Hamiltonicity of 3-arc graphs

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Abstract

An arc of a graph is an oriented edge and a 3-arc is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two. The 3-arc graph of a graph G is defined to have vertices the arcs of G such that two arcs uv, xy are adjacent if and only if (v, u, x, y) is a 3-arc of G. We prove that any connected 3-arc graph is hamiltonian, and all iterative 3-arc graphs of any connected graph of minimum degree at least three are hamiltonian. As a corollary we obtain that any vertex-transitive graph which is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three must be hamiltonian. This confirms the conjecture, for this family of vertex-transitive graphs, that all vertex-transitive graphs with finitely many exceptions are hamiltonian. We also prove that if a graph with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs.

Key words: 3-Arc graph, Hamilton cycle, Hamiltonian graph, Hamilton-connected graph, Vertex-transitive graph

1 Introduction

A path or cycle which contains every vertex of a graph is called a *Hamilton path* or *Hamilton cycle* of the graph. A graph is *hamiltonian* if it contains a Hamilton cycle, and is *Hamilton-connected* if any two vertices are connected by a Hamilton path. The hamiltonian problem, that of determining when a graph is hamiltonian, is a classical problem in graph theory with a long history. The reader is referred to [3], [4, Chapter 18], [8, Chapter 10] and [10] for results on Hamiltonicity of graphs.

In this paper we present a large family of hamiltonian graphs. Such graphs are defined by means of a graph operator, called the 3-arc graph construction, which bears some similarities with the line graph operator. This construction was first introduced in [17, 24] in studying a family of arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is *arc-transitive* if its automorphism group is transitive on the set of oriented edges.) It was used in classifying or characterizing certain families of arc-transitive graphs [9, 12, 17, 18, 23, 25].

All graphs in this paper are finite and undirected without loops. We use the term *multigraph* when parallel edges are allowed. An *arc* of a graph $G = (V(G), E(G))$ is an ordered pair of adjacent vertices, or equivalently an oriented edge. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, vu ($\neq uv$) the arc from v to u, and $\{u, v\}$ the edge between u and v. A *3-arc* of G is a 4-tuple of vertices (v, u, x, y), possibly with $v = y$, such that both (v, u, x) and (u, x, y) are paths of G.

Notation: We follow [4] for graph-theoretic terminology and notation. The degree of a vertex v in a graph G is denoted by $d(v)$, and the minimum degree of G is denoted by $\delta(G)$. The set of arcs of G with tail v is denoted by $A(v)$, and the set of arcs of G is denoted by $A(G)$.

The general 3-arc construction [17, 24] involves a self-paired subset of the set of 3-arcs of a graph. The following definition is obtained by choosing this subset to be the set of all 3-arcs of the graph.

Definition 1 Let G be a graph. The 3-arc graph of G, denoted by $X(G)$, is defined to have vertex set $A(G)$ such that two vertices corresponding to two arcs uv and xy are adjacent if and only if (v, u, x, y) is a 3-arc of G.

It is clear that $X(G)$ is an undirected graph with $2\,|E(G)|$ vertices and $\sum_{\{u,v\}\in E(G)}(d(u)-1)(d(v)-1)$ edges. We can obtain $X(G)$ from the line graph $L(G)$ of G by the following operations [14]: split each vertex $\{u, v\}$ of $L(G)$ into two vertices, namely uv and vu; for any two vertices $\{u, v\}, \{x, y\}$ of $L(G)$ that are distance two apart in $L(G)$, say, u and x are adjacent in G, join uv and xy by an edge. On the other hand, the quotient graph of $X(G)$ with respect to the partition $\mathcal{P} = \{\{uv, vu\} : \{u, v\} \in E(G)\}$ of $A(G)$ is isomorphic to the graph obtained from the square of $L(G)$ by deleting the edges of $L(G)$. The reader is referred to [14, 13, 2] respectively for results on the diameter and connectivity, the independence, domination and chromatic numbers, and the edge-connectivity and restricted edge-connectivity of 3-arc graphs.

The following is the first main result in this paper.

Theorem 1 *Let G be a graph without isolated vertices. The 3-arc graph of G is hamiltonian if and only if*

(a) $\delta(G) \geq 2$*;*

(b) *no two degree-two vertices of G are adjacent; and*

(c) *the subgraph obtained from G by deleting all degree-two vertices is connected.*

We remark that Theorem 1 can not be obtained from known results on the hamiltonicity of line graphs, though $X(G)$ and $L(G)$ are closely related as mentioned above. As a matter of fact, even if $L(G)$ is hamiltonian, $X(G)$ is not necessarily hamiltonian, as witnessed by stars $K_{1,t}$ with $t \geq 3$.

We define the *iterative 3-arc graphs* of G by

$$X^1(G) = X(G), \quad X^{i+1}(G) = X(X^i(G)), \quad i \geq 1.$$

Theorem 1 together with [14, Theorem 2] implies the following result.

Theorem 2 (a) *A 3-arc graph is hamiltonian if and only if it is connected.*

(b) *If G is a connected graph with $\delta(G) \geq 3$, then $X^i(G)$ is hamiltonian for every integer $i \geq 1$.*

We will prove Theorems 1 and 2 in Section 3. In Section 4 we will prove the following result.

Theorem 3 *Let G be a 2-edge connected graph with $\delta(G) \geq 3$. If G contains a path of odd length between any two distinct vertices, then its 3-arc graph is Hamilton-connected.*

A basic strategy in the proof of Theorems 1 and 3 is to find an Eulerian tour or an open Eulerian trail in a properly defined multigraph that produces the required Hamilton cycle or path. This is similar to the observation [5] that an Eulerian tour of a graph produces a Hamilton cycle of its line graph.

Theorem 3 implies the following result.

Theorem 4 *If a graph G with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs $X^i(G)$, $i \geq 1$.*

Given vertex-disjoint graphs G and H, the join $G \vee H$ of them is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. Theorem 3 implies the following result.

Corollary 5 *Let G and H be graphs such that* $\max\{\delta(G), \delta(H)\} \geq 2$. *Then $X(G \vee H)$ is Hamilton-connected.*

In the case when G has a large order but small maximum degree, $X(G)$ has a large order but relatively small maximum degree. In this case the Hamiltonicity of $X(G)$ may not be derived from known sufficient conditions for Hamilton cycles such as the degree conditions in the classical Dirac's or Ore's Theorem (see [3, 4, 8, 10]).

In spirit, Theorems 1 and 2 are parallel to the well-known conjecture of Thomassen [20] which asserts that every 4-connected line graph is hamiltonian. This conjecture is still open; see [6, 10, 11, 16, 22]. In contrast, Theorem 1 solves the hamiltonian problem for 3-arc graphs completely.

A well-known conjecture due to Lovász, formulated by Thomassen [21], asserts that all connected vertex-transitive graphs, with finitely many exceptions, are hamiltonian. Since the 3-arc graph of an arc-transitive graph is vertex-transitive, Theorem 2 implies the following result, which confirms this conjecture for a large family of vertex-transitive graphs. (The family of arc-transitive graphs is large from a group-theoretic point of view [19].)

