# Nordhaus-Gaddum-type theorem for rainbow connection number of graphs 

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#### Abstract

An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of $G$, denoted $r c(G)$, is the minimum number of colors that are used to make $G$ rainbow connected. In this paper we give a Nordhaus-Gaddum-type result for the rainbow connection number. We prove that if $G$ and $\bar{G}$ are both connected, then $4 \leq r c(G)+r c(\bar{G}) \leq n+2$. Examples are given to show that the upper bound is sharp for all $n \geq 4$, and the lower bound is sharp for all $n \geq 8$. For the rest small $n=4,5,6,7$, we also give the sharp bounds.


Keywords: edge-colored graph, rainbow connection number, Nordhaus-Gaddumtype.

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## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined terminology and notations can be found in [1. Let $G$ be a nontrivial connected graph with an edge coloring $c: E(G) \rightarrow\{1,2, \cdots, k\}, k \in \mathbb{N}$, where adjacent edges may be colored the same. A path $P$ of $G$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G$ is called rainbow-connected if for any two vertices $u$ and $v$ of $G$, there is a rainbow $u-v$ path. The minimum number of colors for which there is an edge coloring of $G$ such that $G$ is rainbow connected is called the rainbow connection number, denoted by $r c(G)$. Clearly, if a graph is rainbow connected, then it is also connected. Conversely,
any connected graph has a trivial edge coloring that makes it rainbow connected, just by assigning each edge a distinct color. An easy observation is that if $G$ has $n$ vertices then $r c(G) \leq n-1$, since one may color the edges of a spanning tree with distinct colors, and color the remaining edges with one of the colors already used. It is easy to see that if $H$ is a connected spanning subgraph of $G$, then $r c(G) \leq r c(H)$. It is easy to see that $\operatorname{rc}(G)=1$ if and only if $G$ is a clique, and $r c(G)=n-1$ if and only if $G$ is a tree, as well as that a cycle with $k>3$ vertices has a rainbow connection number $\lceil k / 2\rceil$. Also notice that $r c(G) \geq \operatorname{diam}(G)$, where $\operatorname{diam}(G)$ denotes the diameter of $G$.

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus-Gaddum-type" is so given because it is Nordhaus and Gaddum [3] who first established the following type of inequalities for chromatic number of graphs in 1956. They proved that if $G$ and $\bar{G}$ are complementary graphs on $n$ vertices whose chromatic numbers are $\chi(G), \chi(\bar{G})$, respectively, then

$$
2 \sqrt{n} \leq \chi(G)+\chi(\bar{G}) \leq n+1
$$

Since then, many analogous inequalities of other graph parameters are concerned, such as diameter [4], domination number [5, Wiener index and some other chemical indices [6], and so on. In this paper, we are concerned with analogous inequalities involving the rainbow connection number of graphs, we prove that

$$
4 \leq r c(G)+r c(\bar{G}) \leq n+2
$$

The rest of this paper is organized as follows. First, we give the upper bound, and show that it is sharp for all $n \geq 4$. Then we give the lower bound, and show that it is also sharp for $n \geq 8$. Finally, for the rest small $n=4,5,6,7$, we give the sharp bound, respectively.

## 2 Upper bound on $r c(G)+r c(\bar{G})$

We know that if $G$ is a connected graph with $n$ vertices, then the number of the edges in $G$ must be at least $n-1$. So if both $G$ and $\bar{G}$ are connected then $n$ is not less than 4, since

$$
\begin{equation*}
2(n-1) \leq e(G)+e(\bar{G})=e\left(K_{n}\right)=\frac{n(n-1)}{2} . \tag{*}
\end{equation*}
$$

In the rest of the paper, we always assume that all graphs have at least 4 vertices, and both $G$ and $\bar{G}$ are connected.

Lemma $1 \operatorname{rc}(G)+r c(\bar{G}) \leq n+2$ for $n=4,5$, and the bound is sharp.

Proof. Note that $r c(G) \leq n-1$, equality holds if and only if $G$ is a tree. So

$$
r c(G)+r c(\bar{G}) \leq 2(n-1)
$$

equality holds if and only if both $G$ and $\bar{G}$ are trees. Then (*) must holds with equality. That is, $n$ have to be 4, and

$$
r c(G)+r c(\bar{G})=2(n-1)=6=4+2
$$

Then

$$
r c(G)+r c(\bar{G}) \leq 2 n-3
$$

for $n \geq 5$.
For $n=5$, let $G$ be a tree obtained from $S_{4}$ by attaching a pendent edge to one of the vertices of degree one. Then $\operatorname{rc}(G)=4$. We observe that $\operatorname{diam}(\bar{G})=3$ and it can be colored by three colorings to make it rainbow connected. Thus $r c(\bar{G})=3$. Therefore, we have

$$
r c(G)+r c(\bar{G})=7=2 n-3=5+2 .
$$

Lemma 2 Let $G$ be a nontrivial connected graph of order $n$, and $r c(G)=k$. Let $c$ : $E(G) \rightarrow\{1,2, \cdots, k\}$ be a rainbow $k$-coloring of $G$. Add a new vertex $P$ to $G, P$ is adjacent to $q$ vertices of $G$, the resulting graph is denoted by $G^{\prime}$. Then if $q \geq n+1-k$, we have $r c\left(G^{\prime}\right) \leq k$.

