

# Saturating Sperner families

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**Abstract:** A family  $\mathcal{F} \subseteq 2^{[n]}$  saturates the monotone decreasing property  $\mathcal{P}$  if  $\mathcal{F}$  satisfies  $\mathcal{P}$  and one cannot add any set to  $\mathcal{F}$  such that property  $\mathcal{P}$  is still satisfied by the resulting family. We address the problem of finding the minimum size of a family saturating the  $k$ -Sperner property and the minimum size of a family that saturates the Sperner property and that consists only of  $l$ -sets and  $(l + 1)$ -sets.

**Keywords:** extremal set theory, Sperner property, saturation

## 1 Introduction

One of the most basic and most studied problems of extremal combinatorics is that how many edges a (hyper)graph can have if it possesses some prescribed property  $\mathcal{P}$ . If this property  $\mathcal{P}$

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is monotone decreasing (i.e. if  $G$  possesses  $\mathcal{P}$ , then  $F \subseteq G$  implies  $F$  possesses  $\mathcal{P}$ ), then there exists a “dual” problem to the above one: we say that a (hyper)graph  $G$  *saturates* property  $\mathcal{P}$  if it possesses property  $\mathcal{P}$  but adding any (hyper)edge  $E$  to  $G$  would result in a graph not having property  $\mathcal{P}$ . The problem is to determine the minimum size that such a saturating (hyper)graph can have. Many researchers have dealt with this kind of problems both for graphs [1, 2, 3, 4, 5, 6, 13, 14, 17, 21, 22, 28, 29, 30] and hypergraphs [12, 19, 20]. To our knowledge all papers so far have considered Turán-type properties (with the exception of [9]): properties defined through some forbidden (hyper)graphs.

In the present paper we investigate the saturation of Sperner-type properties. To introduce our main definitions let  $k$  and  $n$  be positive integers and let  $\mathcal{F} \subseteq 2^{[n]}$  be a family of sets such that,

1. there do not exist  $k + 1$  distinct sets  $F_1, \dots, F_{k+1} \in \mathcal{F}$  that form a  $(k + 1)$ -chain, i.e.  $F_1 \subset F_2 \subset \dots \subset F_{k+1}$  holds,
2. for every set  $S \in 2^{[n]} \setminus \mathcal{F}$  there exist  $k$  distinct sets  $F_1, \dots, F_k \in \mathcal{F}$  such that  $S$  and the  $F_i$ 's form a  $(k + 1)$ -chain.

A family  $\mathcal{F}$  is called *k-Sperner* if it satisfies Property 1, *weakly saturating k-Sperner* if it satisfies 2 and *(strongly) saturating k-Sperner* if it satisfies both. The maximum size of a  $k$ -Sperner family  $\mathcal{F} \subset 2^{[n]}$  was determined by Sperner [26] in the special case  $k = 1$  and by Erdős [10] for arbitrary  $k$ .

In Section 2 we will derive bounds on  $\text{sat}(n, k)$  ( $\text{wsat}(n, k)$ ) the minimum number of sets that strongly saturating  $k$ -Sperner (weakly saturating  $k$ -Sperner) family  $\mathcal{F} \subset 2^{[n]}$  can contain. By definition, we have  $\text{wsat}(n, k) \leq \text{sat}(n, k)$ . The following *product construction* shows that there is an upper bound on both of these numbers that is independent of  $n$ , namely  $\text{wsat}(n, k) \leq \text{sat}(n, k) \leq 2^{k-1}$ . Let  $\mathcal{F} \subset 2^{[n]}$  be defined by

$$\mathcal{F} := 2^{[k-2]} \times \{\emptyset, [n] \setminus [k-2]\} = 2^{[k-2]} \cup \{F \in 2^{[n]} : [n] \setminus [k-2] \subseteq F\}.$$

It is easy to see that  $\mathcal{F}$  is indeed strongly saturating  $k$ -Sperner. It is natural to formulate the following conjecture.

**Conjecture 1.** *For every positive integer  $k$  there exists an  $n_0 = n_0(k)$  such that for any  $n \geq n_0$  we have  $\text{sat}(n, k) = 2^{k-1}$ .*

It is trivial to verify that  $n_0(k) = k$  for  $k = 1, 2, 3$ . By giving constructions we will prove the following two upper bounds.

**Theorem 2.** *For integers  $6 \leq k \leq n$  we have the following inequalities:*

- (i)  $\text{sat}(k, k) \leq \frac{15}{16} 2^{k-1}$ ,
- (ii)  $\text{wsat}(n, k) = O\left(\frac{\log k}{k} 2^k\right)$ .

We will also obtain lower bounds on the size of saturating  $k$ -Sperner families. All our lower bounds will apply both for  $\text{wsat}(n, k)$  and  $\text{sat}(n, k)$ .

