A Turán-type problem on distances in graphs

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Abstract

We suggest a new type of problem about distances in graphs and make several conjectures. As a first step towards proving them, we show that for sufficiently large values of n and k, a graph on n vertices that has no three vertices pairwise at distance k has at most $(n - k + 1)^2/4$ pairs of vertices at distance k.

1 Introduction

In [8], Bollobás and Tyomkyn determined the maximum number of paths of length k in a tree T on n vertices. Here we suggest an extension of this problem to general graphs.

The 'obvious' extension, counting paths of a given length in a graph G, has been studied since 1971, see, e.g., [1, 3, 4, 5, 6, 7, 9, 11, 12] and the references therein. On the other hand, counting paths of length k in *trees* can be interpreted as counting pairs of vertices at distance k. Therefore, a natural question to ask is the following.

Question. For a graph G on n vertices, what is the maximum possible number of pairs of vertices at distance k?

To the best of our knowledge, this question has not been considered previously. Our aim in this paper is to formulate several conjectures and to prove one of them in the first non-trivial special case.

For a graph G, define the distance-k graph G_k to be the graph with vertex set V(G) and $\{x, y\} \in E(G_k)$ if and only if x and y are at distance k in G, that is, the shortest path between x and y has length k. We call such vertices x and yk-neighbours and the pair $\{x, y\}$ a k-distance. We call $d_{G_k}(x)$ the k-degree of x.

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Figure 1: A 5-broom for k = 8 and a 5-broom for k = 7.

Observe that if H is an induced subgraph of G, then H_2 is a subgraph of G_2 . This need not be the case when $k \geq 3$. It is clear that $G_k \cong H_k$ does not imply that $G \cong H$. If $G_k \cong H_k$, then we say that G is *k*-isomorphic to H.

We wish to maximise the number of edges in G_k over all graphs G on n vertices. One attempt to construct a graph with many k-neighbours would be to consider what we call t-brooms. For even $k \ge 4$ and for $t \ge 2$, define a t-broom to be a graph consisting of a central vertex v with t 'brooms' attached, each consisting of a path on (k-2)/2 vertices with leaves attached to the ends opposite v. In this way, the leaves of different brooms will be at distance k. For odd $k \ge 3$, to define a t-broom, take a copy of K_t and attach a broom to each vertex, adjusting the length of the path. (See Figure 1.) As in Turán's theorem, the number of k-distances in a t-broom will be maximised when the numbers of leaves in the brooms are as equal as possible.

In [8] Bollobás and Tyomkyn proved that if G is a tree, then $e(G_k)$ is maximal when G is a t-broom for some t.

Theorem 1. Let $n \ge k$. If G is a tree on n vertices, then $e(G_k)$ is maximal when G is a t-broom. If k is odd, then t = 2. If k is even, then t is within 1 of

$$\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{n-1}{k-2}}.$$

These results prompt us to make the following conjecture.

Conjecture 2. Let $k \ge 3$. There exists h = h(k) such that if $n \ge h(k)$, then $e(G_k)$ is maximised over all G with |G| = n when G is k-isomorphic to a t-broom for some t.

For small values of n there exist better constructions. For example, if k = 3 and n = 7, the 7-cycle has more 3-distances than any t-broom.

We firmly believe Conjecture 2 to be true, but are unable to prove it. In this paper, we approach Conjecture 2 by placing a restriction on $\omega(G_k)$, the clique number of G_k , which is the maximal number of vertices at pairwise distance k. We formulate the following natural analogue of Conjecture 2 under this condition.

Conjecture 3. Let $k \ge 3$ and $t \ge 2$. There is a function $h_2: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if $n \ge h_2(k, t)$, then $e(G_k)$ is maximised over all G with |G| = n and $\omega(G_k) \le t$ when G is k-isomorphic to a t-broom for some t.

In this paper, we shall discuss the case t = 2 of Conjecture 3, that is, the case when no three vertices of G are pairwise at distance k. Note that in this case the parity of k matters little, as the conjectured optimal example is just a path of length k - 2 with equally many leaves attached to each of its endvertices. The number of k-distances in such a graph is simply $|(n - k + 1)^2/4|$.

We prove Conjecture 3 for t = 2 and sufficiently large k. More precisely, we prove the following assertion.

Theorem 4. There is a constant k_0 and a function $n_0: \mathbb{N} \to \mathbb{N}$ such that for all $k \geq k_0$, all $n \geq n_0(k)$ and all graphs G of order n with no three vertices pairwise at distance k,

$$e(G_k) \le \frac{(n-k+1)^2}{4}.$$

Moreover, if equality holds, then G is k-isomorphic to the double broom.