Proof. Let $X=\left\{x_{1}, x_{2}, \cdots, x_{q}\right\}$ be the vertices adjacent to $P, V \backslash X=\left\{y_{1}, y_{2}, \cdots, y_{n-q}\right\}$. If $q \geq n+1-k, n-q \leq k-1$.

Since $G$ rainbow connected under the coloring $c$, for any $y_{i}, i \in\{1,2, \cdots, n-q\}$, there is a rainbow $x_{1}-y_{i}$ path, say $P_{x_{1} y_{1}}, P_{x_{1} y_{2}}, \cdots, P_{x_{1} y_{n-q}}$. For each $P_{x_{1} y_{i}}$, we find out the last vertex on the path that belongs to $X$, and the subpath between this vertex to $y_{i}$ of $P_{x_{1} y_{i}}$ is denoted by $P_{i}$. Then $P_{i}$ is a rainbow path whose vertices are in $Y$ except the first vertex.

Let $G_{x_{i}}$ be the union of the paths in $P_{1}, P_{2}, \cdots, P_{n-q}$ whose origin vertex is $x_{i}, 1 \leq$ $i \leq q$. If there is no path with origin vertex $x_{i}$, let $G_{x_{i}}$ be a trivial graph with the vertex $x_{i}$. Then $G_{x_{i}}$ is a subgraph of $G$, and $v\left(G_{x_{i}}\right) \leq n-q+1 \leq k$. First, we consider the subgraph $G_{x_{1}}$, and let $V\left(G_{x_{1}}\right)=\left\{x_{1}, y_{i_{1}}, y_{i_{2}}, \cdots, y_{i_{l}}\right\}$.

Case 1: The number of colors appeared in $G_{x_{1}}$ is $k$. Then $e\left(G_{x_{1}}\right) \geq k$.
Subcase 1.1: $e\left(G_{x_{1}}\right)=k \geq v\left(G_{x_{1}}\right)$.
In this case, $G_{x_{1}}$ contains a cycle, and no two edges of $G_{x_{1}}$ are colored the same. Thus, $G_{x_{1}}$ is rainbow connected. Let $e$ be an edge in the cycle. Then, by deleting $e$ and coloring
the edge $P x_{1}$ with the color $c(e)$, we have that, for any $j \in\{1,2, \cdots, l\}$, there is a rainbow $P-y_{i_{j}}$ path.

Subcase $1.2 e\left(G_{x_{1}}\right)>k \geq v\left(G_{x_{1}}\right)$.
In this case, $G_{x_{1}}$ contains a cycle, and there are two edges $e_{1}, e_{2}$ with $c\left(e_{1}\right)=c\left(e_{2}\right)$.
If one of the edges, say $e_{1}$, is contained in a cycle of $G_{x_{1}}$. Then, by deleting it, we obtain a spanning subgraph $G_{x_{1}}^{\prime}$ of $G_{x_{1}}$ with the same number of colors appearing in it, but $e\left(G_{x_{1}}^{\prime}\right)=e\left(G_{x_{1}}\right)-1$. If $e\left(G_{x_{1}}^{\prime}\right)=k$, by a similar operation as in Subcase 1.1, we can obtain a coloring of $P x_{1}$ such that for any $j \in\{1,2, \cdots, l\}$, there is a rainbow $P-y_{i_{j}}$ path. If $e\left(G_{x_{1}}^{\prime}\right)>k$, we consider the graph $G_{x_{1}}^{\prime}$ other than $G_{x_{1}}$.

If both $e_{1}, e_{2}$ are not in a cycle, they must be cut edges of $G_{x_{1}}$. Then, contract one of them, say $e_{1}$, and denote the resultant graph by $G_{x_{1}}^{\prime \prime}$. The number of colors appeared in $G_{x_{1}}$ is still $k$, and $v\left(G_{x_{1}}^{\prime \prime}\right)=v\left(G_{x_{1}}\right)-1, e\left(G_{x_{1}}^{\prime \prime}\right)=e\left(G_{x_{1}}\right)-1$. If $e\left(G_{x_{1}}^{\prime \prime}\right)=k$, by a similar operation as in Subcase 1.1, we can obtain a coloring of $P x_{1}$ such that for any $y_{k}$ in $G_{x_{1}}^{\prime \prime}$, there is a rainbow $P-y_{k}$ path. It is easy to check that there still exists a rainbow $P-y_{i_{j}}$ path in $G_{x_{1}}$ for any $j \in\{1,2, \cdots, l\}$. If $e\left(G_{x_{1}}^{\prime \prime}\right)>k$, we consider the graph $G_{x_{1}}^{\prime \prime}$ other than $G_{x_{1}}$.