Corollary 6 *If a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is hamiltonian.*

The Lovász conjecture has been confirmed for several families of vertex-transitive graphs [15], including connected vertex-transitive graphs of order kp, where $k \leq 4$, (except for the Petersen graph and the Coxeter graph) of order p^j, where $j \leq 4$, and of order $2p^2$, where p is prime, and some families of Cayley graphs. Tools from group theory were used in the proof of almost all these results. Corollary 6 has a different flavour and its proof does not rely on group theory.

There has also been considerable interest on Hamilton-connectedness of vertex-transitive graphs. Theorem 4 implies that if a vertex-transitive graph (with at least four vertices) is Hamilton-connected, then so are its iterative 3-arc graphs. For example, it is known that every connected non-bipartite Cayley graph of degree at least three on a finite abelian group [7] or a Hamiltonian group [1] is Hamilton-connected. (A finite non-abelian group in which every subgroup is normal is called a Hamiltonian group.) From this and Theorem 4 we know immediately that all iterative 3-arc graphs of such a Cayley graph are also Hamilton-connected.

2 Preliminaries

Let G^* be a multigraph. A *walk* in G^* of length l is a sequence $v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, whose terms are alternately vertices and edges of G^* (not necessarily distinct), such that v_{i-1} and v_i are the end-vertices of e_i, $1 \leq i \leq l$. A walk is *closed* if its initial and terminal vertices are identical, is a *trail* if all its edges are distinct, and is a *path* if all its vertices are distinct. Often we present a trail by listing its sequence of vertices only, with the understanding that the edges used are distinct. A trail that traverses every edge of G^* is called an *Eulerian trail* of G^*, and a closed Eulerian trail is called an *Eulerian tour.* A multigraph is *Eulerian* if it admits an Eulerian tour. It is well known that a multigraph is Eulerian if and only if all its vertices have even degrees.

A *2-trail* of G^* is a trail of length two (and so is a path or cycle of length two). We call a 2-trail (u, x, v) with mid-vertex x a *visit to* x (if $u = v$, then (u, x, u) is thought as entering and leaving x on parallel edges). When there is no need to make distinction between (u, x, v) and (v, x, u), or the orientation of the visit is unknown, we write $[u, x, v]$. Two visits (u, x, v) and (u', x, v') are called *twin visits* if $\{u, v\} = \{u', v'\}$ and the four edges involved are distinct. In particular, when $u = v$, two twin visits (u, x, u) and (u, x, u) use four parallel edges between u and x.

Denote by $E^*(x)$ the set of edges of G^* incident with $x \in V(G^*)$, and $d^*(x) = |E^*(x)|$ the degree of x in G^*. In the case when $d^*(x)$ is even, a decomposition of $E^*(x)$ into a set of visits to x is called a *visit-decomposition* of $E^*(x)$ (at x). In this definition the orientations of the visits in the decomposition are not important in our subsequent discussion. So we may view each visit (u, x, v) in such a visit-decomposition as a non-oriented path (if $u \neq v$) or cycle (if $u = v$) of length two. As an example, if $E^*(x) = \{\{x, y\}, \{x, y\}, \{x, z\}, \{x, z\}\}$, where $\{x, y\}$ and $\{x, y\}$ are viewed as distinct edges between x and y, then both $\{[y, x, y], [z, x, z]\}$ and $\{[y, x, z], [y, x, z]\}$ are visit-decompositions of $E^*(x)$.

Definition 2 *Given a visit-decomposition $J(x)$ of $E^*(x)$, define $H(x)$ to be the bipartite graph with vertex bipartition $\{J(x), A(x)\}$ such that $p \in J(x)$ and $xy \in A(x)$ are adjacent if and only if y is not in p, where $A(x)$ is the set of arcs of the underlying simple graph of G^* with tail x.*

We emphasize that $H(x)$ relies on $J(x)$. One can verify the following result by using Hall's marriage theorem.

Lemma 7 *Suppose x is a vertex of G^* such that $d^*(x) \geq 6$ is even and either x is joined to every neighbour of x by exactly two parallel edges, or x is joined to one of its neighbours by exactly three parallel edges, another neighbour by a single edge, and each of the remaining neighbours by exactly two parallel edges. Let $J(x)$ be a visit-decomposition of $E^*(x)$. Then the bipartite graph $H(x)$ with respect to $J(x)$ has no perfect matchings if and only if $d^*(x) = 6$ and $J(x)$ contains two twin visits.*

Proof We have $|J(x)| = |A(x)| = d^*(x)/2$ and $\delta(H(x)) \geq (d^*(x)/2) - 2 \geq 1$. One can show that, if $d^*(x) \geq 8$, then the neighbourhood $N_{H(x)}(S)$ in $H(x)$ of each $S \subseteq J(x)$ has size at least $|S|$. Thus, by Hall's marriage theorem, $H(x)$ has a perfect matching when $d^*(x) \geq 8$.

Suppose $H(x)$ has no perfect matchings, so that $d^*(x) = 6$ and $|J(x)| = |A(x)| = 3$. Then there exists $S \subseteq J(x)$ such that $|N_{H(x)}(S)| < |S|$. This implies $|S| = 2$ and so $|N_{H(x)}(S)| \leq 1$. Denote $S = \{(u, x, v), (y, x, z)\}$, where $u, v, y, z \in N(x)$ (the neighbourhood of x in G^*). Then $N_{H(x)}(S) = (A(x) - \{xu, xv\}) \cup (A(x) - \{xy, xz\}) = A(x) - (\{xu, xv\} \cap \{xy, xz\})$. Since $|N(x)| = 3$ and $|N_{H(x)}(S)| \leq 1$, it follows that $\{u, v\} = \{y, z\}$, and therefore (u, x, v) and (y, x, z) are twin visits.

Conversely, if $d^*(x) = 6$ and $J(x)$ contains twin visits, then $H(x)$ consists of two paths of length two and hence has no perfect matchings. $\square$

Definition 3 *Let $C : v_0, e_1, v_1, e_2, v_2, \ldots, v_{l-2}, e_{l-1}, v_{l-1}, e_l, v_l$ be an Eulerian trail of G^*, possibly with $v_l = v_0$. The visit (v_{i-1}, v_i, v_{i+1}) to v_i is said to be induced by C, $1 \leq i \leq l - 1$. In addition, if C is an Eulerian tour, then (v_{l-1}, v_0, v_1) is also a visit to v_0 induced by C.*

Denote by $C(x)$ the set of visits to $x \in V(G^*)$ induced by C.

Define $H_C(x)$ to be the bipartite graph at x as defined in Definition 2 with respect to the visit-decomposition $C(x)$ of $E^*(x)$. (We leave $H_C(v_0)$ and $H_C(v_l)$ undefined if C is an open Eulerian trail.)