We do not make any effort to determine k_0 and n_0 exactly; on the contrary, we are rather generous about them. However, we conjecture that k = 2 and (n, k) = (7, 3) are the only exceptions to the optimality of the double-broom.

Conjecture 5. In the setting of Theorem 4 we can take $k_0 = 3$, $n_0(3) = 8$ and $n_0(k) = k + 1$ otherwise.

For k = 2, we can do a little better than the bound in Theorem 4, as the following construction shows.

Example 6. Let X and Y be cliques on (n+1)/2 vertices each, with a vertex z in common. Take vertices $x \in X$ and $y \in Y$. Remove the edges $\{x, z\}$ and $\{y, z\}$ and add the edge $\{x, y\}$; call the resulting graph G. Then G_2 , the distance-2 graph of G, is a complete bipartite graph with one edge subdivided, and thus has a total of $(n-1)^2/4 + 1$ edges.

We believe that for $n \ge 5$, a triangle-free G_2 can have no more than $(n-1)^2/4+1$ edges. This is clearly true for n = 5, and a computer search verifies that it also holds for $6 \le n \le 11$. However, we cannot prove that it is true in general. If this is indeed so, then it shows that for k = 2 the quantity $(n-k+1)^2/4$ is within 1 of the maximum. The same would hold for k = 3 and n = 7, when the aforementioned 7-cycle wins by 1 over the double broom.

In the general case, with no restriction on $\omega(G_2)$, the maximum value of $e(G_2)$ is straightforward.

Proposition 7. Let G be a graph on n vertices. Then

$$e(G_2) \le \binom{n-1}{2}.$$

Moreover, if equality holds, then G is a star.

Proof. The result is clear for n = 3. We suppose that for some n > 3, the result holds for all graphs on at most n - 1 vertices. Let G be a graph on n vertices. We

may assume that G is connected. Otherwise, if G has $\ell \geq 2$ components of sizes c_1, \ldots, c_{ℓ} , say, then, by hypothesis,

$$e(G_2) \le \sum_{i=1}^{\ell} {\binom{c_i - 1}{2}} < {\binom{n-1}{2}}.$$

Observe that, setting $d = \operatorname{diam}(G)$,

$$\binom{n}{2} = e(G) + e(G_2) + \dots + e(G_d)$$

Thus,

$$e(G_2) \le \binom{n}{2} - e(G). \tag{1.1}$$

Since G is connected, $e(G) \ge n-1$, and hence $e(G_2) \le {\binom{n-1}{2}}$. Note that if equality holds, then G must be a tree. Moreover, in this case, by (1.1), diam(G) = 2, so G must be a star.

2 Preliminaries

In this section, we shall prove a straightforward bound on $e(G_k)$. We shall then discuss possible ways of extending this result to the upper bound in Theorem 4. We shall also discuss a useful property of spanning trees.

Since G_k is triangle-free by our assumption, Mantel's theorem implies that $e(G_k) \leq n^2/4$. In fact, we can do somewhat better by adapting a standard proof of Mantel's Theorem (see, e.g., [2]) to k-distances.

Lemma 8. If G_k is triangle-free, then

$$e(G_k) \le \frac{n(n-k+1)}{4}.$$

Proof. For a vertex x, let $\nu(x)$ be the number of k-neighbours of x, or equivalently, the degree of x in G_k . We may bound $e(G_k)$ as follows: for each pair of vertices x, y at distance k, count the k-neighbours of x and of y. Note that since x and y have no common k-neighbours (otherwise there would be a triangle in G_k), we have $\nu(x) + \nu(y) \leq n$. In fact, we can claim that $\nu(x) + \nu(y) \leq n - k + 1$, since none of the k-1 internal vertices on the shortest path between x and y are k-neighbours of x or of y. Summing over all such pairs $\{x, y\}$, we obtain

$$(n-k+1)e(G_k) \ge \sum_{\{x,y\}\in E(G_k)} (\nu(x) + \nu(y)).$$
(2.1)

Observe that for each $x \in V(G)$, the quantity $\nu(x)$ appears $\nu(x)$ times on the right-hand side of (2.1). By the Cauchy-Schwarz inequality, we have

$$\sum_{\{x,y\}\in E(G_k)} \left(\nu(x) + \nu(y)\right) = \sum_{x\in V(G)} \nu(x)^2 \ge \frac{1}{n} \left(\sum_{x\in V(G)} \nu(x)\right)^2 = \frac{4}{n} \left(e(G_k)\right)^2.$$
(2.2)

It follows from (2.1) and (2.2) that

$$e(G_k) \le \frac{n(n-k+1)}{4},$$
 (2.3)

as claimed.

