Simultaneous Domination in Graphs

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Abstract

Let F_1, F_2, \ldots, F_k be graphs with the same vertex set V. A subset $S \subseteq V$ is a simultaneous dominating set if for every $i, 1 \leq i \leq k$, every vertex of F_i not in S is adjacent to a vertex in S in F_i ; that is, the set S is simultaneously a dominating set in each graph F_i . The cardinality of a smallest such set is the simultaneous domination number. We present general upper bounds on the simultaneous domination number. We investigate bounds in special cases, including the cases when the factors, F_i , are r-regular or the disjoint union of copies of K_r . Further we study the case when each factor is a cycle.

Keywords: Factor domination. AMS subject classification: 05C69

1 Introduction

Given a collection of graphs F_1, \ldots, F_k on the same vertex set V, we consider a set of vertices which dominates all the graphs simultaneously. This was first explored by Brigham and Dutton [3] who defined such a set as a factor dominating set and by Sampathkumar [13] who used the name global dominating set. The natural question is what is the minimum size of a simultaneous dominating set. This question has been studied in [2, 6, 7, 8] and [10, Section 7.6] and elsewhere. In this paper we will use the term "simultaneous domination" rather than "global domination" (see [2, 13]) or "factor domination" (see [3, 7, 8]).

 $^{^{*}\}mbox{Research}$ supported in part by the University of Johannes burg and the South African National Research Foundation.

A dominating set of G is a set S of vertices of G such that every vertex outside S is adjacent to some vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For $k \ge 1$, a k-dominating set of G is a set S of vertices of G such that every vertex outside S is adjacent to at least k vertices in S. For a survey see [10, 11].

Following the notation in [7], we define a *factoring* to be a collection F_1, F_2, \ldots, F_k of (not necessarily edge-disjoint) graphs with common vertex set V (the union of whose edge sets is not necessary the complete graph). The *combined graph* of the factoring, denoted by $G(F_1, \ldots, F_k)$, has vertex set V and edge set $\bigcup_{i=1}^k E(F_i)$. We call each F_i a *factor* of the combined graph.

A subset $S \subseteq V$ is a simultaneous dominating set, abbreviated SD-set, of $G(F_1, \ldots, F_k)$ if S is simultaneously a dominating set in each factor F_i for all $1 \leq i \leq k$. We remark that in the literature a SD-set is also termed a factor dominating set or a global dominating set. The minimum cardinality of a SD-set in $G(F_1, \ldots, F_k)$ is the simultaneous domination number of $G(F_1, \ldots, F_k)$, denoted by $\gamma_{sd}(F_1, F_2, \ldots, F_k)$. We remark that the notion of simultaneous domination is closely related to the notion of colored domination studied, for example, in [12] and elsewhere.

For $k \geq 2$ and $\delta \geq 1$, let $\mathcal{G}_{k,\delta,n}$ be the family of all combined graphs on n vertices consisting of k factors each of which has minimum degree at least δ and define

$$\gamma_{\rm sd}(k,\delta,n) = \max\{\gamma_{\rm sd}(G) \mid G \in \mathcal{G}_{k,\delta,n}\}$$

For notational convenience, we simply write $\gamma_{sd}(k,n) = \gamma_{sd}(k,1,n)$.

1.1 Graph Theory Notation and Terminology

For notation and graph theory terminology, we in general follow [10]. Specifically, let G be a graph with vertex set V(G) of order n = |V(G)| and edge set E(G) of size m = |E(G)|. The open neighborhood of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. The degree of v is $d_G(v) = |N_G(v)|$. Let $\delta(G)$, $\Delta(G)$ and $\overline{d}(G)$ denote, respectively, the minimum degree, the maximum degree and the average degree in G. If $d_G(v) = k$ for every vertex $v \in V$, we say that G is a k-regular graph. If the graph G is clear from the context, we simply write N(v), N[v], N(S), N[S] and d(v) rather than $N_G(v)$, $N_G[v]$, $N_G(S)$, $N_G[S]$ and $d_G(v)$, respectively.

