

The *chromatic gap* is defined as

$$\text{gap}(n) := \max\{\chi(G) - \omega(G) : |V(G)| = n\}.$$

Gyárfás, Sebő, and Trotignon [4] showed that $\text{gap}(n) = \lceil n/2 \rceil - \omega(n)$ for almost all n . Our results are closely related, we use similar tools, but ours can be considered as a strengthening of theirs because, obviously, $\text{gap}(n) = \max\{c - Q(n, c)\}$.

4. GRAPHS WITH INDEPENDENCE NUMBER 2

The aim of this section is to prove that in the definition of $q(k)$, we may suppose that $s \leq 3$.

Theorem 4.1. *Let k be a positive integer. Then there is an integer s , $s \leq 3$, such that $q(k) = \sum_{i=1}^s \omega(2k_i + 1) - 1$ where $k_1 + \dots + k_s = k$, k_i 's are positive integers.*

Conjecture 4.2. *The previous statement holds with $s = 2$. Even more, $q(k) = \omega(2k + 1) - 1$ for all k except for $k = 4$.*

Lemma 4.3 (Xiaodong Xu, Zheng Xie, S. Radziszowski [11]). *Assume that $\omega_1 \geq \omega_2 \geq 1$. Then we have*

$$(4) \quad (R(3, \omega_1 + 1) - 1) + (R(3, \omega_2 + 1) - 1) + \omega_2 \leq R(3, \omega_1 + \omega_2 + 1) - 1.$$

Since this is our main tool, for completeness, we include their construction.

Proof. Consider two vertex disjoint graphs G_1 and G_2 with $|V(G_i)| = R(3, \omega_i + 1) - 1$, $\alpha(G_i) \leq 2$ and $\omega(G_i) = \omega_i$ for $i = 1, 2$. Let $R := \{r_1, \dots, r_{\omega_2}\}$ be a set disjoint to $V(G_1)$ and $V(G_2)$ and suppose that $V_1 := \{v_1, \dots, v_{\omega_2}\}$ and $U_2 := \{u_1, \dots, u_{\omega_2}\}$ are forming cliques in G_1 and G_2 respectively. Define the graph H with $V(H) = V(G_1) \cup V(G_2) \cup R$ as follows. $H|V(G_i) = G_i$, $H|(R \cup V_1)$ and $H|(R \cup U_2)$ are complete graphs of sizes $2\omega_2$. Connect every vertex in $v \in V(G_1)$ to every vertex in $u \in V(G_2)$ except if $v \in V_1$ and $u \in U_2$. Finally, we have that r_i and v_i have the same neighbors in $V(G_1) \setminus V_1$, similarly $N_H(r_i) \cap (V(G_2) \setminus U_2) = N_{G_2}(u_i) \setminus U_2$. We obtain $|V(H)| = |V(G_1)| + |V(G_2)| + \omega_2$, $\omega(H) = \omega_1 + \omega_2$ and $\alpha(H) \leq 2$. \square

Proof of Theorem 4.1. Assume that s is the minimum integer such that $q(k) = \sum_{1 \leq i \leq s} (\omega(2k_i + 1) - 1)$ and $k_1 + \dots + k_s = k$, $k_i \geq 1$. Let $\omega_i := \omega(2k_i + 1)$. By definition

$$(5) \quad 2k_i + 1 \leq R(3, \omega_i + 1) - 1 \quad \text{for all } i.$$

We may suppose that $s > 1$, otherwise there is nothing to prove.

Our first observation is, that the multiset k_1, \dots, k_s is *not reducible*. This means that we cannot replace a set of k_i 's by their sum, i.e., for any subset $L \subset \{1, 2, \dots, s\}$, $2 \leq |L| \leq s$ we have that

$$\sum_{i \in L} (\omega(2k_i + 1) - 1) = \left(\sum_{i \in L} \omega_i \right) - |L| < \omega \left(2 \left(\sum_{i \in L} k_i \right) + 1 \right) - 1.$$

This implies

$$R \left(3, \left(\sum_{i \in L} \omega_i \right) - |L| + 2 \right) \leq 2 \left(\sum_{i \in L} k_i \right) + 1,$$

which together with (5) give

$$(6) \quad R \left(3, \left(\sum_{i \in L} \omega_i \right) - |L| + 2 \right) \leq \left(\sum_{i \in L} R(3, \omega_i + 1) \right) - 2|L| + 1.$$