The bound that we have just proved is about halfway between the trivial $n^2/4$ and the desired $(n - k + 1)^2/4$. There are two natural ways in which one could try to improve (2.3). One is to try to find vertices that have no k-neighbours at all. We say that such a vertex is an *interior* vertex; otherwise, a vertex is called an *exterior* vertex. The inspiration for this terminology is as follows. If $k \leq \text{diam}(G) \leq 2k-1$, then a vertex at or near the centre of the graph has no k-neighbours, while a vertex that is far from the centre of the graph will have one or more k-neighbours. If G has at least r interior vertices, then adapting the proof of Lemma 8 improves the bound in (2.3) to

$$e(G_k) \le \frac{(n-r)(n-k+1)}{4}.$$
 (2.4)

Thus, we are done if we can find at least k-1 interior vertices (which holds in the case when G is the double-broom). So, we may assume that r < k-1.

The other way to improve (2.3) would be to find many pairs of vertices $\{u, v\}$ such that many vertices z are k-neighbours of neither u nor v. We say that a vertex $v \in V$ is k-unaffiliated with a vertex $u \in V$ if $d(u, v) \neq k$. A vertex v is kunaffiliated with a set $U \subseteq V$ if it is k-unaffiliated with each $u \in U$. Otherwise, we say that v is k-affiliated with U. Thus, an interior vertex is k-unaffiliated with V. If G has r interior vertices and each $\{u, v\} \in E(G_k)$ has p k-unaffiliated vertices, then (2.3) improves to

$$e(G_k) \le \frac{(n-r)(n-p)}{4} \le \frac{\left(n - \frac{r+p}{2}\right)^2}{4}.$$
 (2.5)

In particular, we are done if $p \ge 2k - r - 2$, i.e., if every $\{u, v\} \in E(G_k)$ has at least k - r - 1 k-unaffiliated vertices other than those on the shortest path between u and v.

In order to prove Theorem 4, we shall show that for every pair of k-neighbours in $G, p \ge 2k - r - 2$. To do so, we shall need to study paths in G. In connection with this, we shall need the following easy result about the lengths of paths in a spanning tree. If P is a path, we write |P| to denote the number of vertices in P.

Lemma 9. Let $r \ge 2$ and let G be a graph on at least r + 1 vertices. If G has at most r interior vertices, then every spanning tree of G either contains no path of length at least r + 1 or contains a path of length 2k - r.

Proof. Let T be a spanning tree of G such that T contains a path of length at least r+1. Let P be a longest path in T and let u and v be its endpoints. Suppose that |P| < 2k - r. Let x_1, \ldots, x_{r+1} be the r+1 central vertices of P, indexed consecutively in order of increasing distance from u. (If |P| is even, then there are two choices for the r+1 central vertices of P; choose one arbitrarily.) Suppose that some x_i had a k-neighbour in G, called y. Let Q be the path in T from x_i

to y. Since, for all $x, y \in G$, $d_G(x, y) \leq d_T(x, y)$, we know that $|Q| \geq k + 1$. Let z be the furthest vertex from x_i at which P and Q coincide; by symmetry, we may assume that z is closer to v than x_i is. We shall construct a path in T that is longer than P, in contradiction to the assumption. Define P_+ to be the path formed by Q and the portion of P between u and x_i . Then

$$|P_{+}| = |Q| + d_{T}(x_{1}, x_{i}) + d_{T}(u, x_{1}) \ge (k+1) + (i-1) + d_{T}(u, x_{1}).$$

Because x_1, \ldots, x_{r+1} are the r+1 central vertices of P, we have

$$d_T(u, x_1) \ge \left\lfloor \frac{|P| - 1 - d_T(x_1, x_{r+1})}{2} \right\rfloor \ge \frac{|P| - r - 2}{2},$$

from which we deduce that

$$|P_+| \ge k + i + \frac{|P| - r - 2}{2} \ge \frac{|P| + 2k - r}{2} > |P|.$$

Thus, no x_i can have a k-neighbour in G, which means that all of the x_i must be interior vertices.

The following concepts will play key roles in the proof of Theorem 4. Let v and w be two vertices at distance k. Recall that a *geodesic* is a shortest path between two vertices in a graph. Define the vw-path P to be a shortest path between v and w. Order the vertices of G and conduct a breadth-first search starting from v such that the resulting tree T_v contains P. We define the v-path P_v to be the longest path in T_v . (If there is more than one longest path in T_v , then we choose one arbitrarily.) Similarly, we define T_w to be breadth-first tree with respect to w containing P, and the w-path P_w to be the longest path in T_w .

