

Subset-Sum Representations of Domination Polynomials

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Abstract

The domination polynomial $D(G, x)$ is the ordinary generating function for the dominating sets of an undirected graph $G = (V, E)$ with respect to their cardinality. We consider in this paper representations of $D(G, x)$ as a sum over subsets of the edge and vertex set of G . One of our main results is a representation of $D(G, x)$ as a sum ranging over spanning bipartite subgraphs of G .

Let $d(G)$ be the number of dominating sets of G . We call a graph G *conformal* if all of its components are of even order. Let $\text{Con}(G)$ be the set of all vertex-induced conformal subgraphs of G and let $k(G)$ be the number of components of G . We show that

$$d(G) = \sum_{H \in \text{Con}(G)} 2^{k(H)}.$$

1 Introduction

Let $G = (V, E)$ be an undirected graph. All graphs considered in this paper are assumed to be finite and simple. The *closed neighborhood* $N_G[v]$ of a vertex $v \in V$ is the set consisting of v and all its neighbor vertices in G . For any subset $W \subseteq V$, we denote by $N_G[W]$ the *closed neighborhood* of W in G , that is

$$N_G[W] = \bigcup_{v \in W} N_G[v].$$

If the graph is clear from the context, then we write $N[v]$ and $N[W]$ instead of $N_G[W]$ and $N_G[v]$, respectively. A *dominating set* of G is a vertex subset $W \subseteq V$ such that $N[W] = V$. Let $W \subseteq V$ be a given vertex subset of the graph $G = (V, E)$. We denote by $\partial(W)$ the set of all edges of G that have exactly one of their end vertices in W , that is

$$\partial(W) = \{\{u, v\} \in E \mid u \in W, v \in V \setminus W\}.$$

The edges of $\partial(W)$ link vertices of W with vertices of $V \setminus W$. Whether a given set W is a dominating set of G depends neither on edges lying completely inside W nor on edges that have no end vertex in W , which gives the following statement.

Proposition 1 *Let $G = (V, E)$ be a graph, $W \subseteq V$, and $F \subseteq E$. Then W is a dominating set of (V, F) if and only if W is dominating in $(V, F \cap \partial(W))$, i.e.*

$$N_{(V, F)}[W] = V \iff N_{(V, F \cap \partial(W))}[W] = V.$$

Definition 2 *Let $G = (V, E)$ be an undirected graph and $d_k(G)$ the number of dominating sets of cardinality k in G for $k = 0, \dots, n = |V|$. The domination polynomial of G is*

$$D(G, x) = \sum_{k=0}^n d_k(G) x^k.$$

We denote by $d(G)$ the number of dominating sets of G . Consequently, we find $d(G) = D(G, 1)$.

The domination polynomial of a graph has been introduced by Arocha and Llano in [5]. More recently it has been investigated with respect to special graphs, zeros, and applications in network reliability, see [1, 2, 3, 4, 7].

The domination polynomial can also be represented as a sum over vertex subsets of G ,

$$D(G, x) = \sum_{\substack{U \subseteq V \\ N[U] = V}} x^{|U|}.$$

The domination polynomial is multiplicative with respect to components, see [5]. Let G_1, \dots, G_k be the components of a given graph G , then

$$D(G, x) = \prod_{i=1}^k D(G_i, x). \quad (1)$$

2 Spanning Subgraphs

In this section, we provide a representation of the domination polynomial as a sum ranging over all bipartite spanning subgraphs of a graph.

2.1 Connected Bipartite Graphs

Alternating sums of domination polynomials of spanning subgraphs of a given graph yield a particularly simple result in case of connected bipartite graphs.

Lemma 3 Let $G = (V, E)$ be a connected bipartite graph with bipartition $V = Y \cup Z$, $Y \neq \emptyset$, $Z \neq \emptyset$. Then

$$\sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) = (-1)^{|Y|} x^{|Z|} + (-1)^{|Z|} x^{|Y|}.$$

Proof. Let W be a dominating set of G ; then we can distinguish three cases, namely

- (a) $W \cap Y \neq \emptyset$ and $W \cap Z \neq \emptyset$,
- (b) $W = Y$,
- (c) $W = Z$.