If G is a disjoint union of k copies of a graph F, we write G = kF. For a subset $S \subseteq V$, the subgraph induced by S is denoted by G[S]. If $S \subseteq V$, then by G - S we denote the graph obtained from G by deleting the vertices in the set S (and all edges incident with vertices in S). If $S = \{v\}$, then we also denote $G - \{v\}$ simply by G - v. A component in G is a maximal connected subgraph of G. If G is a disjoint union of k copies of a graph F, we write G = kF. A star-forests is a forest in which every component is a star.

2 Known Results

Directly from the definition we obtain the following result first observed by Brigham and Dutton [3].

Observation 1 ([3]) If G is the combined graph of $k \ge 2$ factors, F_1, F_2, \ldots, F_k , then

$$\max_{1 \le i \le k} \gamma(F_i) \le \gamma_{\rm sd}(G) \le \sum_{i=1}^k \gamma(F_i).$$

That the lower bound of Observation 1 is sharp, may be seen by taking the k factors, F_1, F_2, \ldots, F_k , to be equal. To see that the upper bound of Observation 1 is sharp, let $k \ge 2$ and let F_1, F_2, \ldots, F_k be factors with vertex V, where |V| = n > k, defined as follows. Let $V = \{v_1, v_2, \ldots, v_n\}$ and let F_i be a star $K_{1,n-1}$ centered at the vertex $v_i, 1 \le i \le k$. Then, $\{v_1, v_2, \ldots, v_k\}$ is a minimum SD-set of the combined graph $G(F_1, F_2, \ldots, F_k)$, implying that

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) = \sum_{i=1}^{k} \gamma(F_i) = k.$$

Brigham and Dutton [3] were also the first to observe the following bound.

Observation 2 ([3]) $\gamma_{sd}(k, \delta, n) \leq n - \delta$.

The following bounds on $\gamma_{\rm sd}(k, n)$ are established in [7, 8].

Theorem 3 The following holds. (a) ([8]) For k = 2, $\gamma_{sd}(k, n) \leq 2n/3$, and this is sharp.

(b) ([7]) For $k \ge 3$, $\gamma_{sd}(k, n) \le (2k - 3)n/(2k - 2)$, and this is sharp for all k.

Values of $\gamma_{sd}(k, n)$ in Theorem 3 for small k are shown in Table 1.

Caro and Yuster [6] considered a combined graph consisting of k factors F_1, F_2, \ldots, F_k . In the language of the current paper, they were interested in finding a minimum subset D of vertices with the property that the subgraph induced by D is a connected r-dominating set in each of the factors F_i , $1 \le i \le k$, where $r \le \delta = \min\{\delta(F_i) \mid i = 1, 2, \ldots, k\}$. As a special consequence of their main result, we have the following asymptotic result.

Theorem 4 ([6]) Let F_1, F_2, \ldots, F_k be factors on n vertices and let $\delta = \min\{\delta(F_i) \mid i = 1, 2, \ldots, k\}$. If $\delta > 1$ and $\ln \ln \delta > k$, then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \left(\frac{(\ln \delta)(1 + o_\delta(1))}{\delta}\right) n.$$

Dankelmann and Laskar [8] established the following upper bound on the simultaneous domination number of k factors, depending on the smallest minimum degree of the factors.

Theorem 5 Let F_1, F_2, \ldots, F_k be factors on n vertices. Let $\delta = \min\{\delta(F_i) \mid i = 1, 2, \ldots, k\}$. If $\delta \geq 2$ and $k \leq e^{\delta + 1}/(\delta + 1)$, then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(\frac{\ln(\delta+1) + \ln k + 1}{\delta+1}\right) n.$$

We close this section with a construction showing that the upper bound in Theorem 3(a), which was originally demonstrated by *star-forests*, can be realized by trees. Let F_1 and F_2 be factors on n = 3k vertices constructed as follows. Let F_1 be obtained from the path $u_1u_2...u_k$ by adding for each $i, 1 \le i \le k$, two new vertices v_i and z_i and joining u_i to v_i and z_i . Further let F_2 be obtained from the path $z_1z_2...z_k$ by adding for each $i, 1 \le i \le k$, for each $i, 1 \le i \le k$, add two new vertices u_i and v_i and joining z_i to u_i and v_i . We note that both factors F_1 and F_2 are trees.