Consider the breadth-first tree T_v and the v-path P_v . Let x and y be the endpoints of P_v . Moving along P_v from x to y, or in fact along any path in T_v , the distance from v will first decrease, then increase — this is a fundamental property of breadth-first search trees, for the depth of a vertex w in such a tree equals $d_G(v, w)$. Thus, P_v can be divided into two geodesics. Let z be a vertex of P_v at minimal distance from v. Let P_1 denote the portion of P_v between x and z and P_2 the portion of P_v between z and y; one of these may be empty.

By our assumption that r < k - 1, if v and w are k-neighbours, then T_v must contain a path of length at least r + 1. Hence, by Lemma 9, P_v contains at least 2k - r vertices. Note also that at most two vertices on P_v can be at distance k from v.

3 Proof of Theorem 4

From now on, we shall assume that G satisfies $e(G_k) \ge (n-k+1)^2/4$. As noted above, we shall also assume that r < k - 1.

Let us briefly discuss the proof of Theorem 4. First, for k and n large enough, we shall prove a simple condition under which a pair of k-neighbours must have at least 2k - r - 2 k-unaffiliated vertices. We shall deduce from this that each pair of k-neighbours in G has almost enough k-unaffiliated vertices to achieve the desired

bound on $e(G_k)$. Second, we shall deduce our key lemma, which says that if some pair of k-neighbours $\{v, w\}$ does not have enough k-unaffiliated vertices, then all geodesics in G must have only a few vertices apart from each of P_v and P_w . Third, we shall show that in this case, every other k-neighbour of v is at distance o(k) from w, and vice-versa. Moreover, for some $\delta = o(1)$, we shall show that all vertices that are at distance at least δk from both v and w have very few k-neighbours, which, by Turán's theorem, will contradict our assumption that $e(G_k) \ge (n - k + 1)^2/4$. Finally, we shall deduce that in this case, G has at most as many k-distances as the double broom, which will imply that $e(G_k) = (n - k + 1)^2/4$. Moreover, we shall show that in this case G is k-isomorphic to the double broom.

Lemma 10. For every $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that for all $k \ge K(\varepsilon)$, if some geodesic in G contains εk vertices that are k-affiliated with either $v \in E(G_k)$ or $w \in E(G_k)$, then we can find 2k vertices that are k-unaffiliated with both v and w.

Proof. Suppose that Q is a geodesic and that εk vertices of Q are k-affiliated with either v or w. By the pigeonhole principle, we can assume that $m \ge \varepsilon k/2$ of them are at distance k from w. Index them consecutively by x_1, \ldots, x_m . For $1 \le i \le m$, let Q_i denote a shortest path in G from x_i to w. Let us 7-colour the vertices of Qso that x_i is coloured with colour j if $i \equiv j \mod 7$ and choose a colour (red, say) that belongs to at least m/7 of the x_i . For each red x_i , move two steps along the path Q_i . In this way, we obtain $m' \ge m/7$ vertices at distance k - 2 from w and at distance at least 7 - 2 - 2 = 3 from each other; let us call them $y_1^1, y_2^1, \ldots, y_{m'}^1$. If a vertex y_i^1 is not at distance k from v, set $z_i^1 = y_i^1$. If y_i^1 is at distance k from v, take z_i^1 to be the vertex obtained by moving from y_i^1 one step towards v (see Figure 2). Since the y_i^1 were at pairwise distance at least 3, all of the z_i^1 will be distinct vertices at distance between k - 1 and k - 3 from w and not at distance kfrom v, i.e., they are k-unaffiliated with both v and w.

Similarly, by 13-colouring Q, defining 'red' to be the largest colour class, and moving 5 steps along the paths Q_i , we find $m'' \ge m/13$ vertices $y_1^2, y_2^2, \ldots, y_{m''}^2$ at distance k - 5 from w and at distance at least 13 - 5 - 5 = 3 from each other. This gives rise to m/13 distinct k-unaffiliated vertices z_i^2 at distance between k - 4 and k - 6 from w and not at distance k from v, i.e., the z_i^2 are disjoint from the previously constructed z_i^1 .

Repeating this procedure for all $6\ell + 1$ -colourings up to $\ell = \lfloor m/6 \rfloor$, we obtain a total of at least

$$\left(\frac{1}{7} + \frac{1}{13} + \dots + \frac{1}{6\lfloor m/6 \rfloor + 1}\right) m \ge \frac{1}{12} \cdot \frac{\varepsilon k}{2} \log \frac{\varepsilon k}{2}$$

k-unaffiliated vertices, which is greater than 2k for k large enough.

Lemma 10 has the following important corollary.