Suppose, on the contrary, that $s \geq 4$ and $\omega_1 \geq \omega_2 \geq \omega_3 \geq \omega_4$. We have $\omega_4 \geq \omega(3) = 2$. Using the Erdős–Szekeres inequality $R(r, s) \leq R(r-1, s) + R(r, s-1)$ with $r = 3$, we get

$$(7) \quad R(3, \omega_2 + 1) - 1 \leq R(3, \omega_2) + \omega_2,$$

$$(8) \quad R(3, \omega_3 + 1) - 1 \leq R(3, \omega_3) + \omega_3,$$

$$(9) \quad R(3, \omega_4 + 1) - 1 \leq R(3, \omega_4) + \omega_4.$$

Repeated applications of Lemma 4.3 gives

$$(10) \quad (R(3, \omega_2) - 1) + (R(3, \omega_4) - 1) + \omega_4 - 1 \leq R(3, \omega_2 + \omega_4 - 1) - 1,$$

$$(11) \quad (R(3, \omega_1 + 1) - 1) + (R(3, \omega_3) - 1) + \omega_3 - 1 \leq R(3, \omega_1 + \omega_3) - 1,$$

$$(12) \quad (R(3, \omega_1 + \omega_3) - 1) + (R(3, \omega_2 + \omega_4 - 1) - 1) + \omega_2 + \omega_4 - 2 \\ \leq R(3, \omega_1 + \omega_2 + \omega_3 + \omega_4 - 2) - 1.$$

Substitute $L := \{1, 2, 3, 4\}$ into (6) and add to the seven inequalities (6)–(12). We obtain

$$(13) \quad \omega_4 \leq 3.$$

If here equality holds then equality must hold in each of the 7 inequalities we added up. However, (9) does not hold with equality for $\omega_4 = 3$ since $R(3, 4) - 1 = 9 - 1 < 6 + 3 = R(3, 3) + 3$.

From now on, we may suppose that $\omega_4 = 2$. Substitute this to (10) and add (7) to (10). After rearrangement we get $2 \leq \omega_2$. So in the case $\omega_2 \geq 4$ the sum of the right hand sides of (6)–(12) exceeds the left hand sides by at least two, contradicting (13). We obtain $\omega_2 \leq 3$.

In the case of $\omega_2 = 3$, taking $L = \{2, 4\}$, $\omega_2 = 3$, $\omega_4 = 2$ into (6) we get the contradiction $R(3, 5) = 14 \leq R(3, 3) + R(3, 4) - 2 \times 2 + 1 = 12$.

The last case to consider is $\omega_4 = \omega_3 = \omega_2 = 2$. Substitute $L = \{2, 3, 4\}$ into (6) to obtain the final contradiction. \square

5. CONSTRUCTION

In the case of $n \geq 2k + 3$ the upper bound $n - 2k + q(k)$ for $Q(n, n - k)$ follows immediately from the definition of q and Theorem 4.1. Suppose that $k = \sum_{i=1}^s k_i$ such that $q(k) = \sum_{i=1}^s (\omega(2k_i + 1) - 1)$ where $k \geq s \geq 1$ and $s \leq 3$. There is a graph H_i with $2k_i + 1$ vertices and with $\omega(H_i) = \omega(2k_i + 1)$ and with no three independent vertices. Define the n -vertex graph G in the following way. Consider the vertex disjoint union of H_1, \dots, H_s and $n - \sum_{i=1}^s (2k_i + 1)$ extra vertices. Then put an edge between any two vertices u, v , unless $u, v \in H_i$ for some $1 \leq i \leq s$. We have that $\omega(G) = n - \sum_{i=1}^s (2k_i + 1) + \sum_{i=1}^s \omega(H_i)$, as stated.

Concerning the chromatic number of G , $\alpha(H_i) \leq 2$ implies that $\chi(H_i) \geq k_i + 1$, hence

$$\chi(G) = n - \sum_{i=1}^s (2k_i + 1) + \sum_{i=1}^s \chi(H_i) \geq n - \sum_{i=1}^s k_i = n - k.$$

To obtain an example with chromatic number exactly $n - k$ delete edges arbitrarily from G one by one until we obtain a subgraph G' with $\chi(G') = n - k$. Since edge deletion does not increase the clique number we have

$$Q(n, n - k) \leq \omega(G') \leq \omega(G) = n - 2k + q(k).$$

6. LOWER BOUND BY INDUCTION

We will use induction on k , the cases $k = 0, 1$ are easy. From now on, we suppose that $k \geq 2$. The definition of q immediately implies that

$$q(a) + q(k - a) \geq q(k)$$

for every integer $0 \leq a \leq q$. In particular we have

$$(14) \quad 1 + q(k - 2) \geq q(k)$$

(for $k \geq 2$) and for all $0 \leq x \leq q$

$$(15) \quad x + q(k - x) \geq q(x) + q(k - x) \geq q(k).$$

Let G be an n -vertex graph with $\chi(G) = n - k$, $n \geq 2k + 3$, $k \geq 2$. We will show a lower bound for $\omega(G)$. We distinguish two cases.

