

Rainbow connection number, bridges and radius¹

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Abstract

Let G be a connected graph. The notion *the rainbow connection number* $rc(G)$ of a graph G was introduced recently by Chartrand et al. Basavaraju et al. showed that for every bridgeless graph G with radius r , $rc(G) \leq r(r+2)$, and the bound is tight. In this paper, we prove that if G is a connected graph, and D^k is a connected k -step dominating set of G , then G has a connected $(k-1)$ -step dominating set $D^{k-1} \supset D^k$ such that $rc(G[D^{k-1}]) \leq rc(G[D^k]) + \max\{2k+1, b_k\}$, where b_k is the number of bridges in $E(D^k, N(D^k))$. Furthermore, for a connected graph G with radius r , let u be the center of G , and $D^r = \{u\}$. Then G has $r-1$ connected dominating sets $D^{r-1}, D^{r-2}, \dots, D^1$ satisfying $D^r \subset D^{r-1} \subset D^{r-2} \dots \subset D^1 \subset D^0 = V(G)$, and $rc(G) \leq \sum_{i=1}^r \max\{2i+1, b_i\}$, where b_i is the number of bridges in $E(D^i, N(D^i)), 1 \leq i \leq r$. From the result, we can get that if for all $1 \leq i \leq r, b_i \leq 2i+1$, then $rc(G) \leq \sum_{i=1}^r (2i+1) = r(r+2)$; if for all $1 \leq i \leq r, b_i > 2i+1$, then $rc(G) = \sum_{i=1}^r b_i$, the number of bridges of G . This generalizes the result of Basavaraju et al.

Keywords: edge-colored graph, rainbow connection number, bridge, radius.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined terminology and notations can be found in [2]. Let G be a graph, and $c : E(G) \rightarrow \{1, 2, \dots, k\}, k \in \mathbb{N}$ be an edge-coloring, where adjacent edges may be colored the same. A graph G is *rainbow connected* if for any pair of distinct vertices u and v of G , G has a $u-v$ path whose edges are colored with distinct colors. The minimum number of colors required to make G rainbow connected is called its *rainbow connection number*, denoted by

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$rc(G)$. These concepts were introduced by Chartrand et al. in [4], where they determined the rainbow connection numbers of wheels, complete graphs and all complete multipartite graphs. Many results involving some graph parameters were obtained. Results involving the minimum degree were obtained in [3, 8, 7, 5]. Results involving the parameters σ_2 and $\sigma_k(G)$ were obtained in [9, 6]. In [1], Basavaraju et al. showed that for every bridgeless graph G with radius r , $rc(G) \leq r(r+2)$, and the bound is tight. As one can see, they did not consider graphs with bridges. In this paper, we will consider graphs with bridges, and $rc(G)$ is bounded by the number of bridges and radius of the graphs. The following are our main results.

Theorem 1 *If G is a connected graph, and D^k is a connected k -step dominating set of G , then G has a connected $(k-1)$ -step dominating set $D^{k-1} \supset D^k$ such that $rc(G[D^{k-1}]) \leq rc(G[D^k]) + \max\{2k+1, b_k\}$, where b_k is the number of bridges of G in $E(D^k, N(D^k))$.*

Theorem 2 *For a connected graph G with radius r , let u be the center of G , and $D^r = \{u\}$. Then G has $r-1$ connected dominating sets $D^{r-1}, D^{r-2}, \dots, D^1$ satisfying $D^r \subset D^{r-1} \subset D^{r-2} \dots \subset D^1 \subset D^0 = V(G)$, and $rc(G) \leq \sum_{i=1}^r \max\{2i+1, b_i\}$, where b_i is the number of bridges in $E(D^i, N(D^i))$, $1 \leq i \leq r$.*

Note that if for all $1 \leq i \leq r$, $b_i \leq 2i+1$, then $rc(G) \leq \sum_{i=1}^r (2i+1) = r(r+2)$; if for all $1 \leq i \leq r$, $b_i > 2i+1$, then $rc(G) = \sum_{i=1}^r b_i$, the number of bridges of G . This generalizes the result of Basavaraju et al.