Let D be a SD-set of the combined graph $G(F_1, F_2)$. On the one hand, if $u_1 \in D$, then in order to dominate the vertex v_1 in F_2 , we have that at least one of v_1 and z_1 belong to D. On the other hand, if $u_1 \notin D$, then in order to dominate the vertices v_1 and z_1 in F_2 , both v_1 and z_1 belong to D. In both cases, $|D \cap \{u_1, v_1, z_1\}| \geq 2$. Analogously, $|D \cap \{u_i, v_i, z_i\}| \geq 2$ for all $i, 1 \leq i \leq k$, implying that $|D| \geq 2k = 2n/3$. Since D was an arbitrary SD-set of $G(F_1, F_2)$, we have that $\gamma_{sd}(F_1, F_2) \geq 2n/3$. Conversely the set $\bigcup_{i=1}^k \{u_i, v_i\}$ is a SD-set of $G(F_1, F_2)$, and so $\gamma_{sd}(F_1, F_2) \leq 2n/3$. Consequently, $\gamma_{sd}(F_1, F_2) = 2n/3$ in this case. Further, $\gamma(F_1) = \gamma(F_1) = n/3$. Hence we have the following statement.

Observation 6 For $n \equiv 0 \pmod{3}$, there exist factors F_1 and F_2 on n vertices, both of which are trees, such that $\gamma_{sd}(F_1, F_2) = 2n/3 = \gamma(F_1) + \gamma(F_2)$.

3 Outline of Paper

In this paper we continue the study of simultaneous domination in graphs. In Section 4 we provide general upper bounds on the simultaneous domination number of a combined graph in terms of the generalized vertex cover and independence numbers. Using a hypergraph and probabilistic approach we provide an improvement on the bound of Theorem 5. In Section 5 we provide general upper bounds on the simultaneous domination number of a combined graph when each factor consists of vertex disjoint union of copies of a clique. We close in Section 6 by studying the case when each factor is a cycle or a disjoint union of cycles.

4 General Upper Bounds

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all the edges of G. We remark that a cover is also called a *transversal* or *hitting set* in the literature. Thus a vertex cover T has a nonempty intersection with every edge of G. The vertex covering number $\tau(G)$ of G is the minimum cardinality of a vertex cover in G. A vertex cover of size $\tau(G)$ is called a $\tau(G)$ -cover. More generally for $t \ge 0$ a *t*-vertex cover in G is a set of vertices S such that the maximum degree in the graph $G[V \setminus S]$ induced by the vertices outside S is at most t. The *t*-vertex covering number $\tau_t(G)$ of G is the minimum cardinality of a *t*-vertex cover in G. A vertex cover of size $\tau_t(G)$ is called a $\tau_t(G)$ -cover. In particular, we note that a 0-vertex cover is simply a vertex cover and that $\tau(G) = \tau_0(G)$.

The independence number $\alpha(G)$ of G is the maximum cardinality of an independent set of vertices of G. More generally, for $k \geq 0$ a k-independent set in G is a set of vertices Ssuch that the maximum degree in the graph G[S] induced by the vertices of S is at most k. The k-independence number $\alpha_k(G)$ of G is the maximum cardinality of a k-independent set of vertices of G. In particular, we note that a 0-independent set is simply an independent set and that $\alpha(G) = \alpha_0(G)$.

Since the complement of a t-vertex cover is a t-independent set and conversely, we have the following observation.

Observation 7 For a graph G of order n and an integer $t \ge 0$, we have $\alpha_t(G) + \tau_t(G) = n$.

We recall the following well-known Caro-Wei lower bound on the independence number in terms of the degree sequence of the graph.

Theorem 8 ([4, 14]) For every graph G of order n,

$$\alpha(G) \ge \sum_{v \in V(G)} \frac{1}{1 + d_G(v)} \ge \frac{n}{\overline{d}(G) + 1}.$$

We will also need the following recent result by Caro and Hansberg [5] who established the following lower bound on the k-independence number of a graph.

Theorem 9 ([5]) For $k \ge 0$ if G is a graph of order n with average degree \overline{d} , then

$$\alpha_k(G) \ge \left(\frac{k+1}{\lceil \overline{d} \rceil + k + 1}\right) n.$$

We begin by establishing the following upper bound on the simultaneous domination number of a combined graph in terms of the *t*-vertex cover number and also in terms of the sum of the average degrees from each factor.