Case 1. $\alpha(G) \geq 3$. Let $S \subset V(G)$ be a three-element independent set. The chromatic number of the restricted graph $G \setminus S := G|(V \setminus S)$ is at least $\chi(G) - 1$. We have that $|V(G \setminus S)| - \chi(G \setminus S) \leq k - 2$. Since $n - 3 \geq 2(k - 2) + 3$ we can use induction to $G \setminus S$.

$$\begin{aligned} \omega(G) &\geq \omega(G \setminus S) \geq Q(n - 3, (n - 3) - (k - 2)) \\ &\geq (n - 3) - 2(k - 2) + q(k - 2) = n - 2k + (q(k - 2) + 1). \end{aligned}$$

Then, (14) yields the desired lower bound.

Case 2. $\alpha(G) = 2$. Consider \overline{G} , the complement of G . The chromatic number of G is $|V(G)|$ minus the matching number of \overline{G} , $\nu(\overline{G})$. So $\nu(\overline{G}) = k$. According to the Berge-Tutte formula, more exactly by the Edmonds-Gallai structure theorem (see, e.g., [9]) we have that there exists a partition of $V(\overline{G}) = V_0 \cup V_1 \cup \dots \cup V_{a+b+c} \cup X$ such that

- V_0 is the set of isolated vertices of \overline{G} ,
- $\overline{G} \setminus (X \cup V_0)$ has $a + b + c \geq 1$ components, namely V_1, \dots, V_{a+b+c} ,
- the sets V_1, \dots, V_a are singletons, we define $k_h = 0$ for $1 \leq h \leq a$,
- the sizes of V_{a+1}, \dots, V_{a+b} are odd, $|V_i| = 2k_i + 1$, $k_i \geq 1$ for $a < i \leq a + b$,
- the sizes of $V_{a+b+1}, \dots, V_{a+b+c}$ are even, $|V_j| = 2k_j$, $k_j \geq 1$ for $a + b < j \leq a + b + c$,
- the matching numbers $\nu(\overline{G}|V_i) = k_i$ for all $1 \leq i \leq a + b + c$,
- $0 \leq |X| \leq a + b$, and finally
- $k = \nu(\overline{G}) = \left(\sum_{i=1}^{a+b+c} k_i \right) + |X|$.

We obtain

$$\begin{aligned} \omega(G) &\geq |V_0| + \sum_{i=1}^{a+b+c} \omega(G|V_i) \\ &\geq \left(n - |X| - a - \sum_{i=a+1}^{a+b} (2k_i + 1) - \sum_{j=a+b+1}^{a+b+c} 2k_j \right) \\ &\quad + a + \sum_{i=a+1}^{a+b} \omega(2k_i + 1) + \sum_{j=a+b+1}^{a+b+c} \omega(2k_j) \\ &= n - |X| - 2 \left(\sum_{i=a+1}^{a+b+c} k_i \right) - b + \sum_{i=a+1}^{a+b} \omega(2k_i + 1) + \sum_{j=a+b+1}^{a+b+c} \omega(2k_j) \\ &= (n - 2k) + |X| + \sum_{i=a+1}^{a+b} (\omega(2k_i + 1) - 1) + \sum_{j=a+b+1}^{a+b+c} \omega(2k_j) \\ &\geq (n - 2k) + |X| + \sum_{i=a+1}^{a+b+c} (\omega(2k_i + 1) - 1) \\ &\geq (n - 2k) + |X| + q(k - |X|) \geq n - 2k + q(k). \end{aligned}$$

In the last step we used (15), and in the previous one we used the obvious inequality $\omega(x) + 1 \geq \omega(x + 1)$, which holds for every $x \geq 0$. This completes the proof of the lower bound for $\omega(G)$, and also the proof of the Theorem. \square

7. OPEN PROBLEMS AND RELATED QUESTIONS

The original motivation of this research was an analogue problem for partially ordered sets (posets).

A *realizer* is a set of linear extensions of the poset P , such that their intersection (as relations) is P . The minimum cardinality of a realizer is the dimension of the poset, a central notion in poset theory. The “standard example” S_n is the poset formed by considering the 1-element subsets and the $n - 1$ element subsets of a set of n elements, ordered by inclusion. It is well known that $\dim(S_n) = n$, but there are posets of arbitrarily large dimensions without including even S_3 as a subposet.