Corollary 11. Let $\{v, w\} \in G_k$. Let P_v and P_w be geodesics as defined above. Then either $|P_v| - \varepsilon k \ge (2k - r) - \varepsilon k$ vertices on P_v are k-unaffiliated with v and wor we can find 2k vertices that are k-unaffiliated with v and w elsewhere. The same is true of P_w .



Figure 2: The set of vertices z_i^1 that are k-unaffiliated with both v and w.

Corollary 11 and equation (2.5) have the following immediate consequence, which is an approximate version of the bound in Theorem 4.

Corollary 12. For every $\varepsilon > 0$ there is a constant $K(\varepsilon)$ such that for all $k \ge K(\varepsilon)$, if G_k is triangle-free then

$$e(G_k) \le \frac{\left(n - (1 - \varepsilon)k\right)^2}{4}.$$

Lemma 10 and Corollary 11 also give us very useful information about the structure of the graph. The following lemma is the main tool in the remainder of the proof of Theorem 4.

Lemma 13. Let $\varepsilon > 0$ and let $k \ge K(\varepsilon)$, where $K(\varepsilon)$ is as in Lemma 10. Suppose that v and w are k-neighbours in a graph G that have fewer than 2k - r - 2k-unaffiliated vertices. Then any geodesic in G contains fewer than $2\varepsilon k$ vertices disjoint from P_v , and similarly for P_w .

Proof. Suppose that Q is a geodesic in G with at least $2\varepsilon k$ vertices disjoint from P_v . First, if εk of these vertices are k-unaffiliated with v and w, then, by Corollary 11, v and w have at least $2k \ge 2k - r - 2$ k-unaffiliated vertices. Second, if not, then $Q \setminus P_v$ must contain at least εk vertices that are k-affiliated with v and w. In this case, by Lemma 10, we again obtain at least $2k \ge 2k - r - 2$ k-unaffiliated vertices. In either case, we reach a contradiction.

Let P denote the vw-path and let P' denote the vw'-path, as defined in Section 2. Recall that the v-path P_v splits into two geodesics, which we call P_1 and P_2 , along each of which the distance from v is strictly monotone. Similarly, the w-path P_w splits into two geodesics, which we call P_3 and P_4 . Note that since P_v was defined on a tree T_v that contains P, we have that $P_v \cap P$ is an interval of P, lying entirely in P_1 or in P_2 , and analogously for $P_w \cap P$. For the remainder of the proof, without loss of generality, let $P_v \cap P \subseteq P_1$ and let $P_w \cap P \subseteq P_3$.

As was already mentioned in Section 2, by Lemma 8 we are done if any pair of k-neighbours has 2k - r - 2 k-unaffiliated vertices. So let us assume for the sake of contradiction that some vertices v and w at distance k have fewer than that many k-unaffiliated vertices. Let w' be another k-neighbour of v. How large can the distance between w and w' be? The following lemma shows that this distances is either close to 2k or close to 0.

Lemma 14. Let $\varepsilon > 0$ and let $k \ge K(\varepsilon)$, where $K(\varepsilon)$ is as in Lemma 10. Let v and w be k-neighbours with fewer than 2k - r - 2 k-unaffiliated vertices. Let w' be another k-neighbour of v. Then either d(w, w') = (2 - o(1))k or d(w, w') = o(k).

Proof. Recall that P_v and P_w denote the v-path and the w-path, respectively. By Corollary 11, P_v contains at least $|P_v| - \varepsilon k \ge 2k - r - \varepsilon k$ k-unaffiliated vertices. Since, by assumption, v and w have at most 2k - r - 2 k-unaffiliated vertices, by Lemma 13, we must have

$$|P_v| \le 2k - r + \varepsilon k.$$



Figure 3: When $u' \in P_1$, all k-neighbours of v are close together. The dashed segments represent the path P_1 .

Observe that G contains at most εk vertices that are k-unaffiliated with v and w and are not on P_v , or else we are done by Corollary 11. The same assertions as above hold for P_w in place of P_v . By Lemma 13, for the vertex sets of the paths,

$$|P \cap P_v| \ge (1 - 2\varepsilon)k$$
 and $|P \cap P_w| \ge (1 - 2\varepsilon)k.$ (3.1)

Let u be the point furthest from v at which P and P_v coincide. By hypothesis, $u \in P_1$. Similarly, let u' be the furthest point from v at which P' and P_v coincide. By Lemma 13, $d(u, w) \leq 2\varepsilon k$ and $d(u', w') \leq 2\varepsilon k$. It follows that

$$(1 - 2\varepsilon)k \le d(v, u), d(v, u') \le k.$$

$$(3.2)$$

Now we consider two cases: when $u' \in P_1$ and when $u' \in P_2$.