Theorem 10 Let F_1, F_2, \ldots, F_k be factors on n vertices such that $\delta(F_i) \geq \delta \geq 1$. Let $G = G(F_1, \ldots, F_k)$ be the combined graph of the factoring F_1, F_2, \ldots, F_k , and let $\overline{d}(G) = \overline{d}$ and $\overline{d}(F_i) = \overline{d}_i$ for $i = 1, 2, \ldots, k$. Then the following holds.

(a) $\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \tau_{\delta-1}(G) = n - \alpha_{\delta-1}(G).$

(b)
$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \left(\frac{\lceil \overline{d} \rceil}{\lceil \overline{d} \rceil + \delta}\right) n.$$

(c) If F_1, F_2, \ldots, F_k are regular factors on n vertices each of degree δ , then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(\frac{k}{k+1}\right) n.$$

Proof. Let $G = G(F_1, \ldots, F_k)$ denote the combined graph of the factoring F_1, F_2, \ldots, F_k and let G have vertex set V. By definition of the average degree, we have

$$\overline{d} = \frac{2m(G)}{n} \le 2\sum_{i=1}^{k} \frac{m(F_i)}{n} = \sum_{i=1}^{k} \frac{2m(F_i)}{n} = \sum_{i=1}^{k} \overline{d}_i.$$

(a) Let S be a $\tau_{\delta-1}(G)$ -cover. Hence the graph $\Delta(G[V \setminus S]) \leq \delta - 1$ and $|S| = \tau_{\delta-1}(G)$. Let F be an arbitrary factor of G, and so $F = F_i$ for some $i \in \{1, 2, \ldots, k\}$. Since $\delta(F) \geq \delta$ and since every vertex in $V \setminus S$ is adjacent to at most $\delta - 1$ other vertices in $V \setminus S$, the set S is a dominating set of F. This is true for each of the k factors in $G(F_1, \ldots, F_k)$. Therefore, S is a SD-set of G, and so $\gamma_{\rm sd}(G) \leq |S| = \tau_{\delta-1}(G)$. By Observation 7, recall that $\tau_{\delta-1}(G) = n - \alpha_{\delta-1}(G)$.

(b) Since $\delta \geq 1$, we note that $\alpha_{\delta-1}(G) \geq \alpha_0(G) = \alpha(G)$, implying by Observation 7 and Theorem 9 that

$$\tau_{\delta-1}(G) = n - \alpha_{\delta-1}(G) \le n - \left(\frac{\delta}{\lceil \overline{d} \rceil + \delta}\right)n = \left(\frac{\lceil \overline{d} \rceil}{\lceil \overline{d} \rceil + \delta}\right)n.$$

The desired result now follows from Part (a).

(c) Let F_1, F_2, \ldots, F_k be regular factors of degree δ . Then, $\overline{d}_i = \delta$ for $1 \leq i \leq k$, and so $\overline{d} \leq \sum_{i=1}^k \overline{d}_i = k\delta$. Therefore by Part (b) above, we have

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \left(\frac{\lceil \overline{d} \rceil}{\lceil \overline{d} \rceil + \delta}\right) n \leq \left(\frac{k\delta}{(k+1)\delta}\right) n = \left(\frac{k}{k+1}\right) n.$$

This establishes Part (c), and completes the proof of Theorem 10. \Box

We next use a hypergraph and probabilistic approach to improve upon a bound already obtained using this approach in [7]. Let H be a hypergraph. A *k*-edge in H is an edge of size k. The rank of H is the maximum cardinality among all the edges in H. If all edges have the same cardinality k, the hypergraph is said to be *k*-uniform. A subset T of vertices in H is a transversal (also called vertex cover or hitting set in many papers) if Thas a nonempty intersection with every edge of H. The transversal number $\tau(H)$ of H is the minimum size of a transversal in H. For $r \geq 2$, if H is an r-uniform hypergraph with n vertices and m edges, then it is shown in [7] that $\tau(H) \leq n \leq n(\ln(rm/n) + 1)/r$. We improve this bound as follows.