Case 2: The number of colors appeared in $G_{x_{1}}$ is less than $k$. Then we color the edge $P x_{1}$ with a color not appeared in $G_{x_{1}}$.

No matter which cases happen, we can always color the edge $P x_{1}$ with one of the colors $\{1,2, \cdots, k\}$, such that for any $j \in\{1,2, \cdots, l\}$, there is a rainbow $P-y_{i_{j}}$ path.

For $G_{x_{2}}, G_{x_{3}}, \cdots, G_{x_{q}}$, we use the same way to color the edges $P x_{2}, P x_{3}, \cdots, P x_{q}$. Then we get a $k$-coloring of $G^{\prime}$. Since for each $y_{i}$, there is an $x_{j}$, such that $y_{i} \in G_{x_{j}}$. Then the path $P x_{j} P_{i}$ is a rainbow path connecting $P$ and $y_{i}$. Thus in this coloring, $G^{\prime}$ is rainbow connected. Therefore $r c\left(G^{\prime}\right) \leq k$.

Theorem $1 \operatorname{rc}(G)+r c(\bar{G}) \leq n+2$ for all $n \geq 4$, and this bound is best possible.
Proof. We use induction on $n$. From Lemma 1, the result is true for $n=4,5$. We assume that $r c(G)+r c(\bar{G}) \leq n+2$ holds for complementary graphs on $n$ vertices. To the union of connected graphs $G$ and $\bar{G}$, a complete graph on the $n$ vertices, we adjoin a new vertex $P$. Let $q$ be the number of vertices of $G$ which are adjacent to $P$, then the number of vertices of $\bar{G}$ which are adjacent to $P$ is $n-q$. If $G^{\prime}$ and $\overline{G^{\prime}}$ are the resultant graphs (each of order $n+1$ ), then

$$
r c\left(G^{\prime}\right) \leq r c(G)+1, r c\left(\overline{G^{\prime}}\right) \leq r c(\bar{G})+1
$$

These inequalities are evident from the fact that if given a rainbow $r c(G)$-coloring $(r c(\bar{G})$ coloring) of $G(\bar{G})$, we assign a new color to the edges added from $P$ to $G(\bar{G})$, the resulting
coloring makes $G^{\prime}\left(\overline{G^{\prime}}\right)$ rainbow connected. Then $r c\left(G^{\prime}\right)+r c\left(\overline{G^{\prime}}\right) \leq r c(G)+r c(\bar{G})+2 \leq$ $n+4$. And $r c\left(G^{\prime}\right)+r c\left(\overline{G^{\prime}}\right) \leq n+3$ except possibly when

$$
r c\left(G^{\prime}\right)=r c(G)+1, r c\left(\overline{G^{\prime}}\right)=r c(\bar{G})+1 .
$$

In this case, by Lemma 2, $q \leq n-r c(G), n-q \leq n-r c(\bar{G})$, thus $r c(G)+r c(\bar{G}) \leq n$, from which $r c\left(G^{\prime}\right)+r c\left(\overline{G^{\prime}}\right) \leq n+2$. This completes the induction.

To see the bound can be attained, let $G$ be a tree obtained by joining the centers of two stars $S_{p}$ and $S_{q}$ by an edge $u v$, where $u$ and $v$ are the centers of $S_{p}$ and $S_{q}$, and $p+q=n$. Then $\operatorname{rc}(G)=n-1$. To compute the rainbow connection number of the complement graph of $G$, we assume that $X=V\left(S_{p} \backslash u\right) \cup\{v\}, Y=V\left(S_{q} \backslash v\right) \cup\{u\}$. Then $G$ is a bipartite graph with bipartition $(X, Y)$. Thus $\bar{G}[X], \bar{G}[Y]$ is complete. We assign color 1 to $\bar{G}[X], 2$ to $\bar{G}[Y]$ and 3 to the edges between $X$ and $Y$. The resulting coloring makes $\bar{G}$ rainbow connected, thus $r c(\bar{G}) \leq 3$. On the other hand $\operatorname{diam}(G)=d(u, v)=3$, it follows that $r c(\bar{G})=3$. Then we have $r c\left(G^{\prime}\right)+r c\left(\overline{G^{\prime}}\right)=n+2$.

## 3 Lower bound on $r c(G)+r c(\bar{G})$

As we have noted that $r c(G)=1$ if and only if $G$ is a complete graph. In this case, $\bar{G}$ is not connected. Thus if both $G$ and $\bar{G}$ are connected, $\operatorname{rc}(G) \geq 2, r c(\bar{G}) \geq 2$. That is, $r c(G)+r c(\bar{G}) \geq 4$.