Suppose that u' lies on P_1 . Suppose first that u is closer to v than u' is (see Figure 3). Then $d(u, w') = d(u, w) \leq 2\varepsilon k$, thus,

$$d(w, w') \le d(w, u) + d(u, w') \le 4\varepsilon k.$$

If u' is closer to v than u is, then by a similar argument, $d(u', w) = d(u', w') \le 2\varepsilon k$, and so

$$d(w, w') \le d(w, u') + d(u', w') \le 4\varepsilon k$$

If, however, $u' \in P_2$, then d(w, w') depends on the length of $P_w \setminus P$. Because $u' \in P_2$, we have $|P_2| \ge (1 - 2\varepsilon)k$. Since $|P_1| + |P_2| = |P_v| + 1 \le 2k + \varepsilon k$ (and similarly for $|P_3|$, $|P_4|$ and $|P_w|$), we have

$$(1 - 2\varepsilon)k \le |P_i| \le (1 + 3\varepsilon)k \text{ for } i = 1, 2.$$

$$(3.3)$$

Now we consider the geodesics P_3 and P_4 that comprise P_w . We have assumed that P_3 contains $P_w \cap P$. Then, because $P_4 \cap P = \emptyset$, we have $P_1 \cap P_4 \subseteq P_1 \setminus P$, hence,

$$|P_1 \cap P_4| \le |P_1 \setminus P| \le (1+3\varepsilon)k - (1-2\varepsilon)k = 5\varepsilon k.$$

Thus, by Lemma 13,

$$|P_4 \setminus P_2| = |P_4 \cap P_1| + |P_4 \setminus P_v| \le 5\varepsilon k + 2\varepsilon k = 7\varepsilon k.$$



Figure 4: When $u' \in P_2$ and $|P_4|$ is large, all k-neighbours of v are again close together. The dashed segments represent the path P_4 . The vertices a and b denote the endpoints of P_2 .

Now we shall show that if $|P_4|$ is at all large, then d(w, w') = o(k), while if $|P_4|$ is very small, then d(w, w') = 2k - o(k).

Case 1: Suppose first that $|P_4| > 7\varepsilon k$. Then, because $|P_4 \setminus P_2| \le 7\varepsilon k$, we have $P_2 \cap P_4 \neq \emptyset$. Let t denote the vertex at which P_4 meets P. Then by Lemma 13, we have $d(t, w) < 2\varepsilon k$. Let q be the vertex of $P_2 \cap P_4$ that is closest to w (see Figure 4). Then $d(q, w) \le |P_4 \setminus P_2| + d(t, w) < 9\varepsilon k$. Then

$$k + 9\varepsilon k \ge d(q, v) \ge k - 9\varepsilon k.$$

Now we bound d(q, u'). The bound depends on the location of q relative to u'. If q is farther away from v than u' is, then by (3.3),

$$d(q,v) \le |P_2| - d(q,u') \le (1+3\varepsilon)k - (1-2\varepsilon)k = 5\varepsilon k.$$

If, however, q is closer to v than u' is, then the fact that $d(q, v) \ge k - 9\varepsilon k$ implies that $d(q, u') < d(q, w') \le 9\varepsilon k$. Thus, $d(q, u') \le 9\varepsilon k$ and $d(q, w') \le 11\varepsilon k$. We therefore have

$$d(w, w') \le d(w, q) + d(q, w') \le 9\varepsilon k + 11\varepsilon k = 20\varepsilon k,$$

which completes the proof of Case 1.

Case 2: Suppose instead that $|P_4| \leq 7\varepsilon k$. Then, by (3.1) and Lemma 13, we have

$$|P_2 \cap P_3| = |P_2| - |P_2 \setminus P_w| - |P_2 \cap P_4| \ge (k - 2\varepsilon k) - 2\varepsilon k - 7\varepsilon k = k - 11\varepsilon k.$$
(3.4)

Since P_3 contains $P_w \cap P$ and P_2 is edge-disjoint from P, it follows from (3.1) and (3.4) that

$$|P_2 \cap P_3| + |P \cap P_3| \ge 2k - 13\varepsilon k. \tag{3.5}$$

Let q' be the vertex of $P_2 \cap P_3$ that is furthest away from w (see Figure 5). Then, by (3.3) and (3.5),

$$2k + 3\varepsilon k \ge d(q', v) + d(v, w) \ge d(q', w) \ge |P_2 \cap P_3| + |P \cap P_3| \ge 2k - 13\varepsilon k.$$
(3.6)



Figure 5: When $u' \in P_2$ and $|P_4|$ is small, all k-neighbours of v are at distance (2 - o(1))k from one another. The dashed segments represent the portion of the path P_3 that is disjoint from P. The vertices a and b denote the endpoints of P_2 .