2 Preliminaries

For two subsets X and Y of V , an (X, Y) -path is a path which connects a vertex of X and a vertex of Y , and whose internal vertices belong to neither X nor Y . We use $E[X, Y]$ to denote the set of edges of G with one end in X and the other end in Y , and $e(X, Y) = |E[X, Y]|$.

Let G be a connected graph. The eccentricity of a vertex v is $ecc(v) = \max_{x \in V(G)} d_G(v, x)$. The radius of G is $rad(G) = \min_{x \in V(G)} ecc(x)$. The diameter of G is $diam(G) = \max_{x \in V(G)} ecc(x)$. Let $S \subseteq V(G)$. The k -step open neighborhood of S is $N^k(S) = \{v \in V(G) | d(v, S) = k, k \in \mathbb{Z}, k \geq 0\}$. Generally speaking, $N^1(S) = N(S)$, $N^0(S) = S$, $N^k[S] = N^k(S) \cup S$. If every vertex in G is at a distance at most k from S , we say that S is a k -step dominating set. If S is connected, then S is a connected k -step dominating set.

The following definitions are needed in our proof. Let D^k be a connected k -step dominating set. A D^k -ear is a path $P = v_0 v_1 \dots v_p$ in G such that $P \cap D^k = \{v_0, v_p\}$. When

$v_0 = v_p$, P is a closed D^k -ear. Moreover, we say that P is an eager D^k -ear, if P is a shortest D^k -ear containing v_0v_1 . Given $2k + 1$ distinct colors, for convenience, we denote them by $1, 2, 3, \dots, 2k + 1$. We say that P is evenly colored, if either the edges of P are colored in this way: $c(v_0v_1) = 1, c(v_1v_2) = 2, c(v_2v_3) = 3, \dots, c(v_{\lceil \frac{p}{2} \rceil - 1}v_{\lceil \frac{p}{2} \rceil}) = \lceil \frac{p}{2} \rceil, c(v_{\lceil \frac{p}{2} \rceil}v_{\lceil \frac{p}{2} \rceil + 1}) = 2k + 2 - \lfloor \frac{p}{2} \rfloor, c(v_{\lceil \frac{p}{2} \rceil + 1}v_{\lceil \frac{p}{2} \rceil + 2}) = 2k + 3 - \lfloor \frac{p}{2} \rfloor, \dots, c(v_{p-2}v_{p-1}) = 2k, c(v_{p-1}v_p) = 2k + 1$, or the edges of P are colored in another way: $c(v_0v_1) = 2k + 1, c(v_1v_2) = 2k, c(v_2v_3) = 2k - 1, \dots, c(v_{p-\lceil \frac{p}{2} \rceil - 2}v_{p-\lceil \frac{p}{2} \rceil - 1}) = 2k + 2 - \lfloor \frac{p}{2} \rfloor, c(v_{p-\lceil \frac{p}{2} \rceil - 1}v_{p-\lceil \frac{p}{2} \rceil}) = \lceil \frac{p}{2} \rceil, \dots, c(v_{p-2}v_{p-1}) = 2, c(v_{p-1}v_p) = 1$. In the proofs later, for convenience, we say that P is evenly colored, if either the edges of P are colored $1, 2, \dots, \lceil \frac{p}{2} \rceil, 2k + 2 - \lfloor \frac{p}{2} \rfloor, 2k + 1 - \lfloor \frac{p}{2} \rfloor, \dots, 2k, 2k + 1$ in this order, or the edges of P are colored $2k + 1, 2k, \dots, 2k + 2 - \lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil - 1, \dots, 3, 2, 1$ in this order.

3 The proofs of our theorems

The proof of Theorem 1:

If G is a tree, then each edge of G is bridge, The result is obvious. Hence we may assume that G is not a tree.

The following, we let D^k be a connected k -step dominating set of G . Then G has k mutually disjoint subsets $N^1(D^k), N^2(D^k), \dots, N^k(D^k)$ and $V(G) = \bigcup_{i=0}^k N^i(D^k)$.

Claim 1: If $\exists x \in N(D^k), y \in D^k$ such that xy is bridge, then we have $d_{G[N[D^k]]}(x) = 1$.