Note that a vertex may be visited several times by C because the vertices on C may be repeated. Indeed, $C(x)$ is a visit-decomposition of $E^*(x)$ for all vertices x, except v_0 and v_l when $v_0 \neq v_l$.

Definition 4 *Let C be an Eulerian tour of G^* and $J(x)$ a visit-decomposition of $E^*(x)$. We say that C is compatible with $J(x)$, written $C(x) \prec J(x)$, if for every $(a, x, b) \in J(x)$, either $(a, x, b) \in C(x)$ or $(b, x, a) \in C(x)$.*

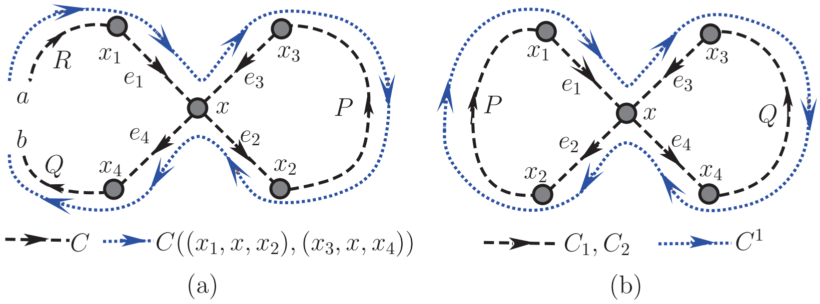

Figure 1: (a) Bow-tie operation; (b) Concatenation operation.

Definition 5 *Let C be a trail of G^* with length at least four. Let $(x_1, x, x_2), (x_3, x, x_4) \in C(x)$ be distinct visits, so that C can be expressed as*

$$C : \overbrace{a, \ldots, x_1}^{R}, e_1, x, e_2, \overbrace{x_2, \ldots, x_3}^{P}, e_3, x, e_4, \overbrace{x_4, \ldots, b}^{Q},$$

possibly with $a = b$.

Define

$$C((x_1, x, x_2), (x_3, x, x_4)) : \overbrace{a, \ldots, x_1}^{R}, e_1, x, e_3^{-1}, \overbrace{x_3, \ldots, x_2}^{P^-}, e_2^{-1}, x, e_4, \overbrace{x_4, \ldots, b}^{Q}$$

where P^- is the trail obtained from P by reversing its direction, and e_2^{-1} and e_3^{-1} are the same edges as e_2 and e_3 but with reversed orientations, respectively. (See Figure 1 (a).)

We call $C \to C((x_1, x, x_2), (x_3, x, x_4))$ the bow-tie operation on C with respect to (x_1, x, x_2) and (x_3, x, x_4).

Definition 6 *Let*

$$C_1 : x_1, e_1, x, e_2, \overbrace{x_2, \ldots, x_1}^{P}; \qquad C_2 : x_3, e_3, x, e_4, \overbrace{x_4, \ldots, x_3}^{Q}.$$

be edge-disjoint closed trails of G^* with x as a common vertex. Define

$$C^1 : x_1, e_1, x, e_3^{-1}, \overbrace{x_3, \ldots, x_4}^{Q^{-1}}, e_4^{-1}, x, e_2, \overbrace{x_2, \ldots, x_1}^{P}.$$

We call $(C_1, C_2) \to C^1$ the concatenation operation with respect to $(C_1, C_2, (x_1, x, x_2), (x_3, x, x_4))$. (See Figure 1 (b).)

Remark 1 *Some of x_1, x_2, x_3, x_4 or even all of them in Definitions 5 and 6 are allowed to be the same vertex. Each of P, Q (and R in Definition 5) may visit some of x, x_1, x_2, x_3, x_4 several times, and they may have common vertices.*

In each operation above, the visits (x_1, x, x_2), (x_3, x, x_4) are replaced by (x_1, x, x_3), (x_4, x, x_2), respectively. All other visits induced by C (in Definition 5) or $C_1 \cup C_2$ (in Definition 6) are retained or with orientation reversed.

In Definition 6, C^1 is a closed trail which covers every edge covered by C_1 and C_2. In particular, if C_1 and C_2 collectively cover all edges of G^*, then C^1 is an Eulerian tour of G^*.

3 Proof of Theorems 1 and 2

Proof of Theorem 1 Denote by S_i the set of vertices of G with degree i, for $i \geq 1$.

Suppose that G has no isolated vertices and $X(G)$ is hamiltonian. We show that (a), (b) and (c) hold. Note first that if G has a degree-one vertex, then the unique arc emanating from it gives rise to an isolated vertex of $X(G)$. Similarly, if $x, y \in S_2$ are adjacent, say, $N(x) = \{u, y\}, N(y) = \{x, v\}$, then the edge of $X(G)$ between xu and yv is an isolated edge no matter whether $u \neq v$ or not. Since $X(G)$ is assumed to be hamiltonian, it follows that G is connected with $\delta(G) \geq 2$ and S_2 is an independent set of G.

It remains to prove that $G - S_2$ is connected. Suppose otherwise. Then we can choose a minimal subset S of S_2 such that $G - S$ is disconnected. Note that $S \neq \emptyset$ as G is connected. Let H be a component of $G - S$. The minimality of S implies that each vertex of S has exactly one neighbour in $V(H)$, and each vertex of S_2 with both neighours in H (if such a vertex exists) is contained in $V(H)$. Denote by A_1 the set of arcs of G with tails in S and heads outside of $V(H)$. Denote by A_2 the set of arcs of G with tails in $V(H)$ (and heads in $V(H)$ or S). One can verify that the subgraph of $X(G)$ induced by $A_1 \cup A_2$ is a connected component of $X(G)$. Since there are arcs of G not in $A_1 \cup A_2$, it follows that $X(G)$ is disconnected, contradicting our assumption. Hence $G - S_2$ is connected.

Suppose that G satisfies (a), (b) and (c). We aim to prove that $X(G)$ is hamiltonian. Note that G is connected by (c). Let G^* be the multigraph obtained from G by doubling each edge. Then the degree $d^*(v) = 2d(v)$ of each $v \in V(G)$ in G^* is even. Hence G^* is Eulerian. We will prove the existence of an Eulerian tour of G^* such that the corresponding bipartite graph (see Definition 3) at each vertex has a perfect matching. We will then exploit such an Eulerian tour to construct a Hamilton cycle of $X(G)$.

We claim first that there exists an Eulerian tour C of G^* such that

$$\text{if } v \in S_2 \text{ with } N(v) = \{u, w\}, \text{ then } C(v) \prec \{(u, v, u), (w, v, w)\}. \tag{1}$$

To construct such an Eulerian tour, we can start from any vertex and travel as far as possible without repeating any edge such that, whenever the tour reaches a vertex of S_2, it returns to the previous vertex immediately. Since $G - S_2$ is connected, an Eulerian tour C of G^* satisfying (1) can be constructed this way. Note that $G^* - S_2$ is Eulerian because it is connected and all its vertices have even degrees.