Also, it follows from (3.3) and (3.4) that

$$k + 3\varepsilon k \ge |P_2| \ge d(q', v) \ge |P_2 \cap P_3| \ge k - 11\varepsilon k.$$

It follows from this and (3.2) that $d(q', u') \leq 13\varepsilon k$ and therefore that $d(q', w') \leq 15\varepsilon k$. We obtain from this and (3.6) that

$$d(w, w') \ge d(q', w) - d(q', w') \ge 2k - 28\varepsilon k.$$

This proves the lemma.

We shall now show that the assumption that d(w, w') = (2 - o(1))k for some w' leads to a contradiction. Let $\delta = O(\varepsilon) = o(1)$. For $v \in V(G)$, define the *cluster* of v to be the set \mathcal{C}_v of vertices at distance at most δk from v.

Lemma 15. Fix $\varepsilon > 0$. For k and n large enough, let G be a graph on n vertices and let v and w be k-neighbours with fewer than 2k - r - 2 k-unaffiliated vertices. Then every k-neighbour of v is at distance $O(\varepsilon k) = o(k)$ from w, and vice versa.

Proof. Suppose that $w' \neq w$ is a k-neighbour of v such that d(w, w') = (2 - o(1))k. In this case, our graph G is 'flat', i.e., it contains a geodesic P_3 of length (2 - o(1))kand every vertex is o(k) away from P_3 . The vertex v lies close to the centre of P_3 , whereas w and w' lie near the opposite ends: every k-neighbour of v lies within o(k)of either w or w'. Since there are no vertices at distance (2 - o(1))k from v, switching v and w in the statement of Lemma 14 yields that all k-neighbours of w are close to v. Since, by assumption, v and w have fewer than 2k k-unaffiliated vertices, all but at most 2k vertices are contained in one of the clusters C_v , C_w and $C_{w'}$. Every vertex on P_3 not lying within $2\delta k$ of either v, w or w' cannot have a k-neighbour in any of these clusters. Therefore any such vertex has at most 2k kneighbours. So, we have c = (2 - o(1))k vertices of k-degree at most 2k. Applying Turán's theorem to the remaining vertices, we obtain

$$e(G_k) \le \left(\frac{n-c}{2}\right)^2 + 2ck < \left(\frac{n-k+1}{2}\right)^2$$

provided that n is sufficiently large compared to k, contradicting our hypothesis that $e(G_k) \ge (n-k+1)^2/4$.



Figure 6: The vertices of S_1 , shown in grey, are just outside of C_v , and so have no k-neighbours in either C_v or C_w .

Since all but at most 2k of the vertices in G are at distance k from either v or w, we can conclude that all but at most 2k vertices lie either in C_v , that is, within distance o(k) of v, or in C_w . This is a fairly strong structural property of G and from here it is a short step to completing the proof of Theorem 4.

Proof of Theorem 4. Observe that by the definition of δ , every vertex on the vw-path P that is at distance more than δk from both v and w cannot have a k-neighbour in either cluster. We shall use these vertices, together with a small set that we shall now construct, to produce a contradiction as in the proof of Lemma 15.

We shall now show that $|P_2|$ must be very small and that $|P_1|$ cannot be much larger than k. It will then follow that $|P_v|$ cannot be much larger than k, either.

If $|P_2| > 10\delta k$, then P_2 contains $4\delta k$ vertices at distance between $2\delta k$ and $6\delta k$ from v. By Lemma 13, at least $4\delta k - 2\varepsilon k > 3\delta k$ of these vertices are contained either in P_3 or in P_4 . We shall consider two cases: when at least δk of these vertices are in P_3 and when at least $2\delta k$ of them are in P_4 . We shall show below that either case produces a contradiction as in the proof of Lemma 15. Suppose first that δk of them are in P_3 . Let $S_1 = \{x \in P_3 : 2\delta k \leq d(v, x) \leq 6\delta k\}$, let y be the closest vertex of P_3 to v and let $s \in P_2$ be such that $d(v, s) = 2\delta k$ (see Figure 6). By Lemma 13 and the triangle inequality, we have $d(y, s) \geq 2\delta k - 2\varepsilon k$. So,

$$d(w,s) \ge k - 2\varepsilon k + 2\delta k - 2\varepsilon k \ge k + \delta k,$$

which means that the vertices of S_1 are all at distance at least $k + \delta k$ from w and therefore have no k-neighbours in the clusters.