Theorem 11 For $r \ge 2$, let H be an r-uniform hypergraph with n vertices and m edges and with average degree d = rm/n and such that $\delta(H) \ge 1$. Then,

$$\tau(H) \le \left(1 - \left(\frac{r-1}{r}\right) \left(\frac{1}{d}\right)^{\frac{1}{r-1}}\right) n \le n(\ln(d) + 1)/r.$$

Proof. For $0 \le p \le 1$, choose each vertex in H independently with probability p. Let X be the set of chosen vertices and let Y be the set of edges from which no vertex was chosen. Then, E(|X|) = np and $E(|Y|) = m(1 - p^r)$. By linearity of expectation, we have that $E(|X| + |Y|) = E(|X|) + E(|Y|) = np + m(1 - p)^r$. Hence if we add to X one vertex from each edge in Y we get a transversal T of H such that $E(|T|) \le np + m(1 - p)^r$, implying that $\tau(H) \le np + m(1 - p)^r$. Let $f(p) = np + m(1 - p)^r$. This function is optimized when

$$p^* = 1 - \left(\frac{1}{d}\right)^{\frac{1}{r-1}}$$

which is a legitimate value for p as $d \ge \delta(H) \ge 1$. Further,

$$f(p^*) = n - n\left(\frac{1}{d}\right)^{\frac{1}{r-1}} + \left(\frac{nd}{r}\right)\left(\frac{1}{d}\right)^{\frac{r}{r-1}} = \left(1 - \left(\frac{r-1}{r}\right)\left(\frac{1}{d}\right)^{\frac{1}{r-1}}\right)n.$$

We also note that $np + m(1-p)^r \leq np + me^{-pr}$. Taking $p = \ln(d)/r = \ln(rm/n)/r \geq 0$, we get $E(|T|) = E(|X| + |Y|) \leq n \ln(rm/n)/r + n/r = n(\ln(d) + 1)/r$. Hence the optimal choice of p, namely $p = 1 - (\frac{1}{d})^{\frac{1}{r-1}}$, implies that

$$\tau(H) \le \left(1 - \left(\frac{r-1}{r}\right) \left(\frac{1}{d}\right)^{\frac{1}{r-1}}\right) n \le n(\ln(d) + 1)/r,$$

which completes the proof of the theorem. \Box

As an application of Theorem 11, we have the following upper bound on the simultaneous domination number of a combined graph that improves the upper bound of Theorem 5. For a graph G, the *neighborhood hypergraph* of G, denoted by NH(G), is the hypergraph with vertex set V(G) and edge set $\{N_G[v] \mid v \in V(G)\}$ consisting of the closed neighborhoods of vertices in G. **Theorem 12** For $k \geq 2$, if F_1, F_2, \ldots, F_k are factors on n vertices, each of which has minimum degree at least δ , then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(1 - \left(\frac{\delta}{\delta+1}\right) \left(\frac{1}{k(\delta+1)}\right)^{\frac{1}{\delta}}\right) n.$$

Proof. Let $G = G(F_1, \ldots, F_k)$ denote the combined graph of the factoring F_1, F_2, \ldots, F_k and let G have vertex set V. Let $\operatorname{NH}(F_i)$ be the neighborhood hypergraph of F_i , where $1 \leq i \leq k$. In particular, we note that $\operatorname{NH}(F_i)$ has vertex set V and rank at least $\delta + 1$. Let H_i be obtained from $\operatorname{NH}(F_i)$ by shrinking all edges of $\operatorname{NH}(F_i)$, if necessary, to edges of size $\delta + 1$ (by removing vertices from each edge of size greater than $\delta + 1$ until the resulting edge size is $\delta + 1$). Let H be the hypergraph with vertex set V and edge set $E(H) = \bigcup_{i=1}^k E(H_i)$. Then, H is a $(\delta + 1)$ -uniform hypergraph with n(H) = n vertices and $m(H) \leq kn$ edges. The average degree of H is $d = (\delta + 1)m(H)/n(H) \leq k(\delta + 1)$, implying by Theorem 11, that

$$\tau(H) \le \left(1 - \left(\frac{\delta}{\delta+1}\right) \left(\frac{1}{k(\delta+1)}\right)^{\frac{1}{\delta}}\right) n.$$

Every transversal in H is a SD-set in G, implying that $\gamma_{sd}(F_1, F_2, \ldots, F_k) \leq \tau(H)$, and the desired result follows. \Box

Let $f(k, \delta)$ denote the expression on the right-hand side of the inequality in Theorem 12. For small k and small δ , the values of $f(k, \delta)$ are given in Table 3 in the Appendix.