Hiraguchi [5] proved that the dimension does not exceed half of the number of elements of the poset. Bogart and Trotter [3] showed that for large n , the only n -dimensional poset on $2n$ points is S_n . But what happens if the dimension is slightly less than half the number of elements, is not known. We conjecture the following.

Conjecture 7.1. *For every $t < 1$, but sufficiently close to 1 there is a $c > 0$, if a poset has $2n$ points, and its dimension is at least tn , then it contains a standard example of dimension cn .*

It is frequently noted that poset problems can be translated to graph theory problems and vice versa by changing chromatic numbers of graphs to dimension of posets, and cliques in graphs to standard examples in posets. Note that the above conjecture would translate to the following statement: For every $t < 1$, but sufficiently close to 1 there is a $c > 0$ such that if a graph has n points, and its chromatic number is at least tn , then it contains a clique of cn points. This graph version is trivial for all $t > 1/2$.

The behavior of the sequence $\{Q(n, \lceil tn \rceil)\}_{n=1}^{\infty}$ is also interesting in case $t \leq 1/2$. On one hand, our result implies that $Q(n, \lceil tn \rceil) \leq Q(n, \lceil (n+3)/2 \rceil) \leq 4 + q(\lceil n/2 \rceil) = O(\sqrt{n \log n})$. On the other hand, $\lim_{n \rightarrow \infty} Q(n, \lceil tn \rceil) \rightarrow \infty$: it follows from the result that a graph with no cliques of size t and n vertices has independence number $\Omega(n^{1/t})$, which is a straightforward consequence of the classical Erdős–Szekeres bound for the Ramsey numbers.

Problem 7.2. *It would be interesting to clarify more precisely what is the exact relation between the (inverse of the) corresponding Ramsey number and $Q(n, \lceil tn \rceil)$.*

A significant first step was done by Liu [8] for the case when $1/t$ is a fixed integer and $n \rightarrow \infty$.

We have reduced the determination of $Q(n, n-k)$ to the classical Ramsey number problem $R(3, \ell)$ whenever $n \geq 2k + 3$. It seems that with a bit of more care one can lower the bound of n to $n \geq 2k + 2$. But below $2k$ one (probably) needs to use $R(4, \ell)$, too.

In the tableaux (3) one can find only a single case when $q(k)$ and $\omega(2k + 1) - 1$ differ from each other, namely $k = 4$. We *conjecture* that this is the only case, i.e., $q(k) = \omega(2k + 1) - 1$ for $k \geq 5$.

We can also observe that $q(k) = \lceil k/2 \rceil$ holds for $k = 0, 1, 2, \dots, 12$ and for $k = 14$ but it does not hold for any other value.

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REFERENCES

- [1] Ajtai, M., Komlós, J., Szemerédi, E.: A note on Ramsey numbers. *J. Combin. Theory Ser. A* **29**(3), 354–360 (1980)
- [2] Biró, C.: Large cliques in graphs with high chromatic number (2011). ArXiv:1107.2630
- [3] Bogart, K.P., Trotter, W.T.: Maximal dimensional partially ordered sets. II. Characterization of $2n$ -element posets with dimension n . *Discrete Math.* **5**, 33–43 (1973)
- [4] Gyárfás, A., Sebő, A., Trotignon, N.: The chromatic gap and its extremes (2011). ArXiv:1108.3444
- [5] Hiraguchi, T.: On the dimension of orders. *Sci. Rep. Kanazawa Univ.* **4**(1), 1–20 (1955)
- [6] Jahanbekam, S., West, D.B.: <http://www.math.uiuc.edu/~west/regs/chromcliq.html>
- [7] Kim, J.H.: The Ramsey number $R(3, t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms* **7**(3), 173–207 (1995)
- [8] Liu, G.: Minimum clique number, chromatic number, and Ramsey numbers. *Electron. J. Combin.* **19**, Research Paper 55, 10 pp. (electronic) (2012). URL <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v19i1p55>
- [9] Lovász, L., Plummer, M.D.: Matching theory, *North-Holland Mathematics Studies*, vol. 121. North-Holland Publishing Co., Amsterdam (1986). *Annals of Discrete Mathematics*, 29
- [10] Radziszowski, S.P.: Small Ramsey numbers. *Electron. J. Combin.* **1**, Dynamic Survey 1, (electronic) (1994). URL <http://www.combinatorics.org/Surveys/index.html>. (version 2011 aug)
- [11] Xu, X.D., Xie, Z., Radziszowski, S.P.: A constructive approach for the lower bounds on the Ramsey numbers $R(s, t)$. *J. Graph Theory* **47**(3), 231–239 (2004)

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