For an Eulerian tour C of G^* satisfying (1), let $Z(C)$ denote the set of vertices x such that $H_C(x)$ has no perfect matchings. Since for every $x \in S_2$, $H_C(x) \cong 2K_2$ is a perfect matching, by Lemma 7 we have $Z(C) \subseteq S_3$.

Now we choose an Eulerian tour C of G^* satisfying (1) such that $|Z(C)|$ is minimum. We claim that $Z(C) = \emptyset$. Suppose otherwise. Then by Lemma 7, $C(x)$ contains twin visits for each $x \in Z(C)$. Denote $N(x) = \{x_1, x_2, x_3\}$ for a fixed $x \in Z(C)$, and assume without loss of generality that $C(x) = \{(x_1, x, x_2), (x_1, x, x_2), (x_3, x, x_3)\}$. Denote $C' = C((x_1, x, x_2), (x_3, x, x_3))$. Then C' is an Eulerian tour of G^* and $C'(x) = \{(x_1, x, x_2), (x_1, x, x_3), (x_2, x, x_3)\}$. One can see that $H_{C'}(x)$ is a perfect matching of three edges, and $H_{C'}(y)$ is isomorphic to $H_C(y)$ for each $y \neq x$. Thus $Z(C')$ is a proper subset of $Z(C)$, and moreover (1) is respected by C' at every $v \in S_2$. Since this contradicts the choice of C, we conclude that $Z(C) = \emptyset$; that is, $H_C(v)$ has a perfect matching for each $v \in V(G)$.

Let C be a fixed Eulerian tour of G^* satisfying (1) such that $Z(C) = \emptyset$. Let us fix a perfect matching of $H_C(v)$ for each $v \in V(G)$. Every traverse of C to v corresponds to a visit to v, say, (u, v, w), and in the chosen perfect matching of $H_C(v)$, (u, v, w) is matched to an arc of $A(v)$ other than vu and vw. Denote this arc by $\phi(u, v, w)$. Then for any two consecutive visits $(u, v, w), (v, w, x)$ induced by C (that is, (u, v, w, x) is a segment of C), $\phi(u, v, w)$ and $\phi(v, w, x)$ are adjacent in $X(G)$. Since C is an Eulerian tour of G^* and a perfect matching of each $H_C(v)$ is used, every arc of G is of the form $\phi(u, v, w)$ for some segment (u, v, w) of C. Therefore, if, say, $C = (u, v, w, x, y, \ldots, a, b, c, u)$, then the sequence

$$\phi(u, v, w), \phi(v, w, x), \phi(w, x, y), \ldots, \phi(a, b, c), \phi(b, c, u), \phi(c, u, v), \phi(u, v, w)$$

of arcs of G gives rise to a Hamilton cycle of $X(G)$. $\square$

We illustrate the proof above by the following example.

Example 1 *Since the Petersen graph PG (see Figure 2) satisfies the conditions in Theorem 1, its 3-arc graph $X(PG)$ is hamiltonian. Let*

$$C : a_1, a_2, a_3, a_4, a_5, a_1, b_1, b_4, b_2, b_5, b_3, b_1, a_1, a_2, b_2, \\ b_5, a_5, a_4, b_4, b_2, a_2, a_3, b_3, b_1, b_4, a_4, a_3, b_3, b_5, a_5, a_1.$$

Then C is an Eulerian tour of the multigraph PG^* obtained from PG by doubling each edge. One can verify that at each a_i or b_i, $H_C(a_i)$ or $H_C(b_i)$ has a perfect matching. In $H_C(a_2)$ the 'vertex' (a_1, a_2, a_3) is matched to the 'vertex' a_2b_2, and in $H_C(a_3)$, (a_2, a_3, a_4) is matched to a_3b_3, and so on. Continuing, one can verify that C gives rise to the following Hamilton cycle of $X(PG)$:

$$a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_1a_2, b_1b_3, b_4a_4, b_2a_2, b_5a_5, b_3a_3, b_1b_4, a_1a_5, a_2a_3, b_2b_4, b_5b_3, \\ a_5a_1, a_4a_3, b_4b_1, b_2b_5, a_2a_1, a_3a_4, b_3b_5, b_1a_1, b_4b_2, a_4a_5, a_3a_2, b_3b_1, b_5b_2, a_5a_4, a_1b_1, a_2b_2.$$

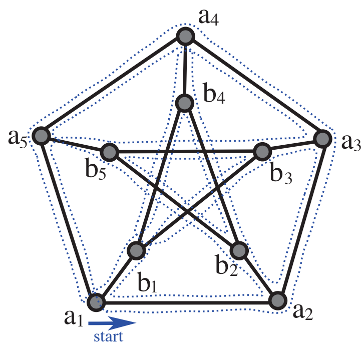

Figure 2: An Eulerian tour of PG^* which produces a Hamilton cycle of the 3-arc graph of the Petersen graph PG.

Proof of Theorem 2 (a) Let G be a graph. Define $\hat{G}$ to be the graph obtained from G by replacing each degree-two vertex v by a pair of nonadjacent vertices each joining to exactly one neighbour of v in G. In [14, Theorem 2] it is proved that, if $\delta(G) \geq 2$, then $X(G)$ is connected if and only if $\hat{G}$ is connected. One can verify that $\delta(G) \geq 2$ and $\hat{G}$ is connected if and only if (a), (b) and (c) in Theorem 1 hold. Thus, by Theorem 1, if $X(G)$ is connected, then it is hamiltonian. The converse of this statement is obvious.

(b) If G is connected with $\delta(G) \geq 3$, then $\hat{G} = G$ and so $X(G)$ is connected by [14, Theorem 2]. Hence, by (a), $X(G)$ is hamiltonian. Since $\delta(G) \geq 3$, we have $\delta(X(G)) \geq 3$. Thus, by applying (a) to $X(G)$, we see that $X^2(G)$ is hamiltonian. Continuing, by induction we can prove that $X^i(G)$ is hamiltonian for every $i \geq 1$. □

4 Proof of Theorems 3 and 4

Let us first introduce an operation that will be used in the proof of Theorem 3. Let G^* be an Eulerian multigraph and C an Eulerian tour of G^*. Let (z_1, x, z_2) be a visit of C to x. Write

$$C : z_1, e_1, x, e_2, \overbrace{z_2, \ldots, z_1}^{T},$$

where e_1 is the oriented edge from z_1 to x, e_2 the oriented edge from x to z_2, and T the segment of C from z_2 to z_1 covering all edges of G^* except e_1 and e_2. Add two new vertices t, t' to G^* and join them to x by edges $e_t, e_{t'}$, respectively, with orientation towards x. Denote the resultant multigraph by $G^*_C(z_1, x, z_2)$. Set

$$W = W_C(z_1, x, z_2) : t, e_t, x, e_2, \overbrace{z_2, \ldots, z_1}^{T}, e_1, x, e_{t'}^{-1}, t'.$$

Since C is an Eulerian tour of G^*, W is an open Eulerian trail of $G^*_C(z_1, x, z_2)$. Denote by $W(x)$ the set of visits to x induced by W. As the first and last visits induced by W, (t, e_t, x, e_2, z_2) and $(z_1, e_1, x, e_{t'}^{-1}, t')$ are members of $W(x)$. Note that $xt, xt' \notin A(x)$.