Proposition 1 Let $G$ and $\bar{G}$ be complementary connected graphs with $r c(G)=r c(\bar{G})=2$. Then
(1) $\operatorname{diam}(G)=\operatorname{diam}(\bar{G})=2$.
(2) $2 \leq \delta(G) \leq \Delta(G) \leq n-3,2 \leq \delta(\bar{G}) \leq \Delta(\bar{G}) \leq n-3$.
(3) A vertex $u$ in $N_{1}(v)$ can not be adjacent to all vertices of $N_{2}(v)$, where $N_{1}(v), N_{2}(v)$ is the first and second neighborhood of a vertex $v$, respectively.

Proof. Since $2 \leq \operatorname{diam}(G) \leq r c(G)=2$, (1) clearly holds.
For (2), first, $\Delta(G) \neq n-1$, otherwise $\bar{G}$ is disconnected.
Second, $\delta(G) \neq 1$. Indeed, if $\delta(G)=1$, let $v$ be a vertex of degree one, and $u$ the vertex adjacent to $v$. Since $\operatorname{diam}(G)=2, u$ must be adjacent to all the other vertices, thus $d(u)=n-1$, a contradiction. Similarly, $\delta(\bar{G}) \neq 1$. That is, $\delta(\bar{G}) \geq 2$. Therefore, $\Delta(G) \leq n-1-\delta(\bar{G}) \leq n-3$, so does $\Delta(\bar{G})$.

For (3), if $u$ is adjacent to all vertices of $N_{2}(v), u$ is not adjacent to them in $\bar{G}$, then the distance between $u$ and $N_{2}(v)$ is at least 2 in $\bar{G}$, and $v$ is adjacent to all vertices of $N_{2}(v)$, but not to the vertices in $N_{1}(v)$ of $\bar{G}$. So $d_{\bar{G}}(u, v) \geq 3$, which contradicts (1).

Theorem 2 For $4 \leq n \leq 7$, there are no graphs $G$ and $\bar{G}$ on $n$ vertices, such that $r c(G)=r c(\bar{G})=2$.

Proof. We consider $n=4,5,6,7$, respectively.
Case 1: $\mathrm{n}=4$.
Then, there is only one pair of complementary connected graphs, each is isomorphic to $P_{4}$, and its rainbow connection number is 3 .

Case 2: $\mathrm{n}=5$.
If $r c(G)=r c(\bar{G})=2$, by Proposition $1,2 \leq \delta(G) \leq \Delta(G) \leq n-3=2$. Then, $G \cong C_{5}, \bar{G} \cong C_{5}$. Since $r c\left(C_{5}\right)=3$, there are no graphs $G$ and $\bar{G}$ on 5 vertices, such that $r c(G)=r c(\bar{G})=2$.

Case 3: $\mathrm{n}=6$.
By Proposition $1,2 \leq \delta(G) \leq \Delta(G) \leq n-3=3$, the possible degree sequences are:
(a) $\left\{\begin{array}{l}d_{G}=(2,2,2,2,2,2) \\ d_{\bar{G}}=(3,3,3,3,3,3) .\end{array}\right.$
(b) $\left\{\begin{array}{l}d_{G}=(3,3,2,2,2,2) \\ d_{\bar{G}}=(2,2,3,3,3,3) .\end{array}\right.$

The graph $G$ with the degree sequence in (a) is a cycle of length 6 , whose rainbow connection is 3 . And the graph $G$ with the degree sequence in (b) satisfying Proposition 1 has to be the graph shown in Figure 1:


G

Figure 1: Graphs with degree sequence $(3,3,2,2,2,2)$ satisfying Proposition 1.

Consider the pair of vertices $\left(v_{2}, v_{4}\right)$. The only 2-path is $v_{2} v_{1} v_{4}$, thus $c\left(v_{1} v_{2}\right) \neq c\left(v_{1} v_{4}\right)$. Similarly, $c\left(v_{1} v_{3}\right) \neq c\left(v_{1} v_{4}\right)$, then $c\left(v_{1} v_{2}\right)=c\left(v_{1} v_{3}\right)$. If we consider the pairs of vertices $\left(v_{2}, v_{6}\right),\left(v_{3}, v_{6}\right)$, we have $c\left(v_{2} v_{5}\right)=c\left(v_{3} v_{5}\right)$. But then there is no rainbow $v_{2}-v_{3}$ path, therefore $r c(G) \neq 2$.

Case 4: $\mathrm{n}=7$.