If $\exists y' \in D^k, y' \neq y$, such that $xy' \in E(G)$. As $G[D^k]$ is connected, $G[D^k]$ has a path connecting y, y' . Hence xy is in a cycle, a contradiction to xy being a bridge. If $\exists x_1 \in N(D^k)$ such that $xx_1 \in E(G)$, as there exists some vertex $y_1 \in D^k$ satisfying $x_1y_1 \in E(G)$ (y_1 may be y), then yx_1y_1 is a path, and $G[D^k]$ has a path connecting y, y_1 , that is, xy is in some cycle, a contradiction. Hence $d_{G[N[D^k]]}(x) = 1$.

Let $x_1y_1, x_2y_2, \dots, x_{b_r}y_{b_r}$ be all the bridges in $E(N(D^k), D^k)$, where $x_i \in N(D^k), y_i \in D^k, 1 \leq i \leq b_r$. Set $B = \{x_1, x_2, \dots, x_{b_r}\}, B_E = \{x_1y_1, x_2y_2, \dots, x_{b_r}y_{b_r}\}, D_1 = D^k \cup B$.

Let D^k be a connected k -step dominating set, we rainbow color $G[D^k]$ with $rc(G[D^k])$ colors. If $N(D^k) = B$, then D_1 is a connected $(k - 1)$ -step dominating set. Set $D^{k-1} = D_1$, we use b_r fresh colors to color these b_r bridges, respectively. Hence $rc(G[D^{k-1}]) \leq rc(G[D^k]) + b_r$, and the theorem follows.

So we may assume $N(D^k) \setminus B \neq \emptyset$. For any vertex $v_1 \in N(D^k) \setminus B$, and any edge $v_0v_1 \in E(v_1, D^k), v_0 \in D^k$, as v_0v_1 is not a bridge, v_0v_1 is in some cycle. Hence we may let $P = v_0v_1v_2 \dots v_tv_{t+1}v_{t+2} \dots v_{t+m}v_{t+m+1} \dots v_{p-1}v_p$ be an eager D^k -ear.

Claim 2: $|P| \leq 2k + 1$.

It mainly depends on the following Claim 2.1 and Claim 2.2.

Claim 2.1: If $v_t, v_{t+1} \in N^t(D^k), v_i \in N^i(D^k), 0 \leq i \leq t-1, t \geq 1$, then $v_{t+2} \in N^{t-1}(D^k)$, and P does not have two vertices v_{t+m}, v_{t+m+1} in some $N^j(D^k)$, where $m \geq 2, 1 \leq j \leq t-1$.

If $v_1, v_2 \in N(D^k), v_1v_2 \in E(G)$, then there exists $v_3 \in D^k$ (v_3 can be v_0) such that $v_0v_1v_2v_3$ is an eager D^k -ear.

So we may assume $t \geq 2$. Suppose that, to the contrary, $v_{t+2} \in N^t(D^k)$ or $v_{t+2} \in N^{t+1}(D^k)$. If $v_{t+1}v_{t-1} \in E(G)$, then we replace P by a shorter path $P' = v_0v_1v_2 \cdots v_{t-2}v_{t-1}v_{t+1}v_{t+2} \cdots v_{t+m}v_{t+m+1} \cdots v_{p-1}v_p$. If $v_{t+1}v_b \in E(G), b \in N^{t-1}(D^k) \cap (P \setminus \{v_{t-1}\})$, then we replace P by a shorter path $P' = v_0v_1v_2 \cdots v_{t-1}v_tv_{t+1}v_bv_{b+1} \cdots v_{t+m}v_{t+m+1} \cdots v_{p-1}v_p$, a contradiction to P being an eager D^k -ear. Hence $v_{t+2} \in N^{t-1}(D^k)$.