5 K_r -Factors

As an application of Theorem 11, we have the following upper bound on the simultaneous domination number of a combined graph when each factor consists of vertex disjoint union of copies of K_r , for some $r \ge 2$.

Theorem 13 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. For $k \ge 2$, if F_1, F_2, \ldots, F_k are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r , then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(1 - \left(\frac{r-1}{r}\right) \left(\frac{1}{k}\right)^{\frac{1}{r-1}}\right) n \le n(\ln(k) + 1)/r.$$

Proof. Let $G = G(F_1, \ldots, F_k)$ denote the combined graph of the factoring F_1, F_2, \ldots, F_k and let G have vertex set V. Let H be the hypergraph with vertex set V and edge set defined as follows: For every copy of K_r in each of the factors F_i , $1 \le i \le k$, add an *r*-edge in *H* defined by the vertices of this copy of K_r . The resulting hypergraph *H* is an *r*-uniform hypergraph on *n* vertices with $m \le kn/r$ edges. The average degree of *H* is therefore $d = rm/n \le k$, implying by Theorem 11, that

$$\tau(H) \le \left(1 - \left(\frac{r-1}{r}\right) \left(\frac{1}{k}\right)^{\frac{1}{r-1}}\right) n \le n(\ln(k) + 1)/r.$$

Every transversal in H is a SD-set in G, implying that $\gamma_{sd}(F_1, F_2, \ldots, F_k) \leq \tau(H)$, and the desired result follows. \Box

Let $g(k, \delta)$ denote the middle term in the inequality chain in Theorem 13. For small k and small δ , the values of $g(k, \delta)$ are given in Table 4 in the Appendix.

Recall that a graph is called *well-dominated graph* if every minimal dominating set in the graph has the same cardinality. This concept was introduced by Finbow, Hartnell and Nowakowski [9]. We remark that if v is an arbitrary vertex of a well-dominated graph G, then the vertex v can be extended to a maximal independent set, which is a minimal dominating set. However, every minimal dominating set in G is a minimum dominating set in G since G is well-dominated. Therefore, every vertex of a well-dominated graph is contained in a minimum dominating set of the graph.

A graph is 1-*extendable-dominated* if every vertex belongs to a minimum dominating set of the graph. We note that every well-dominated graph is a 1-extendable-dominated graph. However, not every 1-extendable-dominated graph is well-dominated as may be seen by taking, for example, a cycle C_6 or, more generally, a cycle C_n , where $n \ge 8$.

Theorem 14 Let F be a 1-extendable-dominated graph of order r. Let n be an integer such that $r \leq n$ and $n \equiv 0 \pmod{r}$. If F_1 and F_2 are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of F, then $\gamma_{sd}(F_1, F_2) \leq \frac{1}{r}(2\gamma(F) - 1)n$.

Proof. We construct a bipartite graph G as follows. Let V_1 and V_2 be the partite sets of G where for $i \in \{1, 2\}$ the vertices of V_i correspond to the n/r copies of F in F_i . An edge in G joins a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$ if and only if the copies of F corresponding to v_1 and v_2 in F_1 and F_2 , respectively, have at least one vertex in common. We observe that $|V_1| = |V_2| = n/r$.

We show that G contains a perfect matching. Let S be a nonempty subset of vertices of V_1 . We consider the corresponding |S| vertex disjoint copies of F in F_1 . These |S| copies of F cover exactly r|S| vertices in F_1 . But the minimum number of copies of F in F_2 needed to cover these r|S| vertices is at least |S| since each copy of F covers r vertices. Every vertex in V_2 corresponding to such a copy of F in F_2 is joined in G to at least one vertex of S, implying that $|N(S)| \ge |S|$. Hence by Hall's Matching Theorem, there is a matching in G that matches V_1 to a subset of V_2 . Since $|V_1| = |V_2|$, such a matching is a perfect matching in G.