By Proposition $1,2 \leq \delta(G) \leq \Delta(G) \leq n-3=4$, the possible degree sequences are: (in the following argument, we use two colors to color the edges of the graphs)
(1) $\left\{\begin{aligned} d_{G} & =(4,4,4,4,4,4,4) \\ d_{\bar{G}} & =(2,2,2,2,2,2,2) .\end{aligned}\right.$

In this case, $\bar{G}$ is a cycle of length $7, \operatorname{rc}(\bar{G})=4$.
(2) $\left\{\begin{array}{l}d_{G}=(4,4,4,4,4,3,3) \\ d_{\bar{G}}=(2,2,2,2,2,3,3) .\end{array}\right.$

The graphs with the degree sequence $(4,4,4,4,4,3,3)$ satisfying Proposition 1 are $G_{1}, G_{2}$ shown in Figure 2. The distance between $v_{2}$ and $v_{5}$ in $\overline{G_{1}}, \overline{G_{2}}$ is larger than 2, thus $r c\left(\overline{G_{1}}\right) \neq 2, r c\left(\overline{G_{2}}\right) \neq 2$.


Figure 2: Graphs with degree sequence $(4,4,4,4,4,3,3)$ satisfying Proposition 1.
(3) $\left\{\begin{aligned} d_{G} & =(4,4,4,3,3,3,3) \\ d_{\bar{G}} & =(2,2,2,3,3,3,3) .\end{aligned}\right.$

The graphs with degree sequence ( $4,4,4,3,3,3,3$ ) satisfying Proposition 1 are subgraphs $G_{1}^{\prime}, G_{2}^{\prime}$ of $G_{1}, G_{2}$ shown in Figure 2 by deleting the edge $v_{2} v_{5}$. We observe that the distance between $v_{3}$ and $v_{4}$ in $\overline{G_{1}^{\prime}}, \overline{G_{2}^{\prime}}$ is larger than 2 , thus $r c\left(\overline{G_{1}^{\prime}}\right) \neq 2 \operatorname{rec}\left(\overline{G_{2}^{\prime}}\right) \neq 2$.
(4) $\left\{\begin{aligned} d_{G} & =(4,3,3,3,3,3,3) \\ d_{\bar{G}} & =(2,3,3,3,3,3,3) .\end{aligned}\right.$

The graphs $G$ with degree sequence ( $4,3,3,3,3,3,3$ ) satisfying Proposition 1 are $G_{1}, G_{2}$ and $G_{3}$ shown in Figure 3.

$G_{1}$

$G_{2}$

$G_{3}$

Figure 3: Graphs with degree sequence $(4,3,3,3,3,3,3)$ satisfying Proposition 1.

Consider $G_{1}$. Since the only 2-path between $v_{3}$ and $v_{4}$ is $v_{3} v_{1} v_{4}, c\left(v_{3} v_{1}\right) \neq c\left(v_{1} v_{4}\right)$. Similarly, $c\left(v_{3} v_{1}\right) \neq c\left(v_{1} v_{5}\right)$. Then $c\left(v_{1} v_{4}\right)=c\left(v_{1} v_{5}\right)$, say color 2. By the same way, $c\left(v_{3} v_{1}\right)=c\left(v_{1} v_{2}\right)=1$. Consider the pairs of vertices $\left(v_{2}, v_{7}\right),\left(v_{3}, v_{7}\right),\left(v_{4}, v_{6}\right),\left(v_{5}, v_{6}\right)$, we have $c\left(v_{2} v_{6}\right)=c\left(v_{3} v_{6}\right)=c\left(v_{4} v_{7}\right)=c\left(v_{5} v_{7}\right)$. If $c\left(v_{2} v_{6}\right)=1$, there is no rainbow $v_{1}-v_{6}$ path, and if $c\left(v_{2} v_{6}\right)=2$, there is no rainbow $v_{1}-v_{7}$ path. Therefore $r c\left(G_{1}\right) \neq 2$.

Consider $G_{2}$, whose rainbow connection number is 2 , where the heavy lines is colored by color 2 , the others are colored by color 1 . So we consider its complement graph, the only 2-path between $v_{6}$ and $v_{7}$ is $v_{6} v_{1} v_{7}$, then $c\left(v_{6} v_{1}\right) \neq c\left(v_{1} v_{7}\right)$. Let $c\left(v_{6} v_{1}\right)=1, c\left(v_{1} v_{7}\right)=2$, thus $c\left(v_{3} v_{7}\right)=c\left(v_{2} v_{7}\right)=1, c\left(v_{4} v_{6}\right)=c\left(v_{5} v_{6}\right)=2$. If $c\left(v_{2} v_{4}\right)=2$, there is no rainbow $v_{2}-v_{6}$ path, and if $c\left(v_{2} v_{4}\right)=1$, there is no rainbow $v_{4}-v_{7}$ path. Therefore, we cannot use two colors to make $\overline{G_{2}}$ rainbow connected, that is $r c\left(\overline{G_{2}}\right) \neq 2$.