Suppose that P has two vertices v_{t+m}, v_{t+m+1} in some $N^j(D^k)$ where $m \geq 2, 1 \leq j \leq t-1$. If $j = 1$, that is $v_{t+m}, v_{t+m+1} \in N(D^k), v_{t+m}v_{t+m+1} \in E(G)$, because there is some vertex $v_{p_1} \in D^k$ (v_{p_1} may be v_0) such that $v_{t+m}v_{p_1} \in E(G)$, then we replace P by a shorter path $P' = v_0v_1v_2 \cdots v_{t-1}v_tv_{t+1} \cdots v_{t+m-1}v_{t+m}v_{p_1}$, a contradiction. So we may assume $2 \leq j \leq t-1$. If $v_{t+m}v_{j-1} \in E(G)$, then we replace P by a shorter path $P' = v_0v_1v_2 \cdots v_{j-1}v_{t+m}v_{t+m+1} \cdots v_{p-1}v_p$. If $v_{t+m}v_a \in E(G)$, where $a \in N^{j-1}(D^k) \cap (P \setminus \{v_{j-1}\})$, then we replace P by a shorter path $P' = v_0v_1v_2 \cdots v_{t-1}v_tv_{t+1} \cdots v_{t+m}v_av_{a+1} \cdots v_{p-1}v_p$, a contradiction to P being an eager D^k -ear. Hence P does not have two vertices v_{t+m}, v_{t+m+1} in some $N^j(D^k)$ where $m \geq 2, 1 \leq j \leq t-1$. Claim 2.1 is true. \blacksquare

Claim 2.2: If $v_i \in N^i(D^k), 0 \leq i \leq t, t \geq 2$ and $v_{t+1} \in N^{t-1}(D^k)$, then P does not have two vertices v_{t+m}, v_{t+m+1} in some $N^j(D^k)$ where $m \geq 1, 1 \leq j \leq t-1$.

Suppose that, to the contrary, P has two vertices v_{t+m}, v_{t+m+1} in some $N^j(D^k)$ where $m \geq 1, 1 \leq j \leq t-1$. The proof of Claim 2.2 is similar to the proof in the latter part of Claim 2.1, and so Claim 2.2 is also true. \blacksquare

By Claim 2.1 and Claim 2.2, we can get $|P| \leq 2k + 1$, in which equality holds if and only if $t = k$ and $v_kv_{k+1} \in N^k(D^k), v_{k+2} \in N^{k-1}(D^k)$, and so Claim 2 is true. \blacksquare

In the following we will construct a connected $(k-1)$ -step dominating set D^{k-1} such that $G[D^{k-1}]$ is rainbow connected. Since $E(D^k, N(D^k) \setminus B)$ has no bridges, for each edge e of $E(D^k, N(D^k) \setminus B)$, e must be in some cycle, and so there exists an eager D^k -ear P containing e . Thus we may construct a sequence of sets $D_1 \subset D_2 \subset D_3 \subset \cdots \subset D_t = D^{k-1}$, where $D_2 = D_1 \cup P_1, D_3 = D_2 \cup P_2, \cdots, D_t = D_{t-1} \cup P_{t-1}, P_1, P_2, \cdots, P_t$ are all eager D^k -ears. We color the new edges in every induced graph $G[D_i]$ such that every $x \in D_i \setminus D_1$ lies in an evenly colored eager D^k -ear in $G[D_i]$ for all $1 \leq i \leq t$.

For $i = 1$, it is obvious. If for some D_i , $N(D^k) \subset D_i$, note that for $1 \leq j \leq i - 1$, $N(D^k) \not\subset D_j$, then D_i is a connected $(k - 1)$ -step dominating set. We stop the procedure and set $D^{k-1} = D_i$, and evenly color the edges of P_{i-1} , and color the remaining uncolored new edges of $G[D_i]$ with the used colors. Otherwise, we will construct D_{i+1} as follows:

We choose any edge $x_0x_1 \in E(D^k, N(D^k) \setminus D_i)$, $x_0 \in D^k, x_1 \in N^1(D^k) \setminus D_i$. If P is an eager D^k -ear containing x_0x_1 , and $P \cap (D_i \setminus D_1) = \emptyset$, then we set $D_{i+1} = D_i \cup P$, and evenly color P , for the uncolored new edges of $G[D_{i+1}]$, we color them randomly with the used colors. Otherwise, the eager D^k -ear P containing x_0x_1 must satisfy $P \cap (D_i \setminus D_1) \neq \emptyset$. Assume $P_1 \subset P$, and let $P_1 = x_0x_1 \cdots x_l$, $P_1 \cap (D_i \setminus D_1) = \{x_l\}$. As $x_l \in D_i \setminus D_1$, x_l is in an evenly colored eager D^k -ear Q . Let Q_1 be the shorter segment of Q respect to x_l . Then $P = P_1 \cup Q_1$ is the eager D^k -ear containing x_0x_1 . We know that Q is evenly colored. If Q_1 is colored by the colors from $\{2k + 1, 2k, 2k - 1, \dots, 2k + 2 - \lfloor \frac{|Q_1|}{2} \rfloor\}$, then we will evenly color P by $1, 2, 3, \dots, \lceil \frac{|P_1|}{2} \rceil, 2k + 2 - \lfloor \frac{|P_1|}{2} \rfloor, \dots, 2k, 2k + 1$ in that order, here $c(x_0x_1) = 1$. If Q_1 is colored by the colors from $\{1, 2, 3, \dots, \lceil \frac{|Q_1|}{2} \rceil\}$, then we will evenly color P by $2k + 1, 2k, 2k - 1, \dots, 2k + 2 - \lfloor \frac{|P_1|}{2} \rfloor, \lceil \frac{|P_1|}{2} \rceil, \dots, 3, 2, 1$ in that order, here $c(x_0x_1) = 2k + 1$. Hence P is evenly colored. Set $D_{i+1} = D_i \cup P$. For the uncolored new edges of $G[D_{i+1}]$, we color them randomly with the used colors. Clearly, every $x \in D_{i+1} \setminus D_1$ lies in an evenly colored eager D^k -ear in $G[D_{i+1}]$.

Thus, we have constructed a connected $(k - 1)$ -step dominating set D^{k-1} , and every edge of $G[D^{k-1} \setminus B]$ is colored.

Claim 3: $G[D^{k-1} \setminus D^k]$ has no bridges.

Suppose that $xy \in G[D^{k-1} \setminus D^k]$ is a bridge.

By Claim 1, we know that if $x \in B$, then $y \notin B$, and if $y \in B$, then $x \notin B$. Hence we will consider the following two cases: If x is in some eager D^k -ear P , y is in some eager D^k -ear Q (P can be Q), then besides xy , there is still another path connecting x and y , so xy is in a cycle. If $x \in B$, y is in some eager D^k -ear Q , then xy is also in some cycle, a contradiction. \blacksquare

Now, we are ready for coloring B_E : If $b_k \leq 2k + 1$, then we use b_k different colors from $\{1, 2, \dots, 2k + 1\}$ to color each edge of B_E , respectively. If $b_k > 2k + 1$, then we first use colors $1, 2, \dots, 2k + 1$ to color any $2k + 1$ edges of B_E , respectively, then we use $b_k - (2k + 1)$ fresh colors to color the remaining uncolored edges, respectively.

In the following we claim that $G[D^{k-1}]$ is rainbow connected. For any two vertices $x, y \in D_1$, we know that x, y is rainbow connected. For $x \in D^{k-1} \setminus D_1, y \in D^k$, as x is in an eager D^k -ear P , let $P \cap D^k = y_1$. In D^k , there exists a rainbow path connecting y, y_1 . For $x \in D^{k-1} \setminus D_1, y \in B$, we know that x is in an evenly eager D^k ear P . If the bridge $yy_1 \in B_E(y_1 \in D^k)$ is colored by c_y which is also in P , then we choose the segment (which

does not contain the color c_y) connecting x to D^k . If the bridge $yy_1 \in B_E, (y_1 \in D^k)$ is colored by c_y which is not in P , then we arbitrarily choose a segment of P connecting x to D^k , we can also find a $x - y$ rainbow path.

For $x \in D^{k-1} \setminus D_1, y \in D^{k-1} \setminus D_1$, since x and y are both in evenly colored eager D^k -ears, let $x \in P, y \in Q, P, Q$ are evenly colored eager D^k -ears. If $P = Q$, then x, y is rainbow connected. Hence we may assume $P \neq Q$. Let $P = x_0x_1 \cdots x_i(x)x_{i+1} \cdots x_p, Q = y_0y_1 \cdots y_j(y)y_{j+1} \cdots y_q$. We distinguish two cases to show that x, y is rainbow connected.