Suppose instead that $2\delta k$ of them are in P_4 . We shall show that this implies that $|P_4| \sim k$. Let x be the closest vertex of P_4 to w and let z be the point at which P_4 meets P_2 . Let a be the closest point of P_2 to v (see Figure 7). We may assume that

$$3\delta k < 4\delta k - 2\varepsilon k \le d(a, z) \le d(v, z) \le 4\delta k.$$

Then, because $a, x \in P$,

$$|P_4| \ge d(x,z) \ge d(a,x) - d(a,z) \ge k - 4\varepsilon k - 4\delta k.$$
(3.7)

Also, since $|P_w| \leq (2 + \varepsilon)k$ and $|P_3| \geq (1 - 2\varepsilon)k$, it follows that

$$|P_4| \le (1+3\varepsilon)k. \tag{3.8}$$



Figure 7: The vertices of S_2 , shown in grey, have no k-neighbours in either cluster.

Then $|P_4| \sim k$. Let S_2 denote the $2\delta k$ central vertices of P_4 . Observe that if $q \in S_2$, then both d(q, w) and d(q, v) are bounded away from both 0 and k. Indeed, after a bit of calculation, it follows from (3.7) and (3.8) that

$$\frac{k-4\varepsilon k-4\delta k}{2}-\delta k\leq d(q,w)\leq \frac{(1+3\varepsilon)k}{2}+\delta k$$

and that

$$3\delta k + \frac{k - 4\varepsilon k - 4\delta k}{2} - \delta k \le d(q, v) \le 4\delta k + \frac{(1 + 3\varepsilon)k}{2} + \delta k.$$

Together with the aforementioned vertices on P, in each case we obtain $c \ge k + \delta k$ vertices that are not in either cluster and whose k-degree is at most 2k. By applying Turán's theorem to these vertices as in the proof of Lemma 15, we obtain a contradiction for large enough n. Thus, $|P_2| \le 10\delta k$.

Similarly, we claim that $|P_1| \leq k + 10\delta k$. If not, then P_1 contains at least $3\delta k$ vertices that are at distance between $k + 2\delta k$ and $k + 5\delta k$ from v, and at distance between $2\delta k - 2\varepsilon k$ and $5\delta k + 2\varepsilon k$ from w, and we obtain a contradiction as above.

We therefore have

$$|P_v| \le k + 20\delta k.$$

Recall that by Lemma 9 we have $|P_v| \geq 2k - r$. If $2k - r > k + 20\delta k$, then we have a contradiction, which means that $r \geq k - 1$, that is, that G has at least k - 1 interior vertices. Hence, by (2.4), we have $e(G_k) = (n - k + 1)^2/4$. Let I denote the set of interior vertices of G, and recall that by definition an interior vertex is isolated in G_k . Because $e(G_k)$ is maximal, it follows from (2.5) that for all $\{x, y\} \in E(G_k)$, the only vertices that are k-unaffiliated with both x and y are those on the (unique) shortest path between x and y. Hence, all of these vertices must be in I. It follows that $G_k \setminus I$ is a complete bipartite graph with balanced parts, which means that G is k-isomorphic to the double broom, as claimed.

Otherwise, for k and n large enough, if any geodesic outside of P has length more than $20\delta k + 4\epsilon k \leq 21\delta k$, then it has at least $2\epsilon k$ vertices disjoint from P_v , and we are done by Lemma 13.

Let m be the midpoint of P, or one of the midpoints if k is odd. Suppose that there exist k-neighbours x and y in G such that every k-geodesic between x and y misses m. Let Q be such a geodesic. Then it must miss either the path P_{vm} between v and m or the path P_{wm} between w and m. In either case, Q will contain a geodesic of length at least k/4 disjoint from P, a contradiction. Hence, every pair of k-neighbours is connected by a geodesic containing m.

It follows that G is k-isomorphic to T_m , the breadth-first tree with respect to m. By Theorem 1, T_m has at most as many k-distances as the double broom does. Thus, $e(G) = (n - k + 1)^2/4$ and G is k-isomorphic to the double broom. The proof of Theorem 4 is complete.

4 Discussion

We believe that Conjecture 3 may be susceptible to a similar approach to the one above. In Section 2, we obtained our first non-trivial bound, Lemma 8, by adapting a proof of Mantel's theorem, which is the simplest case of Turán's theorem. Unfortunately we were not able find a straightforward generalisation of this approach to K_{t+1} -free distance-k graphs when $t \geq 3$. A possible solution might be to adapt a proof of Turán's theorem that works for all t. However, it seems difficult to generalise Lemma 9 for values of t greater than 2.

We also note that in [10], Csikvári asked a similar question about maximising or minimising the number of (closed) walks of length k in a connected graph G on n vertices and m edges. In the same paper Csikvári settled the case of closed walks and m = n - 1, that is, when G is a tree. The answer for general walks on trees was given in [8], but the general case remains open.

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