**Theorem 3.** *For integers  $k, c$  and  $n$  we have the following inequalities:*

- (i)  $2^{k/2-1} \leq \text{wsat}(n, k) \leq \text{sat}(n, k)$  provided  $k \leq n$ ,
- (ii)  $\frac{2^{k+c}}{(k+c)^{c+1}} \leq \text{wsat}(k+c, k) \leq \text{sat}(k+c, k)$  provided  $2 \leq k$  and  $0 \leq c$ .

1-Sperner families are also called *antichains*. Saturating antichains with the simplest structure are the families consisting of all  $l$ -element subsets of the underlying set for any fixed  $l$ . Next, one would consider antichains with two possible set sizes. If the set sizes are consecutive integers, then these families are called *flat antichains*. Grützmüller, Hartmann, Kalinowski, Leck and Roberts [16] proved the following theorem.

**Theorem 4** (Grützmüller, Hartmann, Kalinowski, Leck, Roberts [16]). *If  $\mathcal{F} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$  is a saturating antichain, then the following holds*

$$|\mathcal{F}| \geq \binom{n}{2} - \left\lceil \frac{(n+1)^2}{8} \right\rceil.$$

The authors of [16] also determined all antichains for which equality holds. In Section 3 we obtain the stability version of Theorem 4. Our proof is self-contained and is much shorter than their proof of Theorem 4. Some of its parts generalize to saturating flat antichains with larger set sizes. Unfortunately the lower bounds that we obtain depend on the Turán density of  $K_{l+1}^l$ , the complete  $l$ -graph on  $l+1$  vertices. We will also generalize the construction of [16], but the lower bounds and the size of the construction are quite far apart even if we assume that some famous longstanding conjectures about the above-mentioned Turán densities are true. To state our stability result for a family  $\mathcal{F} \subset \binom{[n]}{l}$  let us write  $\Delta(\mathcal{F}) = \{G \in \binom{[n]}{l-1} : \exists F \in \mathcal{F} \text{ such that } G \subset F\}$  and  $\nabla(\mathcal{F}) = \{G \in \binom{[n]}{l+1} : \exists F \in \mathcal{F} \text{ such that } F \subset G\}$ .

**Theorem 5.** *Let  $\mathcal{F} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$  be a saturating antichain of minimum size. Then the following holds:*

$$|\mathcal{F}| = \left( \frac{3}{8} - o(1) \right) n^2.$$

Moreover, if  $|\mathcal{F}| = (\frac{3}{8} + o(1))n^2$ , then there is a partition  $[n] = A \cup B \cup C$  with  $|A| = |B| = \lfloor n/4 \rfloor$  and a matching  $M$  between  $A$  and  $B$  such that if  $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_3$  is defined by  $\mathcal{G}_3 = \{G \in \binom{[n]}{3} : G \cap C \neq \emptyset \text{ and } \exists m \in M \text{ with } m \subset G\}$  and  $\mathcal{G}_2 = \binom{[n]}{2} \setminus \Delta(\mathcal{G}_3)$ , then  $|\mathcal{G} \Delta \mathcal{F}| = o(n^2)$  holds and  $\mathcal{G}$  is a saturating antichain.

## 2 Bounds on $\text{sat}(n, k)$ and $\text{wsat}(n, k)$

In this section, we prove Theorem 2 and Theorem 3. We start our investigations with an easy lemma stating that we can always assume the empty set and  $[n]$  belong to the family  $\mathcal{F}$ .

**Lemma 6.** *If  $2 \leq k \leq n$ , then there exists a weakly (strongly) saturating  $k$ -Sperner family  $\mathcal{F} \subseteq 2^{[n]}$  of minimum size such that  $\emptyset$  and  $[n]$  belong to  $\mathcal{F}$ .*

*Proof.* Let  $\mathcal{F}$  be of minimum size with  $\emptyset \notin \mathcal{F}$  and let  $\mathcal{F}_m$  denote the minimal sets in  $\mathcal{F}$ . Then  $\mathcal{F} \setminus \mathcal{F}_m \cup \{\emptyset\}$  is weakly (strongly) saturating  $k$ -Sperner and its size is at most the size of  $\mathcal{F}$ . The case of  $[n]$  is completely analogous.  $\square$

*Proof of Theorem 2.* First we give a construction that shows that  $\text{sat}(6, 6) \leq 30 = \frac{15}{16}2^{6-1}$ . We enumerate the sets according to their size:

- $\emptyset$ ,

- four singletons:  $\{3\}, \{4\}, \{5\}, \{6\}$ ,
- six pairs:  $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{5, 6\}$ ,
- eight triples:  $\{1, 2, 5\}, \{1, 2, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 6\}, \{2, 4, 6\}$ ,
- six 4-tuples:  $\{3, 4, 5, 6\}, \{2, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 3, 5\}$ ,
- four 5-tuples:  $\{1, 2, 3, 4, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}$
- and  $\{1, 2, 3, 4, 5, 6\}$ .