Definition 7 *Define $K_C(z_1, x, z_2)$ to be the bipartite graph with bipartition $\{W(x), A(x)\}$ such that an arc in $A(x)$ is adjacent to a visit $p \in W(x)$ if and only if its head does not appear in p. Denote by $L_C(z_1, x, z_2)$ the graph obtained from $K_C(z_1, x, z_2)$ by deleting the vertices (t, e_t, x, e_2, z_2), $(z_1, e_1, x, e_{t'}^{-1}, t')$, xz_1 and xz_2.*

To prove Theorem 3, we need to prove that, for any two distinct arcs xy, uv of G, there exists a Hamilton path of $X(G)$ between xy and uv. We will prove the existence of such a path by constructing a specific Eulerian trail in a certain auxiliary multigraph G^*. We treat the cases $x = u$ and $x \neq u$ separately in the next two lemmas.

Lemma 8 *Under the condition of Theorem 3, for any distinct arcs $xy, xv \in A(G)$ with the same tail, there exists a Hamilton path of $X(G)$ between xy and xv.*

Proof By our assumption there exists a path in G of odd length connecting y and v. Let

$$P : y = x_0, x_1, x_2, \ldots, x_{l-1}, x_l = v$$

be a path in G between y and v with minimum possible odd length $l \geq 1$. Denote $E_0(P) = \{\{x_j, x_{j+1}\} \mid j = 0, 2, \ldots, l-1\}$ and $E_1(P) = \{\{x_j, x_{j+1}\} \mid j = 1, 3, \ldots, l-2\}$.

Case 1. $x \notin V(P)$. In this case let G^* be obtained from G by doubling each edge of $E(G) - (E(P) \cup \{\{x, y\}, \{x, v\}\})$ and tripling each edge of $E_0(P)$.

Case 2. $x \in V(P)$. In this case we have $l \geq 3$ and $x = x_j$ for some $1 \leq j \leq l-1$. If $2 \leq j \leq l-2$, then since l is odd, one of the two paths $y, x_1, \ldots, x_{j-1}, x, v$ and $y, x, x_{j+1}, \ldots, x_{l-1}, v$ would be a path of odd length connecting y and v that is shorter than P, contradicting the choice of P. Therefore, either $x = x_1$ or $x = x_{l-1}$. Assume without loss of generality that $x = x_1$. Define G^* to be the multigraph obtained from G by doubling each edge of $E(G) - [(E(P) - \{\{x, y\}\}) \cup \{\{x, v\}\}]$ and tripling each edge of $E_0(P) - \{\{x, y\}\}$.

In each case above, $d^*(x) = 2d(x) - 2$ and $d^*(z) = 2d(z)$ for every $z \neq x$, and hence G^* is Eulerian.

Set $a = y$ in Case 1 and $a = x_2$ in Case 2. By extending the 2-path a, x, v to an Eulerian tour, we see that there are Eulerian tours of G^* which pass through (a, x, v). Choose C to be an Eulerian tour of G^* with $(a, x, v) \in C(x)$ such that $|Z(C)|$ is minimum, where $Z(C)$ is the set of vertices $w \neq x$ of G^* such that $H_C(w)$ has no perfect matching.

Claim 1. $Z(C) = \emptyset$; that is, $H_C(w)$ has a perfect matching for every $w \neq x$.

Proof of Claim 1. We prove this by way of contradiction. Suppose $H_C(w)$ has no perfect matching for some $w \neq x$. By Lemma 7, $d^*(w) = 6$ and $C(w)$ contains twin visits. Since $w \neq x$, we have $d(w) = 3$ by the construction of G^*. Denote $N(w) = \{w_1, w_2, w_3\}$. In the case when each of w_1, w_2 and w_3 is joined to w by two parallel edges, we apply the bow-tie operation at w with respect to one of the twin visits and the third visit of $C(w)$. Similar to the proof of Theorem 1, for the resultant Eulerian tour C' of G^*, $H_{C'}(w)$ has a perfect matching, and the visit-decomposition at any other vertex is unchanged. Thus $(a, x, v) \in C'(x)$ and $Z(C')$ is a proper subset of $Z(C)$, contradicting the choice of C.

It remains to consider the case where exactly one vertex of $N(w)$ is joined to w by one, two or three (parallel) edges, respectively. Without loss of generality we may assume that there is one edge between w_3 and w, two parallel edges between w_1 and w, and three parallel edges between w_2 and w. Then $C(w) = \{[w_1, w, w_2], [w_1, w, w_2], [w_3, w, w_2]\}$. Reversing the orientation of C when necessary, we may assume $(w_1, w, w_2) \in C(w)$. Denote by e_1, e_3 the oriented parallel edges from w_1 to w, by e_2, e_4, e_6 the oriented parallel edges from w to w_2, and by e_5 the oriented edge from w to w_3.

Case (a): $C(w) = \{(w_1, w, w_2), (w_1, w, w_2), [w_3, w, w_2]\}$. We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \dots, g, w_1, e_3, w, e_4, w_2, h, \dots, k, w_1.$$

Let

$$C' : w_1, e_1, w, e_3^{-1}, w_1, g^{-1}, \dots, f^{-1}, w_2, e_2^{-1}, w, e_4, w_2, h, \dots, k, w_1.$$

Then C' is an Eulerian tour of G^* and $C'(w) = \{(w_1, w, w_1), (w_2, w, w_2), [w_3, w, w_2]\}$. Moreover, $H_{C'}(w)$ has a perfect matching which matches (w_1, w, w_1), (w_2, w, w_2), $[w_3, w, w_2]$ to ww_2, ww_3, ww_1 respectively.

Case (b): $C(w) = \{(w_1, w, w_2), (w_2, w, w_1), [w_3, w, w_2]\}$. We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \dots, g, w_2, e_4^{-1}, w, e_3^{-1}, w_1, h, \dots, k, w_1.$$

Denote

$$C_1 : w_1, e_1, w, e_3^{-1}, w_1, h, \dots, k, w_1; \qquad C_2 : w_2, e_4^{-1}, w, e_2, w_2, f, \dots, g, w_2.$$

Note that each of C_1 and C_2 is a closed trail, and $[w_3, w, w_2]$ is a segment of exactly one of C_1 and C_2.