We decompose the sum according to the above given cases:

$$\begin{aligned} \sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) &= \sum_{F \subseteq E} \sum_{\substack{W \subseteq V \\ N_{(V, F)}[W] = V}} (-1)^{|F|} x^{|W|} \\ &= \sum_{W \subseteq V} x^{|W|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[W] = V}} (-1)^{|F|} \\ &= \sum_{\substack{W \subseteq V \\ W \cap Y \neq \emptyset \\ W \cap Z \neq \emptyset}} x^{|W|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[W] = V}} (-1)^{|F|} & \text{(a)} \\ &+ x^{|Y|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[Y] = V}} (-1)^{|F|} & \text{(b)} \\ &+ x^{|Z|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[Z] = V}} (-1)^{|F|}. & \text{(c)} \end{aligned}$$

We show that the Sum (a) vanishes. According to Proposition 1, a set W is dominating in (V, F) if and only if W is a dominating set of $(V, F \cap \partial(W))$. The evaluation of the Sum (a) yields

$$\begin{aligned} \sum_{\substack{W \subseteq V \\ W \cap Y \neq \emptyset \\ W \cap Z \neq \emptyset}} x^{|W|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[W] = V}} (-1)^{|F|} &= \sum_{\substack{W \subseteq V \\ W \cap Y \neq \emptyset \\ W \cap Z \neq \emptyset}} x^{|W|} \sum_{\substack{F_1 \subseteq E \cap \partial(W) \\ F_2 \subseteq E \setminus \partial(W) \\ N_{(V, F_1)}[W] = V}} (-1)^{|F_1 \cup F_2|} \\ &= \sum_{\substack{W \subseteq V \\ W \cap Y \neq \emptyset \\ W \cap Z \neq \emptyset}} x^{|W|} \sum_{\substack{F_1 \subseteq E \cap \partial(W) \\ N_{(V, F_1)}[W] = V}} (-1)^{|F_1|} \sum_{F_2 \subseteq E \setminus \partial(W)} (-1)^{|F_2|}. \end{aligned}$$

Now assume that $E \setminus \partial(W) = \emptyset$. Let $y \in Y \cap W$ and $z \in Z \cap W$. Then there does not exist a path between y and z in G . This contradicts the assumed connectedness of G ; hence $E \setminus \partial(W) \neq \emptyset$, which gives

$$\sum_{F_2 \subseteq E \setminus \partial(W)} (-1)^{|F_2|} = (1-1)^{|E \setminus \partial(W)|} = 0.$$

Now we turn to the Sum (b),

$$\sum_{\substack{F \subseteq E \\ N_{(V,F)}[Y]=V}} (-1)^{|F|}.$$

An edge subset $F \subseteq E$ satisfies the property “ Y is dominating in (V, F) ” if and only if F contains at least one edge from each vertex of Z . We denote the vertices of Z by v_1, \dots, v_k . For each i , $i = 1, \dots, k$, let E_i be the set of edges of G that are incident to v_i . We define

$$\mathcal{F} = \{A \subseteq E \mid \forall i = 1, \dots, k : |E_i \cap A| \geq 1\}.$$

Now the Sum (b) can be expressed as follows,

$$\begin{aligned} \sum_{\substack{F \subseteq E \\ N_{(V,F)}[Y]=V}} (-1)^{|F|} &= \sum_{F \in \mathcal{F}} (-1)^{|F|} \\ &= \sum_{\substack{F_1 \cup F_2 \cup \dots \cup F_k \in \mathcal{F} \\ \forall i=1, \dots, k: F_i \subseteq E_i}} (-1)^{|F_1 \cup F_2 \cup \dots \cup F_k|} \\ &= \sum_{\forall i=1, \dots, k: \emptyset \neq F_i \subseteq E_i} (-1)^{|F_1| + |F_2| + \dots + |F_k|} \\ &= \sum_{\substack{F_1 \subseteq E_1 \\ F_1 \neq \emptyset}} (-1)^{|F_1|} \sum_{\substack{F_2 \subseteq E_2 \\ F_2 \neq \emptyset}} (-1)^{|F_2|} \dots \sum_{\substack{F_k \subseteq E_k \\ F_k \neq \emptyset}} (-1)^{|F_k|} \\ &= (-1)^k = (-1)^{|Z|}, \end{aligned}$$

which yields

$$x^{|Y|} \sum_{\substack{F \subseteq E \\ N_{(V,F)}[Y]=V}} (-1)^{|F|} = (-1)^{|Z|} x^{|Y|}.$$