Case 1: P and Q are internally disjoint.

We assume that $x_0x_1, \dots, x_{\lfloor \frac{p}{2} \rfloor}$ and $y_0y_1, \dots, y_{\lfloor \frac{q}{2} \rfloor}$ are colored by the colors from $\{1, 2, 3, \dots, k+1\}$, respectively. The other three coloring cases can be discussed in a similar way. We distinguish four subcases to demonstrate that there is an $x - y$ rainbow path.

Subcase 1.1: $i \leq \lfloor \frac{p}{2} \rfloor, j > \lfloor \frac{q}{2} \rfloor$.

We join $x = x_ix_{i-1} \cdots x_0$ to the $x_0 - y_q$ rainbow path in $G[D^k]$ followed by $y_qy_{q-1} \cdots y_j = y$. As the edges of $x = x_ix_{i-1} \cdots x_0$ are colored by the colors from $\{1, 2, \dots, k+1\}$, the edges of $y_qy_{q-1} \cdots y_j = y$ are colored by the colors from $\{2k+1, 2k, \dots, k+2\}$. Hence it is an $x - y$ rainbow path.

Subcase 1.2: $i > \lfloor \frac{p}{2} \rfloor, j \leq \lfloor \frac{q}{2} \rfloor$.

We join $y = y_jy_{j-1} \cdots y_0$ to the $y_0 - x_p$ rainbow path in $G[D^k]$ followed by $x_px_{p-1} \cdots x_i = x$. It is also an $x - y$ rainbow path.

Subcase 1.3: $i \leq \lfloor \frac{p}{2} \rfloor, j \leq \lfloor \frac{q}{2} \rfloor$.

If $i < j$, we join $x = x_ix_{i-1} \cdots x_0$ to the $x_0 - y_q$ rainbow path in $G[D^k]$ followed by $y_qy_{q-1} \cdots y_j = y$. As the edges of $x = x_ix_{i-1} \cdots x_0$ are colored by $i, i-1, \dots, 1$, the edges of $y_qy_{q-1} \cdots y_j = y$ are colored by the colors $2k+1, 2k, \dots, j$. It is an $x - y$ rainbow path. If $i \geq j$, we join $y = y_jy_{j-1} \cdots y_0$ to the $y_0 - x_p$ rainbow path in $G[D^k]$ followed by $x_px_{p-1} \cdots x_i = x$. As the edges of $y = y_jy_{j-1} \cdots y_0$ are colored by the colors $\{j, j-1, \dots, 1\}$, the edges of $x_px_{p-1} \cdots x_i = x$ are colored by the colors $\{2k+1, 2k, \dots, i\}$, it is also an $x - y$ rainbow path.

Subcase 1.4: $i > \lfloor \frac{p}{2} \rfloor, j > \lfloor \frac{q}{2} \rfloor$.

If $p - i \leq q - j$, then we join $x = x_ix_{i+1} \cdots x_p$ to the $x_p - y_0$ rainbow path in $G[D^k]$ followed by $y_0y_1, \dots, y_j = y$. If $p - i > q - j$, we join $y = y_jy_{j+1} \cdots y_q$ to the $y_q - x_0$ rainbow path in $G[D^k]$ followed by $x_0x_1 \cdots x_i = x$. So we find an $x - y$ rainbow path.

Case 2: P and Q are internally joint.

According to the construction and the coloring of D^{k-1} , we may assume that $P \subset D_{i_1}$,

$Q \subset D_{i_2}$, and $i_1 > i_2$, x_l is the first internal vertex of P in Q . If $x_p x_{p-1} \cdots x_{l+1} x_l = y_q y_{q-1} \cdots y_{l+1} y_l$, then the case is similar to Case 1 in essence. So we may assume $x_p x_{p-1} \cdots x_{l+1} x_l = y_0 y_1, \cdots, y_{p-l}$. We also distinguish four subcases to show that there is an $x - y$ rainbow path.

