# Stability Number and f-Factors in Graphs 

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#### Abstract

Let $f: X \longrightarrow N$ be an integer function. An $f$-factor is a spanning subgraph of a graph $G=(X, E)$ whose vertices have degrees defined by $f$. In this paper, we prove a sufficient condition for the existence of a $f$-factor which involves the stability number, the minimun degree of $G$ or the connectivity of the graph.


Keywords: Factor, stability number, connectivity, toughness, minimum degree.

## 1 Introduction

We consider simple graphs without loops. For notation and graph theory terminology we follow in general [10]. Let $G$ be a graph with vertex set $X$ and edge set $E(G)$. Denote by $d_{G}(x)$ the degree of a vertex $x$ in $G$, and by $\delta(G)$ the minimum degree of $G$. A spanning subgraph of $G$ is a subgraph of $G$ with vertex set $X$. Let $f: X \longrightarrow N$ be an integer function. For any subset $A$ of $X$, we denote by $f(A)$ the sum $\sum_{x \in A} f(x)$. A spanning subgraph $H$ of a graph $G$ such for every vertex $x, d_{H}(x)=f(x)$, is called an $f$-factor of $G$. Let $a, b$ be fixed integers. A spanning subgraph $F$ of $G$ is called an [ $a, b]$-factor of $G$ if $a \leq d_{F}(x) \leq b$ for all $x \in X$.

For $S \subseteq X$, let $|S|$ be the number of vertices in $S$ and let $G[S]$ be the subgraph of $G$ induced by $S$. We write $G-S$ for $G[X \backslash S]$. A set $S \subseteq X$
is called independent if $G[S]$ has no edges. Denote by $\alpha(G)$ the stability number of a graph $G$, by $\kappa(G)$ its vertex connectivity. For any vertex $v \in X$, the open neighborhood of $v$ is the set $N(v)=\{u \in X \backslash u v \in E(G)\}$; for a set $A \subseteq X, N_{G}(A)$ denotes the set of neighbors in $G$ of vertices in $A$. Given disjoint subsets $A, B \subseteq X$, we write $e(A, B)$ for the number of edges in $G$ with one extremity in $A$ and the other one in $B$.
If $S$ is a cutset, let $h^{\prime}(G-S)$ be the number of components $C$ of $G-S$ such that $\sum_{x \in C} f(x)$ is odd.

Let $t$ be a nonnegative real number. We say that $G$ is $t$ odd-tough if for each cutset $S, h^{\prime}(G-S) \leq|S| / t$. We remark that if $G$ is $t$ tough then $G$ is $t$ odd-tough.

## 2 Known Results

Given a graph $G=(X, E)$, an application $f$ and a cuple of disjoint subsets of $X$, we recall that an odd component C of $G-(S \cup T)$ is a component $C$ such that odd.

Many authors have investigated $f$-factors, see for example [5]. Tutte ([6]) gave the well-known necessary and sufficient condition for existence of an $f$-factor.
Condition [9] A graph $G=(X, E)$ has an $f$-factor if and only if

1) $\delta(S, T)=f(S)-f(T)+\sum_{v \in T} d_{G \backslash S}(v)-h(S, T) \geq 0$, disjoint subsets $S$ and $T$ of $X$
where $h(S, T)$ is the number of odd components of $G-(S \cup T)$
2) $\delta(S, T) \equiv f(X)(\bmod 2)$.

This condition is also a corollary of the $(g, f)$ factor theorem of Lovász in [6]. However, in practise, this condition remains difficult to verify.

Katerinis and Tsikopoulos established a condition on the minimum degree for the existence of $f$-factors.

Theorem 1 [3] Let $b \geq a$ two positive integers and let $G=(X, E)$ be a graph with the minimum degree $\delta$. Suppose $\delta \geq \frac{b .|X|}{a+b}$, and $|X|>$
$(a+b)(b+a-3) / a$. If $f$ is a function from $X$ to $\{a, a+1, \ldots, b\}$ such that $f(X)$ is even, then $G$ has an $f$-factor.

In [2], Katerinis has a condition on the toughness of the graph.
Only few results are known which relate the stability number and factors. Nishimura had sufficient condition for a k factor.

Theorem 2 Let $r \geq 1$ be an odd integer, and $G$ be a graph of even order. of connectivity $\kappa$. If $\kappa \geq(r+1)^{2} / 2$, and, $\alpha(G) \leq \frac{4 r . \kappa}{(r+1)^{2}}$, then $G$ has an $r$-factor.