In the same vein, we can prove that the sum (c) satisfies

$$x^{|Z|} \sum_{\substack{F \subseteq E \\ N_{(V,F)}[Z]=V}} (-1)^{|F|} = (-1)^{|Y|} x^{|Z|}$$

and the statement follows. ■

2.2 General Bipartite Graphs

Lemma 4 *Let $G = (V, E)$ be a bipartite graph with bipartition $V = Y \cup Z$. Assume that G consists of $k + l$ components such that the k components $G_1 = (V_1, E_1), \dots, G_k = (V_k, E_k)$ have nonempty edge sets and the remaining l components are isomorphic to K_1 . Then*

$$\sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) = x^l \prod_{i=1}^k \left[(-1)^{|Y \cap V_i|} x^{|Z \cap V_i|} + (-1)^{|Z \cap V_i|} x^{|Y \cap V_i|} \right].$$

Proof. For the one-vertex graph $K_1 = (\{v\}, \emptyset)$, we obtain

$$\sum_{F \subseteq \emptyset} (-1)^{|F|} D((\{v\}, F), x) = x.$$

By Equation (1), we obtain

$$\begin{aligned} \sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) &= \sum_{F \subseteq E} (-1)^{|F|} x^l \prod_{i=1}^k D((V_i, F \cap E_i), x) \\ &= x^l \sum_{F \subseteq E} \prod_{i=1}^k (-1)^{|F \cap E_i|} D((V_i, F \cap E_i), x) \\ &= x^l \prod_{i=1}^k \sum_{F \subseteq E_i} (-1)^{|F|} D((V_i, F), x) \\ &= x^l \prod_{i=1}^k \left[(-1)^{|Y \cap V_i|} x^{|Z \cap V_i|} + (-1)^{|Z \cap V_i|} x^{|Y \cap V_i|} \right], \end{aligned}$$

where the last equality is valid due to Lemma 3. ■

Observe that $(-1)^{|Y|} x^{|Z|} + (-1)^{|Z|} x^{|Y|} \neq 0$ for any bipartition $V = Y \cup Z$, which shows together with Lemma 4 that

$$\sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) \neq 0$$

for any bipartite graph $G = (V, E)$. Moreover, we have the following statement.

Theorem 5 *Let $G = (V, E)$ be a graph. Then*

$$\sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) \neq 0$$

if and only if G is bipartite.

Proof. It remains to show that the sum vanishes for non-bipartite graphs. Using Proposition 1, we obtain

$$\begin{aligned}
\sum_{F \subseteq E} (-1)^{|F|} D((V, F), x) &= \sum_{F \subseteq E} \sum_{\substack{W \subseteq V \\ N_{(V, F)}[W] = V}} (-1)^{|F|} x^{|W|} \\
&= \sum_{W \subseteq V} x^{|W|} \sum_{\substack{F \subseteq E \\ N_{(V, F)}[W] = V}} (-1)^{|F|} \\
&= \sum_{W \subseteq V} x^{|W|} \sum_{\substack{F_1 \subseteq \partial(W) \\ N_{(V, F_1)}[W] = V}} (-1)^{|F_1|} \sum_{F_2 \subseteq E \setminus \partial(W)} (-1)^{|F_2|}.
\end{aligned}$$

Since G is not a bipartite graph, the set F_2 is nonempty, which yields

$$\sum_{F_2 \subseteq E \setminus \partial(W)} (-1)^{|F_2|} = 0$$

and hence the statement of the theorem. ■

There is also a “local version” for one direction of Theorem 5, which can be proved by the same method.