To see that these sets indeed form a strongly saturating 6-Sperner family, note that  $\emptyset$ , the singletons and  $\{1, 2\}$  form a strongly saturating 2-Sperner family and so do  $\{1, 2, 3, 4, 5, 6\}$ , the 5-tuples and  $\{3, 4, 5, 6\}$ . The remaining pairs together with  $\{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 6\}, \{2, 4, 6\}$  form a saturating antichain as described in Section 3. Now the remaining sets form a family isomorphic to the complements of the members of the previous family and is therefore saturating antichain. As these four families are disjoint and lie “one below the other”, their union is saturating 6-Sperner.

The following lemma finishes the proof of Theorem 2 (i).

**Lemma 7.** *For any integers  $k, n$  such that  $3 \leq k \leq n$  we have*

$$\text{sat}(n, k) \leq 2\text{sat}(n - 1, k - 1).$$

*Proof.* Let  $\mathcal{F} \subseteq 2^{[n-1]}$  be a strongly saturating  $(k - 1)$ -Sperner family of minimum size such that (by Lemma 6)  $\emptyset, [n - 1] \in \mathcal{F}$ . Then the family  $\mathcal{F}' = \mathcal{F} \times \{\emptyset, \{n\}\} = \mathcal{F} \cup \{F \cup \{n\} : F \in \mathcal{F}\}$  is a strongly saturating  $k$ -Sperner subfamily of  $2^{[n]}$ . Indeed, if there exists a  $(k + 1)$ -chain  $F'_1 \subset \dots \subset F'_{k+1}$  in  $\mathcal{F}'$ , then at least  $k$  out of the sets  $F'_i \cap [n - 1]$  would be distinct and form a  $k$ -chain in  $\mathcal{F}$ . This contradiction shows that  $\mathcal{F}'$  is  $k$ -Sperner. To prove the saturating property of  $\mathcal{F}'$  let us consider a set  $G \in 2^{[n]} \setminus \mathcal{F}'$ . By definition, we know that  $G \cap [n - 1] \notin \mathcal{F}$  holds and thus by the saturating property of  $\mathcal{F}$  there exists  $k - 1$  sets  $F_1, \dots, F_{k-1}$  in  $\mathcal{F}$  together with which  $G \cap [n - 1]$  forms a  $k$ -chain. If  $n \notin G$ , then the  $F_i$ 's,  $G$  and  $[n]$  form a  $(k + 1)$ -chain in  $\mathcal{F}'$ , while if  $n \in G$ , then  $\emptyset, G$  and the  $\{n\} \cup F_i$ 's form a  $(k + 1)$ -chain in  $\mathcal{F}'$  (not necessarily in this order).  $\square$

To prove (ii) we first give a general construction. Let us write  $\mathcal{F}_0 = \{\emptyset\}$  and  $\mathcal{F}_k = \{[k]\}$ . Furthermore, for any  $1 \leq l \leq k - 1$  let  $\mathcal{F}_l \subseteq \binom{[k]}{l}$  be a family satisfying  $\nabla(\mathcal{F}_l) = \binom{[k]}{l+1}$  and  $\Delta(\mathcal{F}_l) = \binom{[k]}{l-1}$ . Then define the family  $\mathcal{F} \subseteq 2^{[n]}$  as follows:

$$\mathcal{F} = \bigcup_{l=0}^k \mathcal{F}_l \times \{\emptyset, [n] \setminus [k]\} = \bigcup_{l=0}^k \mathcal{F}_l \cup \left\{ F \cup ([n] \setminus [k]) : F \in \bigcup_{l=0}^k \mathcal{F}_l \right\}.$$

We claim that  $\mathcal{F}$  is a weakly saturating  $k$ -Sperner family. Indeed, let us consider a set  $G \in 2^{[n]} \setminus \mathcal{F}$  and define  $i = |G \cap [k]|$ . Observe that, by the conditions on the  $\mathcal{F}_l$ 's, there exist sets  $F_l \in \mathcal{F}_l$  for every  $0 \leq l \leq k$ ,  $l \neq i$  such that the  $F_l$ 's together with  $G \cap [k]$  form a  $(k + 1)$ -chain. Then the  $F_l$ 's for  $l < i$ ,  $G$  and the sets  $F_l \cup ([n] \setminus [k])$  for  $l > i$  form a  $(k + 1)$ -chain.

