Stability Number and f-Factors in Graphs

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Abstract

Let $f: X \longrightarrow N$ be an integer function. An *f*-factor is a spanning subgraph of a graph G = (X, E) whose vertices have degrees defined by f. In this paper, we prove a sufficient condition for the existence of a *f*-factor which involves the stability number, the minimum degree of G or the connectivity of the graph.

Keywords: Factor, stability number, connectivity, toughness, minimum degree.

1 Introduction

We consider simple graphs without loops. For notation and graph theory terminology we follow in general [10]. Let G be a graph with vertex set Xand edge set E(G). Denote by $d_G(x)$ the degree of a vertex x in G, and by $\delta(G)$ the minimum degree of G. A spanning subgraph of G is a subgraph of G with vertex set X. Let $f: X \longrightarrow N$ be an integer function. For any subset A of X, we denote by f(A) the sum $\sum_{x \in A} f(x)$. A spanning subgraph H of a graph G such for every vertex x, $d_H(x) = f(x)$, is called an f-factor of G. Let a, b be fixed integers. A spanning subgraph F of G is called an [a, b]-factor of G if $a \leq d_F(x) \leq b$ for all $x \in X$.

For $S \subseteq X$, let |S| be the number of vertices in S and let G[S] be the subgraph of G induced by S. We write G - S for $G[X \setminus S]$. A set $S \subseteq X$

is called independent if G[S] has no edges. Denote by $\alpha(G)$ the stability number of a graph G, by $\kappa(G)$ its vertex connectivity. For any vertex $v \in X$, the open neighborhood of v is the set $N(v) = \{u \in X \setminus uv \in E(G)\}$; for a set $A \subseteq X$, $N_G(A)$ denotes the set of neighbors in G of vertices in A. Given disjoint subsets $A, B \subseteq X$, we write e(A, B) for the number of edges in Gwith one extremity in A and the other one in B.

If S is a cutset, let h'(G-S) be the number of components C of G-S such that $\sum_{x \in C} f(x)$ is odd.

Let t be a nonnegative real number. We say that G is t odd-tough if for each cutset S, $h'(G-S) \leq |S|/t$. We remark that if G is t tough then G is t odd-tough.

2 Known Results

Given a graph G = (X, E), an application f and a cuple of disjoint subsets of X, we recall that an *odd component* C of $G - (S \cup T)$ is a component Csuch that odd.

Many authors have investigated f-factors, see for example [5]. Tutte ([6]) gave the well-known necessary and sufficient condition for existence of an f-factor.

Condition [9] A graph G = (X, E) has an f-factor if and only if

1) $\delta(S,T) = f(S) - f(T) + \sum_{v \in T} d_{G \setminus S}(v) - h(S,T) \ge 0$, disjoint subsets S

and T of X

where h(S,T) is the number of odd components of $G - (S \cup T)$ 2) $\delta(S,T) \equiv f(X) \pmod{2}$.

This condition is also a corollary of the (g, f) factor theorem of Lovász in [6]. However, in practise, this condition remains difficult to verify.

Katerinis and Tsikopoulos established a condition on the minimum degree for the existence of f-factors.

Theorem 1 [3] Let $b \ge a$ two positive integers and let G = (X, E) be a graph with the minimum degree δ . Suppose $\delta \ge \frac{b |X|}{a+b}$, and |X| > (a+b)(b+a-3)/a. If f is a function from X to $\{a, a+1, ..., b\}$ such that f(X) is even, then G has an f-factor.

In [2], Katerinis has a condition on the toughness of the graph.

Only few results are known which relate the stability number and factors. Nishimura had sufficient condition for a k factor.

Theorem 2 Let $r \ge 1$ be an odd integer, and G be a graph of even order. of connectivity κ . If $\kappa \ge (r+1)^2/2$, and, $\alpha(G) \le \frac{4r.\kappa}{(r+1)^2}$, then G has an *r*-factor.

The following result involving the stability number and the minimum degree of a graph was given by M. Kouider and Zbigniew Lonc [4]:

Theorem 3 [4] Let $b \ge a+1$ and let G be a graph with the minimum degree δ . If $\alpha(G) \le 4b(\delta-a+1))/(a+1)^2$, for a odd and $\alpha(G) \le 4b(\delta-a+1)/a(a+2)$, for a even.

then G has an [a, b]-factor.