Theorem 6 *Let $G = (V, E)$ be a graph and $A \subseteq E$ an edge subset such that (V, A) contains an odd cycle. Then*

$$\sum_{F \subseteq A} (-1)^{|F|} D(G - F, x) = 0.$$

2.3 Applications of Spanning Subgraph Expansions

Let $G = (V, E)$ be a given graph. We define for any edge subset F of G ,

$$h(F) = \sum_{A \subseteq F} (-1)^{|A|} D((V, A), x).$$

Möbius inversion yields

$$D((V, F), x) = \sum_{A \subseteq F} (-1)^{|A|} h(A).$$

According to Lemma 3, Lemma 4, and Theorem 5, we define

$$h(F) = \begin{cases} x^l \prod_{i=1}^k (-1)^{|E_i|} [(-1)^{|Y \cap V_i|} x^{|Z \cap V_i|} + (-1)^{|Z \cap V_i|} x^{|Y \cap V_i|}], & \text{if } (V, F) \text{ is bipartite,} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Here the notations are as in Lemma 4. We can now conclude that the domination polynomial of a graph $G = (V, E)$ is a sum of h -function values of spanning bipartite subgraphs, i.e.

$$D(G, x) = \sum_{\substack{B \subseteq E \\ (V, B) \text{ is bipartite}}} h(B). \quad (3)$$

The number of dominating sets of $G = (V, E)$ is $D(G, 1)$. In order to derive this number from Equation (3), we define h_1 by substituting $x = 1$ in h , that is

$$h_1(F) = \prod_{i=1}^k (-1)^{|E_i|} \left[(-1)^{|Y \cap V_i|} + (-1)^{|Z \cap V_i|} \right].$$

Observe that $h_1(\emptyset) = 1$ and $h_1(F) \equiv 0 \pmod{2}$ for $F \neq \emptyset$, which gives the following statement.

Corollary 7 *For any graph G , the number of dominating sets of G is odd.*

For alternative proofs of this corollary, see [6].

Remark 8 *In almost the same manner, by substituting $x = -1$ in h , we can prove that $D(G, -1)$ is odd. Moreover, from the Equations (2) and (3), we obtain*

$$D(G, -1) = (-1)^{|V|} \sum_{\substack{F \subseteq E \\ (V, F) \text{ is bipartite}}} (-1)^{|F|} 2^{c(F)},$$

where $c(F)$ denotes here the number of components of (V, F) that have at least one edge.

3 Vertex Induced Subgraphs

Let $G = (V, E)$ be a graph and $W \subseteq V$. We denote by $G[W]$ the *vertex induced subgraph* of G :

$$G[W] = (W, \{\{u, v\} \in E \mid u \in W \text{ and } v \in W\}).$$

Theorem 9 *Any connected graph $G = (V, E)$ satisfies*

$$\sum_{W \subseteq V} (-1)^{|W|} D(G[W], x) = 1 + (-x)^{|V|}.$$

Proof. By switching the order of summation, we have

$$\begin{aligned}
\sum_{W \subseteq V} (-1)^{|W|} D(G[W], x) &= \sum_{W \subseteq V} (-1)^{|W|} \sum_{\substack{T \subseteq W \\ N_{G[W]}[T]=W}} x^{|T|} \\
&= \sum_{T \subseteq V} x^{|T|} \sum_{\substack{W \supseteq T \\ N_{G[W]}[T]=W}} (-1)^{|W|} \\
&= \sum_{T \subseteq V} x^{|T|} \sum_{W: T \subseteq W \subseteq N_{G[W]}[T]} (-1)^{|W|} \\
&= \sum_{T \subseteq V} x^{|T|} \sum_{W: T \subseteq W \subseteq N_G[T]} (-1)^{|W|} \\
&= \sum_{T \subseteq V} (-x)^{|T|} \sum_{Y \subseteq N_G[T] \setminus T} (-1)^{|Y|}.
\end{aligned}$$

Since G is connected, the set $N_G[T] \setminus T$ is empty if and only if $T = \emptyset$ or $T = V$. Hence we obtain

$$\sum_{T \subseteq V} (-x)^{|T|} \sum_{Y \subseteq N_G[T] \setminus T} (-1)^{|Y|} = 1 + (-x)^{|V|}.$$

■

Definition 10 Let $G = (V, E)$ be a graph with n vertices. The type of G is an integer partition $\lambda_G = (\lambda_1, \dots, \lambda_k) \vdash n$ that gives the sequence of orders of the components of G . We write $i \in \lambda_G$ in order to indicate that i is a part of λ_G . The number of parts of λ_G is denoted by $|\lambda_G|$.

Observe that for all $W \subseteq V$ the relation $|\lambda_{G[W]}| \leq \alpha(G)$ is satisfied, where $\alpha(G)$ denotes the independence number of G . Theorem 9 and Equation (1) immediately imply the following statement.