It is well known [8] that if  $l = \Theta(k)$ , then there exists a family  $\mathcal{F}'_l \subseteq \binom{[k]}{l}$  such that  $\Delta(\mathcal{F}'_l) = \binom{[k]}{l-1}$  and  $|\mathcal{F}'_l| = \Theta\left(\frac{\log k}{k} \binom{k}{l}\right)$  holds. Thus the size of the family  $\mathcal{F}_l = \mathcal{F}'_l \cup \overline{\mathcal{F}'_{n-l}}$  is of the same

order of magnitude and satisfies  $\Delta(\mathcal{F}'_l) = \binom{[k]}{l-1}$ ,  $\nabla(\mathcal{F}'_l) = \binom{[k]}{l+1}$ . Use the general construction with  $\mathcal{F}'_l$ 's as above provided  $k/4 \leq l \leq 3k/4$  and  $\mathcal{F}'_l = \binom{[k]}{l}$  otherwise to obtain the family  $\mathcal{F}$ . Then the size of  $\mathcal{F}$  is  $2 \sum_{i=0}^k |\mathcal{F}'_i| = O(\frac{\log k}{k} 2^k)$ .  $\square$

*Proof of Theorem 3.* Let  $\mathcal{F} \subseteq 2^{[n]}$  be a weakly saturating  $k$ -Sperner family and consider any set  $G \in 2^{[n]} \setminus \mathcal{F}$ . By definition, there exist  $k$  distinct sets  $F_1, \dots, F_k \in \mathcal{F}$  such that  $F_1 \subset \dots \subset F_i \subset S \subset F_{i+1} \subset \dots \subset F_k$  holds. Thus  $G$  is a set from the interval  $I_{F_i, F_{i+1}} = \{S : F_i \subseteq S \subseteq F_{i+1}\}$  which has size at most  $2^{n-k+2}$  as  $|F_{i+1} \setminus F_i| \leq n - k + 2$  holds by the existence of the other  $F_j$ 's. We obtain that  $2^{[n]}$  can be covered by intervals of size at most  $2^{n-k+2}$  and thus we have

$$\frac{|\mathcal{F}|^2}{2} 2^{n-k+2} \geq \binom{|\mathcal{F}|}{2} 2^{n-k+2} \geq 2^n.$$

Now **(i)** follows by rearranging.

To prove **(ii)** let us partition the intervals  $I_{F_i, F_{i+1}}$  in the proof of **(i)** according to the  $F_i$ 's. The intervals belonging to the same  $F_i$  may cover at most the sets  $\{S \supseteq F_i : |S \setminus F_i| \leq c + 1\}$ . As the number of these sets is at most  $\sum_{i=0}^{c+1} \binom{k+c}{i} \leq (k+c)^{c+1}$ , we obtain the inequality  $|\mathcal{F}|(k+c)^{c+1} \geq 2^{k+c}$  and we are done by rearranging.  $\square$

Theorem 3 **(ii)** with  $c = 0$  shows  $wsat(k, k) = \Omega(2^k/k)$ . It is one of the most important questions of the theory of covering codes whether there exist families  $\mathcal{F}_l \subseteq \binom{[n]}{l}$  as in the general construction with size  $O(\binom{[n]}{l}/k)$ . If the answer is positive, then one would obtain a weakly saturating  $k$ -Sperner family with size  $O(2^k/k)$  via the general construction thus  $wsat(k, k) = \Theta(2^k/k)$  would follow.

Let us finish this section with some remarks on strongly saturating  $k$ -Sperner families in the case when  $n$  is large compared to  $k$ . A family  $\mathcal{F} \subseteq 2^{[n]}$  is called *non-separating* if there exist  $x, y \in [n]$  such that for all  $F \in \mathcal{F}$  we have  $x \in F \Leftrightarrow y \in F$ . The family  $\mathcal{F}$  is separating if it is not non-separating. Let us call a strongly saturating  $k$ -Sperner family  $\mathcal{F} \subseteq 2^{[n]}$  *duplicable* if there exists  $x \in [n]$  such that the family  $\mathcal{F}' \subseteq 2^{[n+1]}$  defined as  $\mathcal{F}' := \{F \in \mathcal{F} : x \notin F\} \cup \{F \cup \{n+1\} : x \in F\}$  is strongly saturating  $k$ -Sperner. Finally, a family  $\mathcal{F}$  is *primitive strongly saturating  $k$ -Sperner* if it is separating and duplicable.