The following result involving the stability number and the minimum degree of a graph was given by M. Kouider and Zbigniew Lonc [4]:

Theorem 3 [4] Let $b \geq a+1$ and let $G$ be a graph with the minimum degree $\delta$. If $\alpha(G) \leq 4 b(\delta-a+1)) /(a+1)^{2}$, for $a$ odd and $\alpha(G) \leq 4 b(\delta-a+1) / a(a+2)$, for a even.
then $G$ has an $[a, b]$-factor.
Cai has shown that
Theorem 4 c Let $G$ be a connected $K_{1, n}$-free graph and let $f$ be a nonnegative integer-valued fonction on $V(G)$ such that $1 \leq n-1 \leq a \leq f(x) \leq b$ for every $x \in V(G)$.

If $f(V(G))$ is even, $\delta(G) \geq b+n-1$ and $\alpha(G) \leq \frac{4 a \cdot(\delta-b-n+1)}{(n-1)(b+1)^{2}}$, then $G$ has an $f$ factor.

Note that Cai conjectured that that the condition on the stability $\alpha(G) \leq$ $\frac{4 a .(\delta-b)}{(b+1)^{2}}$ is sufficient in connected graphs. We have the following counterexample.

Suppose $b$ is an odd integer and $a$ an integer strictly less than $b$.
Let $G_{0}$ be a connected graph of minimum degree $\delta$ at least $(b+1)^{3}+b$. Let $p=\frac{4 a \cdot(\delta-b)}{(b+1)^{2}}$. In the graph $G_{0}$, we suppose there exists $S$ be a cutset
of $k<b$ vertices, such that $G(S)$ is complete and $C_{1}, \ldots, C_{p}$, the connected components of $G-S$, form a family of complete subgraphs of order $\delta+1$, mutually independent. Furthermore $G\left(S \cup C_{1}\right)$ is complete, and, for each $i \geq 2$, exactly one edge joins $S$ to $C_{i}$. So $\alpha\left(G_{0}\right)=p=\frac{4 a \cdot(\delta-b)}{(b+1)^{2}}$.

Let us consider the application $f$ on $X$ such that $f(x)=a$ if $x \in S$, and, $f(x)=b$ otherwise.

If a $f$ factor exists we should have $\alpha=c(G-S) \leq a . k$, so $c(G-S)$ should be at most $a b$. This is not satified as $\alpha=\frac{4 a \cdot(\delta-b)}{(b+1)^{2}}>4 a(b+1)$.

One can see the surveys [8] or [5] for other results.

## 3 Main Results

We have established a new sufficient condition for a graph to have an $f$ factor; this condition involves the stability number, the minimum degree of the graph.

Theorem 5 Let $b \geq 2$ be an integer and let $G=(X, E)$ be a connected graph, of minimum degree $\delta$ at least $b$. Let $f$ be a non-negative integer valued function on $X$, such that for each $x \in X, a \leq f(x) \leq b$ and $f(X)$ is even. If $\alpha(G) \leq \frac{4 a .(\delta-b)}{(b+1)^{2}}$, and the odd-toughness of $G$ is at least $1 / a$, then
$G$ contains an $f$-factor.

Furthermore, we get this corollary.
Corollary 1 Let $b \geq 2$ be an integer and let $G=(X, E)$ be a graph, of minimum degree $\delta$ at least $b$ and connectivity $\kappa$. Let $f$ be a non-negative integer valued function on $X$, such that for each $x \in X, a \leq f(x) \leq b$ and $f(x)$ is even. If $\alpha(G) \leq \frac{4 a \cdot(\delta-b)}{(b+1)^{2}}$, then
$G$ contains an $f$-factor.

Corollary 2 Let $b \geq 2$ be an integer and let $G=(X, E)$ be a graph, of minimum degree $\delta$ at least $b$ and connectivity $\kappa$. Let $f$ be a non-negative integer valued function on $X$, such that for each $x \in X, a \leq f(x) \leq b$ and $f(X)$ is even. If $\alpha(G) \leq \min \left(\frac{4 a .(\delta-b)}{(b+1)^{2}}, a \kappa\right)$, then
$G$ contains an $f$-factor •

The condition $\alpha(G)<\frac{4 a \cdot(\delta-b)}{(b+1)^{2}}+1$ is necessary if $b>2 a$. Let $\alpha>\delta>b>r$ be four integers. Let us consider a graph $G_{1}$ composed by the join of a complete graph $A=K_{\delta-r+1}$ and $B$, the disjoint union of $\alpha$ complete graphs of order $r$. Let $f$ be a function such that
$f(x)=a$ if $x \in X(A), f(x)=b$ if $x \in X(B)$. If an $f$ factor exists we get

$$
\alpha(G) \leq \frac{a .(\delta-r+1)}{r .(b+1-r)}
$$

For $b$ odd and $r=(b+1) / 2$, we get $\alpha(G)<\frac{4 a \cdot(\delta-b)}{(b+1)^{2}}+\frac{2 a}{b+1}$.