Let M be a perfect matching in G. For each edge $e \in M$, select a vertex v_e that is common to the copies of F in F_1 and F_2 that correspond to the ends of the edge e. Since F is a 1-extendable-dominated graph, this common vertex v_e extends to minimum dominating set in both copies of F creating a dominating set of these two copies with at most $2\gamma(F) - 1$ vertices. Let D_e denote the resulting dominating set of these two copies of F. Then the set $\cup_{e \in M} D_e$ is a SD-set in the combined graph of F_1 and F_2 , implying that $\gamma_{sd}(F_1, F_2) \leq |M| \cdot (2\gamma(F) - 1) \leq (2\gamma(F) - 1)n/r$. \Box

We remark that the bound in Theorem 14 is strictly better than the bound of Theorem 3 and Theorem 10(c) in the case of k = 2 when $\gamma(F) < (2r+3)/6$. As a consequence of Theorem 14, we have the following results.

Theorem 15 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. If F_1 and F_2 are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r , then $\gamma_{sd}(F_1, F_2) = n/r$.

Proof. We note that K_r is a well-dominated graph. Further, $\gamma(K_r) = 1$. Applying Theorem 14 with the graph $F = K_r$, we have that $\gamma_{sd}(F_1, F_2) \leq n/r$. By Observation 1(a), we know that $\gamma_{sd}(F_1, F_2) \geq \gamma(F_1) = n/r$. Consequently, $\gamma_{sd}(F_1, F_2) = n/r$. \Box

Corollary 16 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. If F_1 and F_2 are factors on n vertices, each of which contain a spanning subgraph that is the vertex disjoint union of n/r copies of K_r , then $\gamma_{sd}(F_1, F_2) \le n/r$.

As an immediate consequence of Corollary 16 and Observation 1, we have the following observation.

Corollary 17 For *n* even, if F_1 and F_2 are factors on *n* vertices both having a 1-factor, then $\gamma_{sd}(F_1, F_2) \leq n/2$. Further, if $\max\{\gamma(F_1), \gamma(F_2)\} = n/2$, then $\gamma_{sd}(F_1, F_2) = n/2$.

We next extend the result of Theorem 15 to more than two factors.

Theorem 18 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. For $k \ge 2$, if F_1, F_2, \ldots, F_k are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r , then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \left(1 - \left(\frac{r-1}{r}\right)^{k-1}\right) n.$$

Proof. We proceed by induction on $k \ge 2$. The base case when k = 2 follows from Theorem 15. Assume, then, that $k \ge 3$ and that the result holds for k' factors, each of

which consist of the vertex disjoint union of n/r copies of K_r , where $2 \leq k' < k$. Let F_1, F_2, \ldots, F_k be factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r . First we consider the combined graph $G(F_1, F_2, \ldots, F_{k-1})$ with only $F_1, F_2, \ldots, F_{k-1}$ as factors. Let D be a $\gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$ -set in $G(F_1, F_2, \ldots, F_{k-1})$, and so $|D| = \gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$. By the inductive hypothesis,

$$|D| \le \left(1 - \left(\frac{r-1}{r}\right)^{k-2}\right)n.$$

We now consider the combined graph $G(F_1, F_2, \ldots, F_k)$. Since each copy of K_r in F_k can have at most r vertices from D, the set D must dominate at least |D|/r copies of K_r from F_k . Therefore in F_k there remains at most n/r - |D|/r copies of F_k that are not dominated by D. We now extend the set D to an SD-set of $G(F_1, F_2, \ldots, F_k)$ by adding to it one vertex from each non-dominated copy of K_r of F_k . Hence,

$$\begin{split} \gamma_{\rm sd}(F_1, F_2, \dots, F_k) &\leq |D| + \frac{n - |D|}{r} \\ &= \frac{1}{r} (n + (r - 1)|D|) \\ &\leq \frac{1}{r} \left(n + (r - 1) \left(1 - \left(\frac{r - 1}{r}\right)^{k - 2} \right) n \right) \\ &\leq \frac{1}{r} \left(r - (r - 1) \left(\frac{r - 1}{r}\right)^{k - 2} \right) n \\ &= \left(1 - \left(\frac{r - 1}{r}\right)^{k - 1} \right) n, \end{split}$$