Clearly, if  $2^{|\mathcal{F}|} < n$ , then  $\mathcal{F}$  is non-separating. As, by the product construction defined in the Introduction, we know that  $\text{sat}(n, k) \leq 2^{k-1}$ , we obtain that any strongly saturating  $k$ -Sperner family of minimum size is non-separating provided  $2^{2^{k-1}} < n$ . Let  $\mathcal{F} \subseteq 2^{[n]}$  be such a family and  $x, y \in [n]$  be the elements showing the non-separating property of  $\mathcal{F}$ . Then it is easy to verify that the family  $\mathcal{F}^* \subseteq 2^{[n] \setminus \{y\}}$  defined as  $\mathcal{F}^* = \{F \cap ([n] \setminus \{y\}) : F \in \mathcal{F}\}$  is strongly saturating  $k$ -Sperner and we have  $|\mathcal{F}| = |\mathcal{F}^*|$ .

Let  $\mathcal{F} \subseteq 2^{[n]}$  be a non-separating strongly saturating  $k$ -Sperner family. We would like to prove that there is only one way to duplicate such a family. Formally, we claim that there do not exist  $x, y, u, v \in [n]$  and  $F' \in \mathcal{F}$  such that  $x \in F \Leftrightarrow y \in F$  and  $u \in F \Leftrightarrow v \in F$  holds for all  $F \in \mathcal{F}$  but exactly one of  $x$  and  $u$  is contained in  $F'$ . Indeed, if that is not the case, then there would exist a set  $S \subseteq [n] \setminus \{x, y, u, v\}$  such that  $S_1 = S \cup \{x, u\} \notin \mathcal{F}$  but at least one of  $S \cup \{x, y\}$  and  $S \cup \{u, v\}$  belongs to  $\mathcal{F}$  which we denote by  $S_2$ . Therefore there would exist sets  $F_1, \dots, F_k \in \mathcal{F}$  that together with  $S_1$  form a  $(k+1)$ -chain. As the family  $\mathcal{F}$  does not separate  $x$

and  $y$  nor  $u$  and  $v$ , all the  $F_i$ 's contain either all four of  $x, y, u, v$  or none of them. But then the  $F_i$ 's and  $S_2$  form a  $(k+1)$ -chain as well which contradicts the  $k$ -Sperner property of  $\mathcal{F}$ .

The above two statements yield the following lemma.

**Lemma 8.** *Let  $n$  and  $k$  be positive integers such that  $2^{2^{k-1}} < n$  holds. Then we have*

$$\text{sat}(n, k) = \min_{n' \leq 2^{2^{k-1}}} \left\{ |\mathcal{F}'| : \mathcal{F}' \subseteq 2^{[n']} \text{ is primitive strongly saturating } k\text{-Sperner} \right\}.$$

Moreover, for any extremal family  $\mathcal{F} \subseteq 2^{[n]}$ , there exist integers  $x \leq n' \leq 2^{2^{k-1}}$  and a primitive family  $\mathcal{F}' \subseteq 2^{[n']}$  such that  $\mathcal{F} = \{F' \in \mathcal{F}' : x \notin F'\} \cup \{F' \cup ([n] \setminus [n']) : x \in F' \in \mathcal{F}'\}$ .

### 3 Saturating flat antichains

In this section we consider saturating flat antichains. Let us start with an easy lemma that gives a necessary and sufficient condition for a family  $\mathcal{F}$  to be a saturating flat antichain.

**Lemma 9.** *A family  $\mathcal{F} = \mathcal{F}_l \cup \mathcal{F}_{l+1}$  with  $\mathcal{F}_l \subset \binom{[n]}{l}, \mathcal{F}_{l+1} \subset \binom{[n]}{l+1}$  is a saturating antichain if and only if we have  $\Delta(\mathcal{F}_{l+1}) = \binom{[n]}{l} \setminus \mathcal{F}_l$  and  $\nabla(\binom{[n]}{l} \setminus \mathcal{F}_l) = \mathcal{F}_{l+1}$ .*

*Proof.* Assume first that  $\mathcal{F}$  is a saturating antichain. The inclusions  $\Delta(\mathcal{F}_{l+1}) \subseteq \binom{[n]}{l} \setminus \mathcal{F}_l$  and  $\nabla(\binom{[n]}{l} \setminus \mathcal{F}_l) \supseteq \mathcal{F}_{l+1}$  follow trivially from  $\mathcal{F}$  being an antichain. The other inclusions follow from the saturating property of  $\mathcal{F}$ . Indeed, if a  $G \in \binom{[n]}{l}$  is not in  $\mathcal{F}$ , then the only reason for this is that there should be an  $F \in \mathcal{F}_{l+1}$  containing  $G$ , similarly if no  $l$ -subsets of a  $(l+1)$ -set  $G$  belong to  $\mathcal{F}_l$ , then  $G$  can be added to  $\mathcal{F}_{l+1}$ .