Corollary 11 For any graph $G = (V, E)$, we have

$$\sum_{W \subseteq V} (-1)^{|W|} D(G[W], x) = \prod_{i \in \lambda_G} (1 + (-x)^i). \quad (4)$$

The application of the Möbius inversion to Equation (4) yields

$$\begin{aligned}
D(G, x) &= \sum_{W \subseteq V} (-1)^{|W|} \prod_{i \in \lambda_{G[W]}} (1 + (-x)^i) \\
&= \sum_{W \subseteq V} \prod_{i \in \lambda_{G[W]}} (x^i + (-1)^i). \quad (5)
\end{aligned}$$

Remark 12 If we substitute $x = 1$ (or $x = -1$) in Equation (5) then all the products are equal to 0 (mod 2). There is only one exception, namely the empty product corresponding to $W = \emptyset$, which is 1. This gives an alternative proof of Corollary 7.

We call a graph G *conformal* if all of its components are of even order. Let $\text{Con}(G)$ be the set of all vertex-induced conformal subgraphs of G and let $k(G)$ be the number of components of G .

Theorem 13 *The number of dominating sets of a graph G satisfies*

$$d(G) = \sum_{H \in \text{Con}(G)} 2^{k(H)}.$$

Proof. The statement follows from Equation (5) by substituting $x = 1$. In this case any component of odd order leads to a zero product, such that only conformal vertex-induced subgraphs count. Observe that the null graph is conformal and has no components, which produces the only odd term of the sum, namely $2^0 = 1$. ■

Equation (5) offers a possibility to derive a decomposition for the domination polynomial.

Theorem 14 *Let $G = (V, E)$ be a graph and $v \in V$. Then*

$$D(G, x) = D(G - v, x) + \sum_{\substack{\{v\} \subseteq W \subseteq V \\ G[W] \text{ is connected}}} \left(x^{|W|} + (-1)^{|W|} \right) D(G - N[W], x).$$

Proof. We start from Equation (5):

$$\begin{aligned} D(G, x) &= \sum_{W \subseteq V} \prod_{i \in \lambda_{G[W]}} (x^i + (-1)^i) \\ &= \sum_{W \subseteq V \setminus \{v\}} \prod_{i \in \lambda_{G[W]}} (x^i + (-1)^i) + \sum_{\{v\} \subseteq W \subseteq V} \prod_{i \in \lambda_{G[W]}} (x^i + (-1)^i) \\ &= D(G - v, x) \\ &+ \sum_{\substack{\{v\} \subseteq W \subseteq V \\ G[W] \text{ is connected}}} \left(x^{|W|} + (-1)^{|W|} \right) \sum_{T \subseteq V \setminus N[W]} \prod_{i \in \lambda_{G[T]}} (x^i + (-1)^i) \\ &= D(G - v, x) + \sum_{\substack{\{v\} \subseteq W \subseteq V \\ G[W] \text{ is connected}}} \left(x^{|W|} + (-1)^{|W|} \right) D(G - N[W], x). \end{aligned}$$

■

The following statement for the number of dominating sets of G is an immediate consequence of Theorem 14.

Corollary 15 *Let $G = (V, E)$ be a graph and $v \in V$. Then the difference $d(G) - d(G - v)$ is even.*

Proof. When we substitute $x = 1$ in Theorem 14, then we obtain

$$d(G) = d(G - v) + 2 \sum_{\substack{\{v\} \subseteq W \subseteq V \\ G[W] \text{ is connected} \\ |W| \text{ is even}}} d(G - N[W]),$$

which gives the desired result. ■

4 Inclusion–Exclusion

We obtain a different representation of the domination polynomial as a sum ranging over vertex subsets by counting all vertex subsets of $G = (V, E)$ that do not dominate the whole vertex set V and applying inclusion-exclusion.