Without loss of generality, assume that the edges of $y_0 y_1 \cdots y_{\lfloor \frac{q}{2} \rfloor}$ are colored by $1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$. According to the coloring of D^{k-1} , the edges of $x_p x_{p-1} \cdots x_{\lfloor \frac{p}{2} \rfloor}$ are also colored by the colors from $\{1, 2, \cdots, k+1\}$, and the edges of $x_0 x_1 \cdots x_{\lceil \frac{p}{2} \rceil}$ are colored by the colors from $\{2k+1, 2k, \cdots, k+2\}$.

Subcase 2.1: $i \leq \lfloor \frac{p}{2} \rfloor, j > \lfloor \frac{q}{2} \rfloor$.

If $i < q - j$, then we join $x = x_i x_{i-1} \cdots x_0$ to the $x_0 - y_0$ rainbow path in $G[D^k]$ followed by $y_0 y_1 \cdots y_j = y$. If $i \geq q - j$, then we join $y = y_j y_{j+1} \cdots y_q$ to the $y_q - x_p$ rainbow path in $G[D^k]$ followed by $x_p x_{p-1} \cdots x_i = x$. We find the required $x - y$ rainbow path.

Subcase 2.2: $i > \lfloor \frac{p}{2} \rfloor, j \leq \lfloor \frac{q}{2} \rfloor$.

If $p - i \leq j$, then we join $x = x_i x_{i+1} \cdots x_p$ to the $x_p - y_q$ rainbow path in $G[D^k]$ with $y_q y_{q-1} \cdots y_j = y$. If $p - i > j$, we join $y = y_j y_{j-1} \cdots y_0$ to the $y_0 - x_0$ rainbow path in $G[D^k]$ followed by $x_0 x_1 \cdots x_i = x$. We also find the required $x - y$ rainbow path.

Subcase 2.3: $i \leq \lfloor \frac{p}{2} \rfloor, j \leq \lfloor \frac{q}{2} \rfloor$.

we join $x = x_i x_{i-1} \cdots x_0$ to the $x_0 - y_0$ rainbow path in $G[D^k]$ followed by $y_0 y_1 \cdots y_j = y$.

Subcase 2.4: $i > \lfloor \frac{p}{2} \rfloor, j > \lfloor \frac{q}{2} \rfloor$.

We join $x = x_i x_{i+1} \cdots x_p$ to the $x_p - y_q$ rainbow path in $G[D^k]$ followed by $y_q y_{q-1} \cdots y_j = y$.

Hence, for any two vertices $x, y \in D^{k-1} \setminus D_1$, there is rainbow path connecting x and y . Thus, we have constructed a connected D^{k-1} from D^k , and $rc(G[D^{k-1}]) \leq rc(G[D^k]) + \max\{2k+1, b_k\}$.

Hitherto, the proof of Theorem 1 has been completed. ■

The proof of Theorem 2:

Let u be the center of G , and set $D^r = \{u\}$. Then D^r is an r -step dominating set of G , and $rc(G[D^r]) = 0$. By making use of Theorem 1, we may construct $D^{r-1}, D^{r-2}, \cdots, D^2, D^1$ such that $D^r \subset D^{r-1} \subset D^{r-2} \cdots \subset D^1 \subset D^0 = V(G)$, and we have

$$\begin{aligned} rc(G[D^{r-1}]) &\leq rc(G[D^r]) + \max\{2r+1, b_r\} \\ rc(G[D^{r-2}]) &\leq rc(G[D^{r-1}]) + \max\{2(r-1)+1, b_{r-1}\} \\ &\dots \\ rc(G[D^0]) &\leq rc(G[D^1]) + \max\{2+1, b_1\} \end{aligned}$$

where $rc(G[D^0]) = rc(G)$, for $1 \leq i \leq r$, b_i is the number of bridges in $E(D^i, N(D^i))$. Thus we get that $rc(G) \leq rc(G[D^r]) + \sum_{i=1}^r \max\{2i + 1, b_i\} = \sum_{i=1}^r \max\{2i + 1, b_i\}$.

This completes the proof of Theorem 2. ■

By Claim 3, the subgraph $G[D^{i-1} \setminus D^i]$ has no bridges. Hence we immediately obtain the following corollary.

Corollary 1 *The number of bridges of G is equal to $\sum_{i=1}^r b_i$.*

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