For $G_{3}$, whose rainbow connection number is also 2, by coloring the heavy lines with color 2 , and the others with color 1 . We consider its complement graph. By the same reason as above for $\overline{G_{2}}$, let $c\left(v_{6} v_{1}\right)=1, c\left(v_{1} v_{7}\right)=2, c\left(v_{3} v_{7}\right)=c\left(v_{2} v_{7}\right)=1, c\left(v_{4} v_{6}\right)=$ $c\left(v_{5} v_{6}\right)=2$. Then, if $c\left(v_{2} v_{5}\right)=2$, there is no rainbow $v_{2}-v_{6}$ path, and if $c\left(v_{2} v_{5}\right)=1$, there is no rainbow $v_{5}-v_{7}$ path. Therefore, we cannot use two colors to make $\overline{G_{2}}$ rainbow connected, that is $r c\left(\overline{G_{2}}\right) \neq 2$.

$$
(5)\left\{\begin{array}{l}
d_{G}=(4,4,4,4,3,3,2) \\
d_{\bar{G}}=(2,2,2,2,3,3,4) .
\end{array}\right.
$$

The graphs $G$ with degree sequence (4,4,4,4,3,3,2) satisfying Proposition 1 are $G_{1}, G_{2}, G_{3}, G_{4}$ shown in Figure 4.


Figure 4: Graphs with degree sequence ( $4,4,4,4,3,3,2$ ) satisfying Proposition 1.

Since $d_{\overline{G_{1}}}\left(v_{4}, v_{7}\right)=3, d_{\overline{G_{2}}}\left(v_{2}, v_{5}\right)=3, d_{\overline{G_{4}}}\left(v_{2}, v_{6}\right)=3, r c\left(\overline{G_{1}}\right) \geq 3, r c\left(\overline{G_{2}}\right) \geq 3$, $r c\left(\overline{G_{4}}\right) \geq 3$.

For $\overline{G_{3}}$, consider the pair of vertices $\left(v_{4}, v_{5}\right)$. the only 2-path is $v_{4} v_{6} v_{5}$, so $c\left(v_{4} v_{6}\right) \neq$ $c\left(v_{5} v_{6}\right)$, and let $c\left(v_{5} v_{6}\right)=2$. Similarly, $c\left(v_{3} v_{5}\right)=c\left(v_{2} v_{5}\right)=1, c\left(v_{7} v_{2}\right)=c\left(v_{7} v_{3}\right)=2$, $c\left(v_{7} v_{1}\right)=c\left(v_{7} v_{4}\right)=1$,. If $c\left(v_{1} v_{6}\right)=2$, there is no rainbow $v_{1}-v_{5}$ path, and if $c\left(v_{1} v_{6}\right)=1$, there is no rainbow $v_{1}-v_{4}$ path. Thus, $r c\left(\overline{G_{3}}\right) \neq 2$.
(6) $\left\{\begin{array}{l}d_{G}=(4,4,3,3,3,3,2) \\ d_{\bar{G}}=(2,2,3,3,3,3,4) .\end{array}\right.$

The graphs $\bar{G}$ with degree sequence ( $2,2,3,3,3,3,4$ ) satisfying Proposition 1 have to be the following three graphs.

For $G_{1}$, consider the pair of vertices $\left(v_{2}, v_{7}\right)$. There is only one 2 -path $v_{2} v_{6} v_{7}$ between them, then let $c\left(v_{2} v_{6}\right)=1, c\left(v_{6} v_{7}\right)=2$. Similarly, consider the pair of vertices $\left(v_{3}, v_{7}\right)$, we have $c\left(v_{3} v_{6}\right)=1$. Consider the pairs of vertices $\left(v_{5}, v_{6}\right),\left(v_{4}, v_{7}\right),\left(v_{4}, v_{6}\right)$, we get $c\left(v_{5} v_{7}\right)=1, c\left(v_{4} v_{5}\right)=2, c\left(v_{3} v_{4}\right)=2$. Consider the pairs of vertices $\left(v_{2}, v_{5}\right),\left(v_{3}, v_{5}\right)$, then $c\left(v_{2} v_{1}\right)=c\left(v_{3} v_{1}\right)$, thus there is no rainbow $v_{2}-v_{3}$ path.

For $G_{2}$, consider the pairs of vertices $\left(v_{4}, v_{6}\right),\left(v_{5}, v_{6}\right)$, we have $c\left(v_{4} v_{7}\right)=c\left(v_{5} v_{7}\right)$, consider the pairs of vertices $\left(v_{3}, v_{4}\right),\left(v_{3}, v_{5}\right)$, we get $c\left(v_{1} v_{4}\right)=c\left(v_{1} v_{5}\right)$, thus there is no rainbow $v_{4}-v_{5}$ path.

For $G_{3}$, its rainbow connection number is 2 by coloring the heavy lines with color 2 , and assign color 1 to the other edges. So we consider its complement graph shown in the figure too. Since the only 2-path between $v_{6}$ and $v_{7}$ is $v_{6} v_{1} v_{7}$, let $c\left(v_{1} v_{7}\right)=1, c\left(v_{1} v_{6}\right)=2$. Thus $c\left(v_{7} v_{2}\right)=c\left(v_{7} v_{3}\right)=2, c\left(v_{6} v_{4}\right)=c\left(v_{6} v_{5}\right)=1, c\left(v_{3} v_{5}\right)=2, c\left(v_{2} v_{4}\right)=1$. If $c\left(v_{2} v_{5}\right)=1$, there is no rainbow $v_{2}-v_{6}$ path. If $c\left(v_{2} v_{5}\right)=2$, there is no rainbow $v_{5}-v_{7}$ path.