In the case when $(w_3, w, w_2) \in C(w)$ and it is in C_2, we first rewrite C_2 to highlight the position of (w_3, w, w_2) in C_2:

$$C_2' : w_3, e_5^{-1}, w, e_6, w_2, \dots, w_3.$$

Applying the concatenation operation to $(C_1, C_2', (w_1, w, w_1), (w_3, w, w_2))$ yields:

$$C' : w_1, e_1, w, e_5, w_3, \dots, w_2, e_6^{-1}, w, e_3^{-1}, w_1, h, \dots, k, w_1.$$

We have $C'(w) = \{(w_1, w, w_3), (w_2, w, w_1), [w_2, w, w_2]\}$. Hence $H_{C'}(w)$ has a perfect matching which matches (w_1, w, w_3), (w_2, w, w_1), $[w_2, w, w_2]$ to ww_2, ww_3, ww_1 respectively.

In the case when $(w_3, w, w_2) \in C(w)$ and it is in C_1, we first rewrite C_1 to highlight the position of $[w_3, w, w_2]$ in C_1:

$$C_1' : w_3, e_5^{-1}, w, e_6, w_2, \dots, w_3.$$

Applying the concatenation operation to $(C_2, C_1', (w_2, w, w_2), (w_3, w, w_2))$ yields:

$$C' : w_2, e_4^{-1}, w, e_5, w_3, \dots, w_2, e_6^{-1}, w, e_2, w_2, f, \dots, g, w_2.$$

Since $C'(w) = \{(w_2, w, w_3), (w_2, w, w_2), [w_1, w, w_1]\}$, $H_{C'}(w)$ has a perfect matching which matches (w_2, w, w_3), (w_2, w, w_2), $[w_1, w, w_1]$ to ww_1, ww_3, ww_2 respectively.

The remaining case when $(w_2, w, w_3) \in C(w)$ can be dealt with similarly.

In all possibilities above we obtain a new Eulerian tour C' of G^* such that $H_{C'}(w)$ has a perfect matching whilst the visit-decomposition at any other vertex is unchanged. Thus $(a, x, v) \in C'(x)$ and $Z(C')$ is a proper subset of $Z(C)$, contradicting the choice of C. This completes the proof of Claim 1.

Claim 2. There exists an Eulerian tour C^* of G^* together with a visit $(u_1, x, u_2) \in C^*(x)$ such that (i) $H_{C^*}(z)$ has a perfect matching for every $z \neq x$, and (ii) the bipartite graph $K_{C^*}(u_1, x, u_2)$ (as defined in Definition 7) has a perfect matching under which the first and last visits induced by $W_{C^*}(u_1, x, u_2)$ are matched to xy and xv resepctively.

Note that, for $z \neq x$, $H_{C^*}(z) = H_W(z)$, where $W = W_{C^*}(u_1, x, u_2)$.

Proof of Claim 2. We will prove the existence of C^* and $(u_1, x, u_2) \in C^*(x)$ based on C as in Claim 1.

Case (a): G^* was constructed in Case 1. Then $(a, x, v) = (y, x, v) \in C(x)$ and all edges of G incident with x except $\{x, y\}$ and $\{x, v\}$ were doubled.

In the case when $d(x) = 3$, let z_1 be the neighbour of x in G other than y and v. One can see that $K_C(z_1, x, z_1)$ has a perfect matching which matches (t, x, z_1), (y, x, v), (z_1, x, t') to xy, xz_1, xv, respectively.

In the case when $d(x) = 4$, let z_1 and z_2 be the neighbours of x in G other than y and v. Since $(y, x, v) \in C(x)$, without loss of generality we may assume $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$ or $\{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$. If $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$, then $K_C(y, x, v)$ has a perfect matching which matches (t, x, v), (z_1, x, z_1), (z_2, x, z_2), (y, x, t') to xy, xz_2, xz_1, xv, respectively. In the case when $C(x) \prec \{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$, by applying the bow-tie operation at x with respect to $((z_1, x, z_2), (y, x, v))$ we obtain a new Eulerian tour $C' = C((z_1, x, z_2), (y, x, v))$ for which $C'(x) = \{[z_1, x, z_2], (z_j, x, y), (z_{j'}, x, v)\}$, where $\{j, j'\} = \{1, 2\}$. Without loss of generality we may assume $C'(x) = \{(z_1, x, z_2), (z_j, x, y), (z_{j'}, x, v)\}$. One can see that $K_{C'}(z_1, x, z_2)$ contains a perfect matching which matches (t, x, z_2), (z_j, x, y), $(z_{j'}, x, v)$, (z_1, x, t') to xy, $xz_{j'}$, xz_j, xv, respectively.

Assume $d(x) \geq 5$. If $L_C(y, x, v)$ has a perfect matching, then adding the edges $\{(t, x, v), xy\}$, $\{(y, x, t'), xv\}$ to it yields a perfect matching of $K_C(y, x, v)$ which matches the first and last visits of $W_C(y, x, v)$ to xy, xv, respectively. Suppose that $L_C(y, x, v)$ has no perfect matchings. Similar to Lemma 7, by using Hall's marriage theorem we can prove that $d(x) = 5$ and $C(x)$ contains twin visits, say, $[z_1, x, z_2]$; that is, $C(x) \prec \{[z_1, x, z_2], [z_1, x, z_2], [z_3, x, z_3], (y, x, v)\}$. Without loss of generality we may assume $(z_1, x, z_2) \in C(x)$. It is not hard to see that $K_C(z_1, x, z_2)$ has a perfect matching which matches (t, x, z_2), (z_1, x, z_2), $[z_3, x, z_3]$, (y, x, v), (z_1, x, t') to xy, xz_3, xz_2, xz_1, xv, respectively.

Case (b): G^* was constructed in Case 2. Then $(x_2, x, v) \in C(x)$ and all edges of G incident with x except $\{x, x_2\}$ and $\{x, v\}$ were doubled.

In the case when $d(x) = 3$, we have $C(x) \prec \{(x_2, x, v), (y, x, y)\}$ and $K_C(x_2, x, v)$ has a perfect matching which matches $(t, x, v), (y, x, y), (x_2, x, t')$ to xy, xx_2, xv, respectively.

In the case when $d(x) = 4$, we have $C(x) \prec \{(x_2, x, v), [z_1, x, y], [z_1, x, y]\}$ or $C(x) \prec \{(x_2, x, v)(z_1, x, z_1),\ (y, x, y)\}$, where z_1 is the neighbour of x other than y, v, x_2. If $C(x) \prec \{(x_2, x, v), [z_1, x, y], [z_1, x, y]\}$, let $(z_1, x, y) \in C(x)$, say. Then $K_C(y, x, z_1)$ has a perfect matching, namely $(t, x, z_1), (x_2, x, v), [z_1, x, y]$, (y, x, t') are matched to xy, xz_1, xx_2, xv, respectively. If $C(x) \prec \{(x_2, x, v)(z_1, x,\ z_1), (y, x, y)\}$, then $K_C(z_1, x, z_1)$ has a perfect matching which matches (t, x, z_1), $(x_2, x, v), (y, x, y), (z_1, x, t')$ to xy, xz_1, xx_2, xv, respectively.