Cai has shown that

Theorem 4 c Let G be a connected $K_{1,n}$ -free graph and let f be a nonnegative integer-valued fonction on V(G) such that $1 \le n - 1 \le a \le f(x) \le b$ for every $x \in V(G)$.

If f(V(G)) is even, $\delta(G) \ge b + n - 1$ and $\alpha(G) \le \frac{4a.(\delta - b - n + 1)}{(n - 1)(b + 1)^2}$, then G has an f factor.

Note that Cai conjectured that the condition on the stability $\alpha(G) \leq \frac{4a.(\delta-b)}{(b+1)^2}$ is sufficient in connected graphs. We have the following counterexample.

Suppose b is an odd integer and a an integer strictly less than b.

Let G_0 be a connected graph of minimum degree δ at least $(b+1)^3 + b$. Let $p = \frac{4a.(\delta - b)}{(b+1)^2}$. In the graph G_0 , we suppose there exists S be a cutset of k < b vertices, such that G(S) is complete and $C_1, ..., C_p$, the connected components of G - S, form a family of complete subgraphs of order $\delta + 1$, mutually independent. Furthermore $G(S \cup C_1)$ is complete, and, for each $i \geq 2$, exactly one edge joins S to C_i . So $\alpha(G_0) = p = \frac{4a.(\delta - b)}{(b+1)^2}$.

Let us consider the application f on X such that f(x) = a if $x \in S$, and, f(x) = b otherwise.

If a f factor exists we should have $\alpha = c(G-S) \leq a.k$, so c(G-S) should be at most *ab*. This is not satified as $\alpha = \frac{4a.(\delta - b)}{(b+1)^2} > 4a(b+1)$.

One can see the surveys [8] or [5] for other results.

3 Main Results

We have established a new sufficient condition for a graph to have an f-factor; this condition involves the stability number, the minimum degree of the graph.

Theorem 5 Let $b \ge 2$ be an integer and let G = (X, E) be a connected graph, of minimum degree δ at least b. Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \le f(x) \le b$ and f(X) is even. If $\alpha(G) \le \frac{4a.(\delta - b)}{(b+1)^2}$, and the odd-toughness of G is at least 1/a, then G contains an f-factor.

Furthermore, we get this corollary.

Corollary 1 Let $b \ge 2$ be an integer and let G = (X, E) be a graph, of minimum degree δ at least b and connectivity κ . Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \le f(x) \le b$ and f(x) is even. If $\alpha(G) \le \frac{4a.(\delta - b)}{(b+1)^2}$, then G contains an f-factor. **Corollary 2** Let $b \ge 2$ be an integer and let G = (X, E) be a graph, of minimum degree δ at least b and connectivity κ . Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \le f(x) \le b$ and f(X) is even. If $\alpha(G) \le \min(\frac{4a.(\delta - b)}{(b+1)^2}, a\kappa)$, then G contains an f-factor \bullet

The condition $\alpha(G) < \frac{4a.(\delta - b)}{(b+1)^2} + 1$ is necessary if b > 2a. Let $\alpha > \delta > b > r$ be four integers. Let us consider a graph G_1 composed by the join of a complete graph $A = K_{\delta - r+1}$ and B, the disjoint union of α complete graphs of order r. Let f be a function such that

f(x) = a if $x \in X(A)$, f(x) = b if $x \in X(B)$. If an f factor exists we get $\alpha(G) \le \frac{a.(\delta - r + 1)}{r.(b + 1 - r)}.$

For *b* odd and r = (b+1)/2, we get $\alpha(G) < \frac{4a.(\delta - b)}{(b+1)^2} + \frac{2a}{b+1}$.

4 Proof of Theorem 4

We set first some usefull lemmas.

Lemma 6 $\delta(S,T)$ is even.