Now let us assume that  $\Delta(\mathcal{F}_{l+1}) = \binom{[n]}{l} \setminus \mathcal{F}_l$  and  $\nabla(\binom{[n]}{l} \setminus \mathcal{F}_l) = \mathcal{F}_{l+1}$  hold. These equations clearly imply that  $\mathcal{F}$  is an antichain. Also,  $\mathcal{F}$  is saturating as any  $l$ -set is either in  $\mathcal{F}$  or is contained in a set in  $\mathcal{F}_{l+1}$  and any  $(l+1)$ -set is either in  $\mathcal{F}$  or it is not in  $\nabla(\binom{[n]}{l} \setminus \mathcal{F}_l)$  and therefore contains an  $l$ -set from  $\mathcal{F}$ .  $\square$

Before we start to prove Theorem 5 we need to introduce our two main tools.

**Theorem 10** (Ruzsa-Szemerédi, [24]). *Let  $G_n$  be a graph on  $n$  vertices such that the number of triangles in  $G_n$  is  $o(n^3)$ . Then there exists a subset  $E$  of  $E(G_n)$  of size  $o(n^2)$  such that if we remove all the edges in  $E$  from  $G_n$ , then the resulting graph is triangle free.*

**Theorem 11** (Erdős-Simonovits, [11, 25]). *Let  $G_n$  be a triangle free graph on  $n$  vertices with  $|E(G_n)| = (\frac{1}{4} - o(1))n^2$ . Then there exists a bipartition  $V(G_n) = X \cup Y$  with  $||X| - |Y|| \leq 1$  such that  $|E(G_n) \Delta E(K_{X,Y})| = o(n^2)$  holds, where  $K_{X,Y}$  is the complete bipartite graph with parts  $X$  and  $Y$ .*

*Proof of Theorem 5.* Lemma 9 shows that the construction of the theorem is indeed a saturating antichain hence the upper bound of the theorem.

To prove the lower bound of the theorem let  $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_3$  with  $\mathcal{F}_2 \subset \binom{[n]}{2}, \mathcal{F}_3 \subset \binom{[n]}{3}$  be a saturating antichain. Sets in  $\mathcal{F}_2$  and  $\mathcal{F}_3$  will be called  $\mathcal{F}$ -edges and  $\mathcal{F}$ -triples, while sets in  $\binom{[n]}{2} \setminus \mathcal{F}_2$  and  $\binom{[n]}{3} \setminus \mathcal{F}_3$  will be called  $\mathcal{F}$ -non-edges and  $\mathcal{F}$ -non-triples. Consider the graph  $H$  of  $\mathcal{F}$ -non-edges, i.e.  $V(H) = [n]$  and  $E(H) = \binom{[n]}{2} \setminus \mathcal{F}_2$ . By Lemma 9, we know that the

triangles in  $H$  are the  $\mathcal{F}$ -triples. As the number of  $\mathcal{F}$ -triples is  $O(n^2) = o(n^3)$  it follows by Theorem 10 that  $H$  can be made triangle free removing a set  $E'$  of edges (i.e.  $\mathcal{F}$ -non-edges of  $\mathcal{F}$ ) with size  $o(n^2)$ . By Turán's theorem [27] we have that  $|E(H)| \leq (\frac{1}{4} + o(1))n^2$  and thus  $|\mathcal{F}_2| = \binom{n}{2} - |E(H)| \geq \binom{n}{2} - (\frac{1}{4} + o(1))n^2 = (\frac{1}{4} - o(1))n^2$ .

Let  $\mathcal{F}'_3 \subset \mathcal{F}_3$  a maximal subfamily of the  $\mathcal{F}$ -triples such that for any  $F_1, F_2 \in \mathcal{F}'_3$  we have  $|F_1 \cap F_2| \leq 1$ . As every  $\mathcal{F}$ -non-edge in  $E'$  is contained in at most one  $\mathcal{F}$ -triple in  $\mathcal{F}'_3$ , we obtain  $|\mathcal{F}'_3| = o(n^2)$ . By Lemma 9 we know that every  $\mathcal{F}$ -non-edge is contained in at least one  $\mathcal{F}$ -triple. But an  $\mathcal{F}$ -triple covers three  $\mathcal{F}$ -non-edges, moreover, any  $\mathcal{F}$ -triple in  $\mathcal{F}_3 \setminus \mathcal{F}'_3$  covers at most two  $\mathcal{F}$ -non-edges that are not covered by any  $\mathcal{F}$ -triple in  $\mathcal{F}'_3$ . Thus we obtain the inequality  $\binom{n}{2} - |\mathcal{F}_2| \leq 3|\mathcal{F}'_3| + 2|\mathcal{F}_3 \setminus \mathcal{F}'_3| \leq 2|\mathcal{F}_3| + o(n^2)$ .