Figure 5: Graphs with degree sequence $(2,2,3,3,3,3,4)$ satisfying Proposition 1.
(7) $\left\{\begin{aligned} d_{G} & =(4,4,4,3,3,2,2) \\ d_{\bar{G}} & =(2,2,2,3,3,4,4) .\end{aligned}\right.$

The graphs $\bar{G}$ with degree sequence $(2,2,2,3,3,4,4)$ satisfying Proposition 1 have to be the following two graphs.

For $G_{1}$, consider the pair of vertices $\left(v_{1}, v_{7}\right)$, the only 2-path between them is $v_{1} v_{5} v_{7}$, thus let $c\left(v_{1} v_{5}\right)=1, c\left(v_{5} v_{7}\right)=2$. Similarly, $c\left(v_{6} v_{7}\right)=1, c\left(v_{2} v_{6}\right)=c\left(v_{3} v_{6}\right)=c\left(v_{4} v_{6}\right)=2$, $c\left(v_{2} v_{1}\right)=c\left(v_{3} v_{1}\right)=c\left(v_{4} v_{1}\right)=2$. Therefore, there is no rainbow $v_{2}-v_{3}$ path.

For $G_{2}$, consider the pairs of vertices $\left(v_{2}, v_{7}\right),\left(v_{3}, v_{7}\right)$, we have $c\left(v_{2} v_{6}\right)=c\left(v_{3} v_{6}\right)$, consider the pairs of vertices $\left(v_{2}, v_{5}\right),\left(v_{3}, v_{5}\right)$, then $c\left(v_{2} v_{1}\right)=c\left(v_{3} v_{1}\right)$, so there is no rainbow $v_{2}-v_{3}$ path.


Figure 6: Graphs with degree sequence $(2,2,2,3,3,4,4)$ satisfying Proposition 1.
(8) $\left\{\begin{aligned} d_{G} & =(4,2,2,2,2,2,2) \\ d_{\bar{G}} & =(2,4,4,4,4,4,4) .\end{aligned}\right.$

There is no graph with degree sequence $(4,2,2,2,2,2,2)$ satisfying Proposition 1.
(9) $\left\{\begin{array}{l}d_{G}=(4,4,2,2,2,2,2) \\ d_{\bar{G}}=(2,2,4,4,4,4,4) .\end{array}\right.$

The graph $G$ with degree sequence $(4,4,2,2,2,2,2)$ satisfying Proposition 1 is the subgraph $G^{\prime}$ of $G_{1}$ depicted in Figure 6 by deleting the edge $v_{3} v_{4}$. Consider the pairs of vertices $\left(v_{2}, v_{7}\right),\left(v_{3}, v_{7}\right)$, we have $c\left(v_{2} v_{6}\right)=c\left(v_{3} v_{6}\right)$, consider the pairs of vertices $\left(v_{2}, v_{5}\right),\left(v_{3}, v_{5}\right)$, we get $c\left(v_{2} v_{1}\right)=c\left(v_{3} v_{1}\right)$. Then there is no rainbow $v_{2}-v_{3}$ path. Therefore $r c\left(G^{\prime}\right) \neq 2$.
$(10)\left\{\begin{array}{l}d_{G}=(4,4,4,2,2,2,2) \\ d_{\bar{G}}=(2,2,2,4,4,4,4) .\end{array}\right.$
There is no graph with degree sequence $(4,4,4,2,2,2,2)$ satisfying Proposition 1.
Theorem 3 For $n \geq 8$, the lower bound $r c(G)+r c(\bar{G}) \geq 4$ is best possible, that is, there are connected graphs $G$ and $\bar{G}$ on $n$ vertices, such that $r c(G)=r c(\bar{G})=2$.

Proof. For $n=8$, see figure $G_{8}$, colored with two colors, the heavy line with color 2 , the others with color 1. It is easy to check that they are rainbow connected.


Figure 7: $r c(G)=r c(\bar{G})=2$ for $n=8$.

If $n=4 k$, let $G$ be the graph with vertex set $X \cup Y \cup\{v\}$, where $X=\left(x_{1}, x_{2}, \cdots, x_{2 k-1}\right)$, $Y=\left\{y_{1}, y_{2}, \cdots, y_{2 k}\right\}$, such that $N(v)=X, X$ is an independent set, $G[Y]$ is a clique, and for each $x_{i}, x_{i}$ is adjacent to $y_{i}, y_{i+1}, \cdots, y_{i+k}$, where the sum is taken modulo $2 k$.