Proof Let \mathcal{I}_1 (respectively \mathcal{I}_2) be the set of even (resp. odd) components of $G - (S \cup T)$. By definition,

$$f(\mathcal{I}_1) \equiv e(\mathcal{I}_1, T), \quad (1)$$
$$f(\mathcal{I}_2) \equiv h(S, T) + e(\mathcal{I}_2, T), \quad (2)$$

so, by (1) and (2),

 $f(X) = f(S) + f(T) + f(\mathcal{I}_1 + f(\mathcal{I}_2) \equiv f(S) - f(T) + e(G - (S \cup T), T) + h(S, T).$ As f(X) is even, the conclusion follows.

Lemma 7 T is non-emptyset.

Proof

If $T = \emptyset$ and $S = \emptyset$, then $\delta(S, T) = -h = 0$ as G is connected and f(X) is even. If $T = \emptyset$ and S is not empty, then h(S, T) is the number of components of G - S such that f(C) is odd.

Either S is not a cutset, then $h(S,T) \leq 1 \leq a|S|$; or S is a cutset, as G is 1/a-tough, $h \leq a|S|$.

As $a|S| \leq f(S)$, then $\delta(S,T) = f(S) - h(S,T) \geq f(S) - a|S| \geq 0$.

Proposition 1 If $\alpha(G) \leq \frac{4a.(\delta - b)}{(b+1)^2}$, then

$$|S| > \delta - b$$

Proof

The proof is by contradiction. As $\delta(S,T) < 0$ and $a \leq f(x) \leq b$ for each x, then

$$(\delta - |S|)|T| + a|S| - b|T| - h < 0,$$

 \mathbf{SO}

$$(\delta - |S| - b)|T| < h - a|S|.$$

If $|S| = \delta - b$, we get $|S| < \frac{h}{a} < \frac{\alpha(G)}{a} < \frac{4(\delta - b)}{9}$. This a contradiction. Now we assume $|S| < \delta - b$, and we get

$$|T| < \frac{h-a|S|}{(\delta-|S|-b)}.$$

If h < a|S|, then |T| = 0. As $h < \alpha$, then

$$|T| < \frac{4a}{(b+1)^2} \cdot \frac{((\delta - |b|) - (b+1)^2 |S|)}{(\delta - |b| - |S|)}$$

We get

$$|T| < \frac{4a}{(b+1)^2} \cdot \left(1 - \frac{((b+1)^2/4 - 1) \cdot |S|}{(\delta - |b| - |S|)}\right)$$
$$T < \frac{4a}{(b+1)^2}.$$

As $b \ge a$, $T < \frac{4a}{(b+1)^2} \le 1$, so |T| = 0. This is in contradiction with Lemma 5.

End of the proof of the theorem

Let h_2 be the number of components of $G - (S \cup T)$ not adjacent to T. As $\delta(S,T) < 0$, we have

$$2|E_T| + |T| + a|S| - (b+1)|T| - h_2 \le 0, \quad (1)$$

As α_T the stability number of T is at least $\frac{|T|^2}{2|E_T| + |T|}$, and
 $\alpha_T \le \alpha(G) - h_2$, we get, using (1),

$$\alpha(G) - h_2 \ge \frac{|T|^2}{(b+1)|T| - a|S| + h_2}$$

Let us set |T| = r.|S|. Then

$$\alpha(G) - h_2 \ge \frac{r^2 |S|^2}{(b+1)r|S| - a|S| + h_2}$$
$$\alpha(G) - h_2 \ge \frac{r^2 |S|}{(b+1)r - a + h_2/|S|}$$

The minimum of the bound as a function of r is for $r = \frac{2(a - h_2/|S|)}{b+1}$. It follows that

$$\alpha(G) - h_2 \ge \frac{4a \cdot |S|}{(b+1)^2} - \frac{4h_2}{(b+1)^2}$$

As by hypothesis $\alpha(G) \leq \frac{4a(\delta - b)}{(b+1)^2}$, and $|S| \geq (\delta - b)$, we get

$$h_2 \le \frac{4h_2}{(b+1)^2}.$$

So $h_2 = 0$. As $\delta \ge b$, then $\delta(S,T) \ge 0$. This is a contradiction with the definition of the cuple S, T.

This ends the proof of the theorem 4 \bullet

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