Theorem 16 ([7]) *Let $G = (V, E)$ be a graph. Then*

$$D(G, x) = \sum_{W \subseteq V} (-1)^{|W|} (1+x)^{|V \setminus N[W]|}. \quad (6)$$

Corollary 17 *The domination polynomial of a graph $G = (V, E)$ with n vertices satisfies*

$$D(G, x) = \sum_{k=0}^n x^k \sum_{\substack{W \subseteq V \\ |N[W]| \leq n-k}} (-1)^{|W|} \binom{n - |N[W]|}{k}.$$

Proof. Using Equation (6), we obtain

$$\begin{aligned} D(G, x) &= \sum_{W \subseteq V} (-1)^{|W|} (1+x)^{|V \setminus N[W]|} \\ &= \sum_{W \subseteq V} (-1)^{|W|} \sum_{k=0}^{|V \setminus N[W]|} \binom{|V \setminus N[W]|}{k} x^k \\ &= \sum_{k=0}^n x^k \sum_{W \subseteq V} (-1)^{|W|} \binom{n - |N[W]|}{k} \\ &= \sum_{k=0}^n x^k \sum_{\substack{W \subseteq V \\ |N[W]| \leq n-k}} (-1)^{|W|} \binom{n - |N[W]|}{k}. \end{aligned}$$

■

Remark 18 *An interesting consequence of Corollary 17 is the characterization of the domination number $\gamma(G)$ of a graph $G = (V, E)$ as the smallest nonnegative integer k such that the sum*

$$\sum_{\substack{W \subseteq V \\ |N[W]| \leq n-k}} (-1)^{|W|} \binom{n - |N[W]|}{k}$$

does not vanish.

We call a vertex subset $W \subseteq V$ of a graph $G = (V, E)$ *essential* in G if W contains the closed neighborhood $N[v]$ of at least one vertex $v \in V$. We denote by $\text{Ess}(G)$ the family of all essential sets of G .

Theorem 19 *Let $G = (V, E)$ be a graph with nonempty vertex set. Then the domination polynomial of G satisfies*

$$D(G, x) = (-1)^{|V|} \sum_{U \in \text{Ess}(G)} (-1)^{|U|} \left[(1+x)^{|\{u \in U \mid N[u] \subseteq U\}|} - 1 \right].$$

Proof. According to Equation (6), we have

$$\begin{aligned} D(G, x) &= \sum_{W \subseteq V} (-1)^{|W|} (1+x)^{|V \setminus N[W]|} \\ &= \sum_{U \subseteq V} (-1)^{|V| - |U|} (1+x)^{|V \setminus N[V \setminus U]|} \\ &= \sum_{U \subseteq V} (-1)^{|V| - |U|} (1+x)^{|\{u \in U \mid N[u] \subseteq U\}|}. \end{aligned}$$

In order to see the last equality, we verify

$$\begin{aligned} N[V \setminus U] &= \bigcup_{v \in V \setminus U} N[v] \\ &= (V \setminus U) \cup \{u \in U \mid N[u] \cap (V \setminus U) \neq \emptyset\} \\ &= (V \setminus U) \cup \{u \in U \mid N[u] \not\subseteq U\} \end{aligned}$$

and consequently,

$$\begin{aligned} V \setminus N[V \setminus U] &= V \setminus [(V \setminus U) \cup \{u \in U \mid N[u] \not\subseteq U\}] \\ &= U \setminus \{u \in U \mid N[u] \not\subseteq U\} \\ &= \{u \in U \mid N[u] \subseteq U\}. \end{aligned}$$

All polynomials of the form $(1+x)^{|\{u \in U \mid N[u] \subseteq U\}|}$ have the constant term 1. As $V \neq \emptyset$, the constant term in

$$\sum_{U \subseteq V} (-1)^{|V| - |U|} (1+x)^{|\{u \in U \mid N[u] \subseteq U\}|}$$

vanishes, which gives

$$D(G, x) = \sum_{U \subseteq V} (-1)^{|V| - |U|} \left[(1+x)^{|\{u \in U \mid N[u] \subseteq U\}|} - 1 \right].$$

If U is a non-essential set of G then we have $\{u \in U \mid N[u] \subseteq U\} = \emptyset$ and hence $(1+x)^{|\{u \in U \mid N[u] \subseteq U\}|} = 1$. Consequently, all non-vanishing terms correspond to essential sets, yielding the statement of the theorem. ■

Another interesting consequence of Theorem 16 is the following relation between $D(G, x)$ and $D(G, \frac{1}{x})$.