Assume $d(x) \geq 5$ hereafter. In the case when $L_C(x_2, x, v)$ has a perfect matching, say, M, let xy be matched to (w_1, x, w_2) by M, where $w_1, w_2 \in N(x) - \{x_2, v, y\}$. Deleting $\{(w_1, x, w_2), xy\}$ from M and then adding $\{(w_1, x, w_2),\ xx_2\}, \{(t, x, v), xy\}$ and $\{(x_2, x, t'), xv\}$ yields a perfect matching of $K_C(x_2, x, v)$ satisfying (ii) in Claim 2.

Suppose $L_C(x_2, x, v)$ has no perfect matchings. Similar to Lemma 7, we can prove that $d(x) = 5$ and $C(x)$ contains twin visits. Denote by $z_1, z_2 \neq y, v, x_2$ the other two neighbours of x. Let (w_1, x, w_2) be one of the twin visits in $C(x)$, where $w_1, w_2 \in \{y, z_1, z_2\}$ are distinct, and let w_3 denote the unique vertex in $\{y, z_1, z_2\} - \{w_1, w_2\}$. Then $C(x) \prec \{(x_2, x, v), (w_1, x, w_2), [w_1, x, w_2], (w_3, x,\ w_3)\}$. Since w_1 and w_2 are distinct, one of them, say, w_2, is not equal to y. Thus $K_C(w_1, x, w_2)$ has a perfect matching which matches (t, x, w_2), (x_2, x, v), $[w_1, x,\ w_2]$, (w_3, x, w_3), (w_1, x, t') to xy, xw_2, xw_3, xx_2, xv, respectively.

Since $H_C(z)$ has a perfect matching for every $z \neq x$, one can see that in all possibilities above, condition (i) in Claim 2 is satisfied by the underlying Eulerian tour (which is C or C'). This proves Claim 2.

Choose an Eulerian tour $C^* : w_l, x, w_1, w_2, w_3, \ldots, w_l$ of G^* together with a visit $(w_l, x, w_1) \in C^*(x)$ satisfying the conditions of Claim 2. Then $W = W_{C^*}(w_l, x, w_1) : t, x, w_1, w_2, w_3, \ldots, w_{l-1}, w_l, x, t'$.

Denote by $\phi(t, x, w_1)$ ($\phi(w_l,\ x, t')$, respectively) the arc of G with tail x that is matched to (t, x, w_1) ((w_l, x, t'), respectively) by a perfect matching of $K_{C^*}(w_l, x, w_1)$ satisfying (ii) in Claim 2. Let $\phi(x, w_1, w_2)$ denote the arc matched to (x, w_1, w_2) in a perfect matching of $H_{C^*}(w_1)$ ($= H_W(w_1)$), and let $\phi(w_1, w_2, w_3)$, $\ldots$, $\phi(w_{l-1}, w_l, x)$ be interpreted similarly. Conditions (i) and (ii) in Claim 2 ensure that

$$xy = \phi(t, x, w_1), \phi(x, w_1, w_2), \phi(w_1, w_2, w_3), \ldots, \phi(w_{l-1}, w_l, x), \phi(w_l, x, t') = xv$$

is a Hamilton path of $X(G)$ connecting xy and xv. $\square$

Lemma 9 *Under the condition of Theorem 3, for distinct $xy, uv \in A(G)$ with $x \neq u$, there exists a Hamilton path of $X(G)$ between xy and uv.*

Proof We have five possibilities to consider: $x = v$ and $y = u$; x, y, u, v are pairwise distinct; $x = v$ and $y \neq u$; $y = v$ and $x \neq u$; $y = u$ and $x \neq v$. The following treatment covers all of them.

By our assumption there exists a path of odd length connecting x and u in G. Let

$$P : x = x_0, x_1, x_2, \ldots, x_{l-1}, x_l = u \tag{2}$$

be such a path with shortest (odd) length $l \geq 1$. (It may happen that $y = x_1$ and/or $v = x_{l-1}$.) Define G^* to be the multigraph obtained from G by doubling each edge of G outside of P and tripling each edge $\{x_j, x_{j+1}\}$ for $j = 1, 3, \ldots, l-2$. Then $d^*(x) = 2d(x) - 1$, $d^*(u) = 2d(u) - 1$ and $d^*(z) = 2d(z)$ for $z \neq x, u$.

Let $G^*_{x,u}(t, t')$ be the multigraph obtained from G^* by adding two new vertices t, t' and joining them to x, u respectively by a single edge. Then all vertices of $G^*_{x,u}(t, t')$ except t and t' have even degrees in $G^*_{x,u}(t, t')$. Hence $G^*_{x,u}(t, t')$ has Eulerian trails connecting t and t'.

Since $\delta(G) \geq 3$, we can choose x' to be a neighbour of x other than y and x_1, and u' a neighbour of u other than v and x_{l-1}. In addition, if $d(x) = d(u) = 3$, $y = x_1$ and $v = x_{l-1}$, say, $N(x) = \{y, x', z\}$ and $N(u) = \{v, u', w\}$, then we can choose x' and u' in such a way that the edges $\{x, z\}$ and $\{u, w\}$ do not form an edge cut of G. In fact, if $\{\{x, z\}, \{u, w\}\}$ is an edge cut of G in this case, then since G is assumed to be 2-edge connected, $G - \{\{x, z\}, \{u, w\}\}$ has two connected components, say, G_0 and G_1 with $z, w \in V(G_0)$ and P in G_1. Since x' is in G_1 and removal of $\{x, x'\}$ does not disconnect G, one can see that $\{\{x, x'\}, \{u, w\}\}$ is not an edge-cut of G. Thus interchanging the roles of x' and z produces the desired x' and u'. (In general, at most one of x' and u' lies on P since P is a path between x and u with minimum odd length.)

With x' and u' as above, let

$$W' : t, x, x', \overbrace{x, x_1, x_2, \ldots, x_{l-1}, u}^{P}, u', u, t',$$

where P is the path given in (2). Then W' is a trail of $G^*_{x,u}(t, t')$. Let W be an Eulerian trail of $G^*_{x,u}(t, t')$ obtained by extending W' to cover all edges of $G^*_{x,u}(t, t')$ while maintaining (t, x, x') and (u', u, t') as its first and last visits respectively. Such a trail W exists because removing the four edges in (t, x, x') and (u', u, t') from $G^*_{x,u}(t, t')$ results in a connected multigraph with x' and u' as the only odd-degree vertices. In addition, if $d(x) = 3$ and $y = x_1$, say, $N(x) = \{y, x', z\}$, since $\{\{x, z\}, \{u, w\}\}$ is not an edge cut of G by our choices of x' and u', we can choose W in such a way that (x', x, x_1) is a visit induced by W; similarly, we can choose W such that (u', u, x_{l-1}) is a visit induced by W, if $d(u) = 3$ and $v = x_{l-1}$, say, $N(u) = \{v, u', w\}$. (Such a W can be constructed as follows: extend W' to an Eulerian trail of the multigraph obtained by deleting the parallel edges between x and z and/or that between u and w, and then insert the visits (z, x, z) and/or (w, u, w) to this trail.) In this way we obtain an Eulerian trail W of $G^*_{x,u}(t, t')$ such that

(A) (t, x, x') and (u', u, t') are its first and last visits, respectively; and

(B) if $d(x) = 3$ and $y = x_1$, say, $N(x) = \{y, x', z\}$, then $(x', x, x_1) \in W(x)$; and, if $d(u) = 3$ and $v = x_{l-1}$, say, $N(u) = \{v, u', w\}$, then $(u', u, x_{l-1}) \in W(x)$.