We define a coloring $c$ for the graph $G$ by the following rules:

$$
c(e)= \begin{cases}2 & \text { if } e=v x_{i} \text { for } k+1 \leq i \leq 2 k-1 \\ 2 & \text { if } e=x_{i} y_{i} \text { for } 1 \leq i \leq 2 k-1, \text { and } e=x_{k} y_{k+1}, \\ 1 & \text { otherwise }\end{cases}
$$

Then $c$ is a rainbow 2-coloring. And it is easy to check that $\bar{G}$ can also be colored by two colors and make it rainbow connected.

If $n=4 k+1, G$ can be obtained by adding a vertex $x_{2 k}$ to the vertex set $X$ in the case $n=4 k$, and joined $x_{2 k}$ to $v, y_{2 k}, y_{1}, \cdots, y_{k-1}$. With the coloring $c$ defined above, in addition with $c\left(v x_{2 k}\right)=c\left(x_{2 k} y_{2 k}\right)=2, G$ is rainbow connected.

If $n=4 k+2, G$ can be obtained by adding two vertices $x_{2 k}, y_{2 k+1}$ to the vertex set $X$ and $Y$, respectively, in the case $n=4 k$, and joined $x_{2 k}$ to $v, y_{2 k+1}$ to each vertex in
$Y$. And for each $x_{i}, x_{i}$ is adjacent to $y_{i}, y_{i+1}, \cdots, y_{i+k}$, where the sum is taken modulo $2 k+1$. With the coloring $c$ defined above, in addition with $c\left(v x_{2 k}\right)=c\left(x_{2 k} y_{2 k}\right)=2, G$ is rainbow connected.

If $n=4 k+3, G$ can be obtained by adding two vertices $x_{2 k+1}, y_{2 k+1}$ to the vertex set $X$ and $Y$, respectively, in the case $n=4 k+1$, and joined $x_{2 k+1}$ to $v, y_{2 k+1}$ to each vertex in $Y$. And for each $x_{i}, x_{i}$ is adjacent to $y_{i}, y_{i+1}, \cdots, y_{i+k}$, where the sum is taken modulo $2 k+1$, we also join $x_{k+1}$ to $y_{2 k+1}$. With the coloring $c$ defined above, in addition with $c\left(v x_{2 k+1}\right)=c\left(x_{2 k+1} y_{2 k+1}\right)=2, G$ is rainbow connected.

Theorem 4 For $n=4,5, r c(G)+r c(\bar{G}) \geq 6$, and $r c(G)+r c(\bar{G}) \geq 5$ for $n=6$, 7. All these bounds are best possible.

Proof. From Theorem 2, we have $r c(G)+r c(\bar{G}) \geq 5$ for $n=4,5,6,7$.
For $n=4$, as we have shown, $r c(G)+r c(\bar{G})=6$. If $n=5$, the possible complementary connected graphs are:


Figure 8: Complementary connected graphs for $n=5$.

For all these cases, $r c(G)+r c(\bar{G}) \geq 6$.
For $n=6$, let $G$ be the cycle $C_{6}$, whose vertices are $\left\{v_{1}, v_{2}, \cdots, v_{6}\right\}$. Then $r c(G)=3$. We color the edges $v_{1} v_{3}, v_{2} v_{4}, v_{3} v_{5}$ in $\bar{G}$ by 2 , and the other edges by 1 . This coloring makes $\bar{G}$ rainbow connected. Therefore, $r c(G)+r c(\bar{G})=5$.

For $n=7$, the graph $G_{2}$ in Figure 3 has rainbow connection number 2. We have shown that $r c\left(\overline{G_{2}}\right) \neq 2$, but we can use three colors to make it rainbow connected, just by assigning the edges $v_{2} v_{4}$ and $v_{3} v_{5}$ with color 3 , the others the same as before. So, $r c\left(\overline{G_{2}}\right)=3$. Thus, $r c(G)+r c(\bar{G})=5$.

## 4 Concluding remark

Given a graph $G$, a set $D \subseteq V(G)$ is called a domination set of $G$, if every vertex in $G$ is at a distance at most 1 from $D$. Further, if $D$ induces a connected subgraph of $G$, it is called a connected dominating set of $G$. The cardinality of a minimum connected dominating set in $G$ is called its connected dominating number, denoted by $\gamma_{c}(G)$. In [7], the authors proved that for every connected graph $G$ with minimum degree $\delta(G) \geq 2$, $r c(G) \leq \gamma_{c}(G)+2$. In [5], the authors introduced a result of Nordhaus-Gaddum type result for the connected dominating number. They showed that if $G$ and $\bar{G}$ are both connected, then $\gamma_{c}(G)+\gamma_{c}(\bar{G}) \leq n+1$. If one uses their results, one can only get that $r c(G)+r c(\bar{G}) \leq \gamma_{c}(G)+\gamma_{c}(\bar{G})+4 \leq n+5$, which is weaker than our result.

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