Theorem 20 *Let $G = (V, E)$ be a graph. Then*

$$D(G, x) = (1+x)^{|V|} \sum_{W \subseteq V} \left(\frac{-x}{1+x} \right)^{|W|} D\left(G[W], \frac{1}{x}\right).$$

Proof. We consider the right-hand side of the equation from the theorem. Substituting $D\left(G[W], \frac{1}{x}\right)$ according to the definition of the domination polynomial yields

$$\begin{aligned} & (1+x)^{|V|} \sum_{W \subseteq V} \left(\frac{-x}{1+x} \right)^{|W|} \sum_{\substack{T \subseteq W \\ N_{G[W]}[T]=W}} x^{-|T|} \\ &= (1+x)^{|V|} \sum_{W \subseteq V} \left(\frac{-x}{1+x} \right)^{|W|} \sum_{T: T \subseteq W \subseteq N_G[T]} x^{-|T|}. \end{aligned}$$

Switching the order of summation gives

$$(1+x)^{|V|} \sum_{T \subseteq V} x^{-|T|} \sum_{W: T \subseteq W \subseteq N_G[T]} \left(\frac{-x}{1+x} \right)^{|W|}.$$

Now we define $U = W \setminus T$ and substitute $W = U \cup T$, yielding

$$\begin{aligned} & (1+x)^{|V|} \sum_{T \subseteq V} x^{-|T|} \sum_{U \subseteq N_G[T] \setminus T} \left(\frac{-x}{1+x} \right)^{|U|+|T|} \\ &= (1+x)^{|V|} \sum_{T \subseteq V} \left(\frac{-1}{1+x} \right)^{|T|} \sum_{U \subseteq N_G[T] \setminus T} \left(\frac{-x}{1+x} \right)^{|U|}, \end{aligned}$$

which simplifies via the binomial theorem to

$$\begin{aligned} & (1+x)^{|V|} \sum_{T \subseteq V} \left(\frac{-1}{1+x} \right)^{|T|} \left(1 - \frac{x}{1+x} \right)^{|N_G[T]|-|T|} \\ &= (1+x)^{|V|} \sum_{T \subseteq V} (-1)^{|T|} (1+x)^{-|N_G[T]|}. \end{aligned}$$

The statement follows now by Theorem 16. \blacksquare

The following statement can be shown by substituting $x = 1$ in Theorem 20.

Corollary 21 *Let $G = (V, E)$ be a graph. The numbers of dominating sets of vertex-induced proper subgraphs of G satisfy*

$$\sum_{W \subseteq V} (-1)^{|W|} \frac{d(G[W])}{2^{|W|}} = 0.$$

5 Conclusions and Open Problems

The domination polynomial of a graph can be expressed as a sum of quite simple polynomials of vertex-induced or spanning subgraphs. In case of spanning subgraphs, we can show that the domination polynomial depends only on bipartite spanning subgraphs.

There remain interesting open problems for further research in this field. The first one concerns the number of dominating sets of a graph given by Theorem 13.

Problem 22 *The simple formula*

$$d(G) = \sum_{H \in \text{Con}(G)} 2^{k(H)}$$

suggests that there is a bijection between subsets of components of conformal graphs and dominating sets of G . Is there a bijective proof for Theorem 13? What is the best way to enumerate the set $\text{Con}(G)$?

In Corollary 11, we showed that the type of a subgraph yields the essential information for a representation of $D(G, x)$ as a sum over vertex-induced subgraphs. Here it seems interesting to investigate whether we need all vertex-induced subgraphs in order to derive the domination polynomial.

Problem 23 *Components of $G[W]$ that have odd order lead to cancellation of terms of the sum in Equation (5),*

$$D(G, x) = \sum_{W \subseteq V} \prod_{i \in \lambda_{G[W]}} (x^i + (-1)^i).$$

Is there a way to identify those cancelling terms?

Problem 24 *In Theorem 19, we showed that the restriction to essential sets is sufficient in order to compute the domination polynomial of a graph. Can we reduce the number of terms needed to derive $D(G, x)$ further?*

Further topics of interest for future research include the investigation of special graph classes with respect to the given representations of the domination polynomial and the application of these representations to special graph classes. Since bipartite graphs play an important role for the representation of the domination polynomial, we conjecture that also matchings have a close relation to dominating sets. However, until now all attempts to find a sum representation of $D(G, x)$ based on matchings of G failed.

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