Let us define  $\alpha = \alpha(n)$  by writing  $|\mathcal{F}_2| = (\frac{1}{4} + \alpha)n^2$ . By what we have so far, we know that  $\liminf \alpha \geq 0$ . Using the inequality above we obtain

$$|\mathcal{F}| = |\mathcal{F}_2| + |\mathcal{F}_3| \geq \left(\frac{1}{4} + \alpha\right)n^2 + \frac{(\frac{1}{4} - \alpha)}{2}n^2 - o(n^2) = \left(\frac{3}{8} + \frac{\alpha}{2} - o(1)\right)n^2.$$

As  $\liminf \alpha \geq 0$ , this completes the proof of the lower bound. Moreover, we obtain that if  $\mathcal{F} \subset \binom{[n]}{2} \cup \binom{[n]}{3}$  is a saturating antichain with  $|\mathcal{F}| = (\frac{3}{8} + o(1))n^2$ , then we have  $|\mathcal{F}_2| = (\frac{1}{4} - o(1))n^2$  and  $|\mathcal{F}_3| = (\frac{1}{8} - o(1))n^2$ .

All what remains is to prove the stability of the extremal family. Note that a saturating antichain  $\mathcal{F}$  is clearly determined by  $\mathcal{F}$ -non-edges. In the case of the conjectured extremal family these are the edges of  $K_{A \cup B, C}$  and the matching  $M$ . Let  $\mathcal{F}$  be a saturating antichain of size  $(\frac{3}{8} - o(1))n^2$ . Then by what we have proved so far, we know that the graph  $H$  of the  $\mathcal{F}$ -non-edges is of size  $(\frac{1}{4} - o(1))n^2$  and is triangle free after removing  $o(n^2)$  edges. Thus, by Theorem 11, after changing at most  $o(n^2)$  edges in  $H$  we obtain the bipartite Turán graph  $K_{X, Y}$  with  $||X| - |Y|| \leq 1$ . Let us put a maximal matching  $M$  into any of the two classes, say to  $X$ , and define the tripartition of  $[n]$  as  $C$  to be the vertices not incident to  $M$  and  $A$  and  $B$  to contain different vertices from all edges of  $M$  and  $\mathcal{G}$  to be the extremal family built on this tripartition. No matter how we chose  $M$ , the  $\mathcal{G}_2$  part of the resulting family  $\mathcal{G}$  will satisfy  $|\mathcal{G}_2 \Delta \mathcal{F}_2| = o(n^2)$ . Note that, as  $X \subseteq C$  or  $Y \subseteq C$ , the bipartition  $A \cup B, C$  is already known up to one vertex possibly moving from one part to the other and thus the graph  $K_{A \cup B, C}$  is known up to a change of at most  $n - 1$  edges.

Let  $H'$  be the graph with  $V(H') = [n]$  and  $E(H') = E(K_{X, Y}) \cap (\binom{[n]}{2} \setminus \mathcal{F}_2)$ . By the above we know that  $|E(H')| = (\frac{1}{4} - o(1))n^2$  and thus with an exception of  $o(n)$  vertices every vertex has degree  $(\frac{1}{2} - o(1))n$ .

**Claim 12.** *Either  $X$  or  $Y$  contains a matching  $M$  that consists of  $(\frac{1}{4} - o(1))n$   $\mathcal{F}$ -non-edges.*

*Proof.* We only consider vertices with degree  $(\frac{1}{2} - o(1))n$  in  $H'$ . Note that any  $\mathcal{F}$ -non-edge between two such vertices in the same vertex class defines  $(\frac{1}{2} + o(1))n$  triangles in  $H$  and thus, by Lemma 9, that many  $\mathcal{F}$ -triples. Also, these  $\mathcal{F}$ -triples are distinct, therefore there can be at most  $(\frac{1}{4} + o(1))n$  such  $\mathcal{F}$ -non-edges as we have already proved that  $|\mathcal{F}_3| = (\frac{1}{8} + o(1))n^2$ . Observe that all but  $o(n)$  vertices in either  $X$  or  $Y$  are contained in at least one  $\mathcal{F}$ -non-edge with the other vertex in the same vertex class of  $H'$ . Indeed, otherwise there would be an edge in  $H'$  between such an  $x \in X$  and such a  $y \in Y$  (if not, then  $\Omega(n^2)$  edges would be missing from  $H'$ ). And since any edge of  $H'$  is an  $\mathcal{F}$ -non-edge, by Lemma 9, it has to be contained in an  $\mathcal{F}$ -triple all three 2-subsets of which are  $\mathcal{F}$ -non-edges.