Similar to Claim 1, one can show that there exists an Eulerian trail of $G^*_{x,u}(t, t')$, denoted by W hereafter, satisfying (A), (B) and

(C) $H_W(z)$ has a perfect matching for every $z \in V(G) - \{x, u\}$.

Note that $|W(z)| = |A(z)| = d(z)$ for every $z \in V(G)$.

Claim 3. There exists an Eulerian trail W^* of $G^*_{x,u}(t, t')$ such that (i) (t, x, x') and (u', u, t') are its first and last visits, respectively; (ii) $H_{W^*}(x)$ has a perfect matching under which (t, x, x') is matched to xy; (iii) $H_{W^*}(u)$ has a perfect matching under which (u', u, t') is matched to uv; and (iv) $H_{W^*}(z)$ has a perfect matching for every $z \in V(G) - \{x, u\}$.

Proof of Claim 3. Let $p = (t, x, x')$ denote the first visit of W, and let $L_W(x) = H_W(x) - \{p, xy\}$ be the subgraph of $H_W(x)$ obtained by deleting vertices p and xy. For $S \subseteq W(x) - \{p\}$, denote by $N_{L_W(x)}(S)$ the neighbourhood of S in $L_W(x)$.

Case (a): $y \neq x_1$. If $d(x) \geq 5$, then $|N_{L_W(x)}(S)| \geq |S|$ for any S, and so $L_W(x)$ contains a perfect matching by Hall's marriage theorem.

Suppose $d(x) = 4$. Then $|N_{L_W(x)}(S)| \geq |S|$ for every S with $|S| = 1$ or 3. Suppose $|S| = 2$ and $S = \{(a, x, b), (a', x, b')\}$. Then $N_{L_W(x)}(S) = [(A(x) - \{xy\}) - \{xa, xb\}] \cup [(A(x) - \{xy\}) - \{xa', xb'\}] = [(A(x) - \{xy\})] - (\{xa', xb'\} \cap \{xa, xb\})$. Thus, if $|\{xa', xb'\} \cap \{xa, xb\}| \leq 1$, then $|N_{L_W(x)}(S)| \geq |S|$. If $|\{xa', xb'\} \cap \{xa, xb\}| = 2$, then $\{a, b\} = \{a', b'\}$ and $\{x', x_1\} \cap \{a, b\} = \emptyset$, which implies $y \in \{a, b\}$ and $|N_{L_W(x)}(S)| = |(A(x) - \{xa, xb\}| = 2$. Hence $L_W(x)$ contains a perfect matching by Hall's theorem.

Suppose $d(x) = 3$. Then $W(x) = \{p, (x', x, y), (y, x, x_1)\}$ or $W(x) = \{p, (x', x, x_1), (y, x, y)\}$. In the former case $L_W(x)$ clearly has a perfect matching. In the latter case, apply the bow-tie operation to W with respect to (x', x, x_1) and (y, x, y) to obtain a new Eulerian trail W_0 such that $L_{W_0}(x)$ has a perfect matching.

Case (b): $y = x_1$. Similar to Case (a), if $d(x) \geq 5$, then $L_W(x)$ has a perfect matching. If $d(x) = 4$, let $N(x) = \{x', x_1, z_1, z_2\}$. Then $|N_{L_W(x)}(S)| \geq |S|$ unless $S = \{(z_1, x, z_2), [z_1, x, z_2]\}$. In this exceptional case, $W(x) = \{p, (x', x, x_1), (z_1, x, z_2), [z_1, x, z_2]\}$, and we apply the bow-tie operation to W with respect to (x', x, x_1) and (z_1, x, z_2) to obtain a new Eulerian trail W_0. One can show that $L_{W_0}(x)$ has a perfect matching.

If $d(x) = 3$, let $N(x) = \{x', x_1, z\}$. By (B), (x', x, x_1) is a visit to x induced by W. Hence $W(x) = \{p, (x', x, x_1), (z, x, z)\}$ and $L_W(x)$ has a perfect matching.

So far we have proved that there exists an Eulerian trail W_1 of $G^*_{x,u}(t, t')$ (which is either W or W_0) satisfying (A) such that $L_{W_1}(x)$ has a perfect matching. This matching together with the edge between (t, x, x') and xy is a perfect matching of $H_{W_1}(x)$. Moreover, since W satisfies (C), from the proof above one can see that W_1 satisfies (C) as well. If $H_{W_1}(u)$ has a perfect matching which matches (u', u, t') to uv, then set $W^* = W_1$ and we are done. Otherwise, beginning with W_1 and using similar arguments as above, we can construct an Eulerian trail W^* of $G^*_{x,u}(t, t')$ satisfying all requirements in Claim 3. This completes the proof of Claim 3.

Similar to the proof of Lemma 8, we can show that the Eulerian trail W^* in Claim 3 produces a Hamilton path in $X(G)$ connecting xy and uv. □

Proof of Theorem 3 This follows from Lemmas 8 and 9 immediately. □

In the proof of Theorem 4 we will use the following lemma which may be known in the literature. We give its proof since we are unable to allocate a reference.

Lemma 10 *In any Hamilton-connected graph with at least four vertices, there exists a path of odd length connecting any two distinct vertices.*

Proof Let G be such a graph. Then for any distinct $u, v \in V(G)$ there exists a Hamilton path $P : u = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = v$, where $n = |V(G)| - 1$. It suffices to consider the case when n is even. Denote $A = \{x_0, x_2, \ldots, x_n\}$ and $B = \{x_1, x_3, \ldots, x_{n-1}\}$. Since $\{A, B\}$ is a partition of $V(G)$ and any bipartite graph other than K_2 is not Hamilton-connected, there exist adjacent vertices x_i, x_j both in A or B, where $j \geq i + 2$. Thus $x_0, x_1, \ldots, x_{i-1}, x_i, x_j, x_{j+1}, \ldots, x_n$ is a path of odd length between u and v. □

Proof of Theorem 4 It can be verified that any Hamilton-connected graph with at least four vertices is 2-edge connected and has minimum degree at least three. Hence Theorem 3 and Lemma 10 together imply that the 3-arc graph of such a graph is Hamilton-connected (with more than four vertices). Applying this iteratively, we obtain Theorem 4. □

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