## 4 Proof of Theorem 4

We set first some usefull lemmas.
Lemma $6 \delta(S, T)$ is even.
Proof Let $\mathcal{I}_{1}$ (respectively $\mathcal{I}_{2}$ ) be the set of even (resp. odd) components of $G-(S \cup T)$. By definition,

$$
\begin{gather*}
f\left(\mathcal{I}_{1}\right) \equiv e\left(\mathcal{I}_{1}, T\right),  \tag{1}\\
f\left(\mathcal{I}_{2}\right) \equiv h(S, T)+e\left(\mathcal{I}_{2},, T\right) \tag{2}
\end{gather*}
$$

so, by (1) and (2),
$f(X)=f(S)+f(T)+f\left(\mathcal{I}_{1}+f\left(\mathcal{I}_{2}\right) \equiv f(S)-f(T)+e(G-(S \cup T), T)+h(S, T)\right.$.
As $f(X)$ is even, the conclusion follows.

Lemma $7 T$ is non-emptyset.
Proof
If $T=\emptyset$ and $S=\emptyset$, then $\delta(S, T)=-h=0$ as $G$ is connected and $f(X)$ is even. If $T=\emptyset$ and $S$ is not empty, then $h(S, T)$ is the number of components of $G-S$ such that $f(C)$ is odd.

Either $S$ is not a cutset, then $h(S, T) \leq 1 \leq a|S|$; or $S$ is a cutset, as $G$ is $1 / a$-tough, $h \leq a|S|$.

As $a|S| \leq f(S)$, then $\delta(S, T)=f(S)-h(S, T) \geq f(S)-a|S| \geq 0$.

Proposition 1 If $\alpha(G) \leq \frac{4 a \cdot(\delta-b)}{(b+1)^{2}}$, then

$$
|S|>\delta-b
$$

Proof
The proof is by contradiction. As $\delta(S, T)<0$ and $a \leq f(x) \leq b$ for each $x$, then

$$
(\delta-|S|)|T|+a|S|-b|T|-h<0,
$$

so

$$
(\delta-|S|-b)|T|<h-a|S|
$$

If $|S|=\delta-b$, we get $|S|<\frac{h}{a}<\frac{\alpha(G)}{a}<\frac{4(\delta-b)}{9}$. This a contradiction.
Now we assume $|S|<\delta-b$, and we get

$$
|T|<\frac{h-a|S|}{(\delta-|S|-b)}
$$

If $h<a|S|$, then $|T|=0$. As $h<\alpha$, then

$$
|T|<\frac{4 a}{(b+1)^{2}} \cdot \frac{\left((\delta-|b|)-(b+1)^{2}|S|\right)}{(\delta-|b|-|S|)}
$$

We get

$$
\begin{gathered}
|T|<\frac{4 a}{(b+1)^{2}} \cdot\left(1-\frac{\left((b+1)^{2} / 4-1\right) \cdot|S|}{(\delta-|b|-|S|)}\right) \\
T<\frac{4 a}{(b+1)^{2}} .
\end{gathered}
$$

As $b \geq a, T<\frac{4 a}{(b+1)^{2}} \leq 1$, so $|T|=0$. This is in contradiction with Lemma 5.

## End of the proof of the theorem

Let $h_{2}$ be the number of components of $G-(S \cup T)$ not adjacent to $T$. As $\delta(S, T)<0$, we have

$$
\begin{equation*}
2\left|E_{T}\right|+|T|+a|S|-(b+1)|T|-h_{2} \leq 0, \tag{1}
\end{equation*}
$$

As $\alpha_{T}$ the stability number of $T$ is at least $\frac{|T|^{2}}{2\left|E_{T}\right|+|T|}$, and $\alpha_{T} \leq \alpha(G)-h_{2}$, we get, using (1),

$$
\alpha(G)-h_{2} \geq \frac{|T|^{2}}{(b+1)|T|-a|S|+h_{2}}
$$

Let us set $|T|=r .|S|$. Then

$$
\begin{aligned}
\alpha(G)-h_{2} & \geq \frac{r^{2} \cdot|S|^{2}}{(b+1) r|S|-a|S|+h_{2}} \\
\alpha(G)-h_{2} & \geq \frac{r^{2} \cdot|S|}{(b+1) r-a+h_{2} /|S|}
\end{aligned}
$$

The minimum of the bound as a function of $r$ is for $r=\frac{2\left(a-h_{2} /|S|\right)}{b+1}$. It follows that

$$
\alpha(G)-h_{2} \geq \frac{4 a \cdot|S|}{(b+1)^{2}}-\frac{4 h_{2}}{(b+1)^{2}}
$$

As by hypothesis $\alpha(G) \leq \frac{4 a(\delta-b)}{(b+1)^{2}}$, and $|S| \geq(\delta-b)$, we get

$$
h_{2} \leq \frac{4 h_{2}}{(b+1)^{2}} .
$$

So $h_{2}=0$. As $\delta \geq b$, then $\delta(S, T) \geq 0$. This is a contradiction with the definition of the cuple $S, T$.

This ends the proof of the theorem $4 \bullet$

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