Now the claim follows as  $(\frac{1}{4} - o(1))n$  edges can cover at least one vertex class with the exception of  $o(n)$  vertices if and only if those edges with  $o(n)$  exceptions form a matching in one of the classes.  $\square$

We extend the matching  $M$  given by Claim 12 to a maximal matching in  $X$  and define the partition  $A, B, C$  accordingly. By the reasoning of Claim 12 there are  $(\frac{1}{8} - o(1))n^2$   $\mathcal{F}$ -triples containing some  $\mathcal{F}$ -non-edge from  $M$  and all these  $\mathcal{F}$ -triples belong to  $\mathcal{G}$ , too. As both  $|\mathcal{F}_3| = (\frac{1}{8} - o(1))n^2$  and  $\mathcal{G}_3 = (\frac{1}{8} - o(1))n^2$  hold, we obtain  $|\mathcal{F}_3 \Delta \mathcal{G}_3| = o(n^2)$ .  $\square$

In the remainder of the section we show how to generalize Theorem 5 to flat antichains with set sizes  $l$  and  $l + 1$ . Let us start by defining the generalization of the construction of [16].

**Construction 13.** *Let us consider the partition  $[n] = A \cup B \cup C$  with  $|A| = |B|$  and let  $\mathcal{M}$  be a complete matching between  $A$  and  $B$ . Let  $\mathcal{G}_{l+1} = \{G \in \binom{[n]}{l+1} : \exists M \in \mathcal{M} \text{ with } M \subset G \text{ and } G \setminus M \subset C\}$  and  $\mathcal{G}_l = \binom{[n]}{l} \setminus \Delta(\mathcal{G}_l)$ . It is easy to see that the conditions of Lemma 9 hold and thus  $\mathcal{G} = \mathcal{G}_l \cup \mathcal{G}_{l+1}$  is a saturating antichain.*

*The number of  $(l + 1)$ -tuples in  $\mathcal{G}_{l+1}$  is  $|A| \binom{n-2|A|}{l-1}$  and the number of  $l$ -tuples not in  $\mathcal{G}_l$  is  $|A| \binom{n-2|A|}{l-2} + 2|A| \binom{n-2|A|}{l-1}$  thus we have  $\mathcal{G} = |A| \binom{n-2|A|}{l-1} + \binom{n}{l} - (|A| \binom{n-2|A|}{l-2} + 2|A| \binom{n-2|A|}{l-1}) = \binom{n}{l} - |A| \binom{n-2|A|+1}{l-1}$ .*

Observe that by replacing Theorem 10 with the hypergraph removal lemma [15, 18, 23], we can use the argument of Theorem 5 to get lower bounds for the size of a saturating flat antichain consisting only of  $l$  and  $(l + 1)$ -sets. Also, Construction 13 gives an upper bound on the minimum size that such a family can have. In order to be able to state the theorem we define  $t_l$  to be the Turán-density of  $K_{l+1}^l$  the complete  $l$ -uniform hypergraph on  $l + 1$  vertices, i.e. if  $ex(n, K_{l+1}^l)$  denotes the most number of edges that an  $l$ -uniform hypergraph on  $n$  vertices can have without containing a copy of  $K_{l+1}^l$ , then  $t_l = \lim ex(n, K_{l+1}^l) / \binom{n}{l}$ . Determining  $t_l$  is one of the most important open problems of extremal combinatorics and even the value of  $t_3$  is unknown. It is conjectured to be  $5/9$  and the current best upper bound is  $\frac{3+\sqrt{17}}{12}$  [7].

**Theorem 14.**

$$\left(1 - \frac{l-1}{l}t_l - o(1)\right) \binom{n}{l} \leq sat(n, l, l+1) \leq \left(1 - \frac{1}{2} \left(1 - \frac{1}{l}\right)^{l-1} + o(1)\right) \binom{n}{l}.$$

*Proof.* The upper bound follows from Construction 13 by setting  $|A| = |B| = \frac{1}{2l}n$  and  $|C| = \frac{l-1}{l}n$ .  $\square$

Note that Theorem 14 would not give the correct asymptotics even in the case  $l = 3$  and with the assumption that Turán's conjecture true.

## 4 Final remarks and open problems

In this section we enumerate the open problems in this topic that we find the most important and interesting.

- What is the correct order of magnitude of  $wsat(k, k)$  and  $sat(k, k)$ ? Do they coincide? Can one find a sequence of families showing  $sat(k, k) = o(2^k)$ ?

- We feel that there do not exist too many primitive strongly saturating  $k$ -Sperner families. A better understanding of these families could help in proving Conjecture 1 via Lemma 8.
- Try to close the gap between the lower and upper bounds on the minimum size of a saturating flat antichain for  $l \geq 3$ . Give any general lower bound which does not use the Turán density of hypergraphs.

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