completing the proof of the theorem. \Box

We remark that the bound in Theorem 18 is strictly better than the bounds of Theorem 3, Theorem 10(c) and Theorem 13 when k = 3 and for all $r \ge 3$. In particular, we remark that when k = 3 and $r \ge 3$, the bound in Theorem 18 is strictly better than the bound of Theorem 13 if

$$1 - \left(\frac{r-1}{r}\right)^2 < 1 - \left(\frac{r-1}{r}\right) \left(\frac{1}{3}\right)^{\frac{1}{r-1}},$$

or, equivalently, if

$$\frac{1}{3} < \left(\frac{r-1}{r}\right)^{r-1}.$$

Since $\left(\frac{r-1}{r}\right)^{r-1}$ attains the value 4/9 when r = 3 and is a decreasing function in r approaching 0.367879 as $r \to \infty$, the above inequality holds. In the special case in Theorem 18 when k = 3, we have the following result.

Corollary 19 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. If F_1, F_2, F_3 are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r , then

$$\gamma_{\mathrm{sd}}(F_1, F_2, F_3) \le \left(\frac{2r-1}{r^2}\right) n.$$

Using Corollary 19, the upper bound of Theorem 18 can be improved slightly as follows.

Theorem 20 Let r and n be integers such that $1 \le r \le n$ and $n \equiv 0 \pmod{r}$. For $k \ge 2$, if F_1, F_2, \ldots, F_k are factors on n vertices, each of which consist of the vertex disjoint union of n/r copies of K_r , then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \begin{cases} \left(\frac{k}{2r}\right)n & \text{if } k \text{ is even} \\ \left(\frac{r(k+1)-2}{2r^2}\right)n & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Suppose first that k is even. Consider the combined graph $G(F_{2i-1}, F_{2i})$ with only F_{2i-1} and F_{2i} as factors, where $1 \le i \le k/2$. For each such i, let D_i be a $\gamma_{sd}(F_{2i-1}, F_{2i})$ -set in $G(F_{2i-1}, F_{2i})$ and note that by Theorem 15, we have $|D_i| = n/r$. Let $D = \bigcup_{i=1}^{k/2} D_i$. Then the set D is a SD-set of $G(F_1, F_2, \ldots, F_k)$, implying that $\gamma_{sd}(F_1, F_2, \ldots, F_k) \le |D| \le kn/2r$.

Suppose next that k is odd. Let D_1 be a $\gamma_{\rm sd}(F_1, F_2, F_3)$ -set in the combined graph $G(F_1, F_2, F_3)$ with only F_1, F_2, F_3 as factors. By Corollary 19, we have $|D_1| \leq (2r-1)n/r^2$. For i with $2 \leq i \leq (k-1)/2$, consider the combined graph $G(F_{2i}, F_{2i+1})$ with only F_{2i} and F_{2i+1} as factors and let D_i be a $\gamma_{\rm sd}(F_{2i}, F_{2i+1})$ -set in $G(F_{2i}, F_{2i+1})$. By Theorem 15, we have $|D_i| = n/r$ for $2 \leq i \leq (k-1)/2$. Let $D = \bigcup_{i=1}^{(k-1)/2} D_i$. Then the set D is a SD-set of $G(F_1, F_2, \ldots, F_k)$, implying that

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le |D| \le \left(\frac{2r-1}{r^2}\right) n + \left(\frac{k-3}{2r}\right) n = \left(\frac{r(k+1)-2}{2r^2}\right) n,$$

which established the desired upper bound in this case when k is odd. \Box

We remark that the bound in Theorem 20 is strictly better than the bounds of Theorem 3 and Theorem 10(c) for $r \ge 3$. Further the bound in Theorem 20 is strictly better than the bound of Theorem 13 for $r \ge 4$.

We close this section by considering the special case when every factor in the combined graph is the disjoint union of copies of K_2 . If G is a graph of even order and if F is a 1-regular spanning subgraph of G, we call F a 1-factor of G. Hence if F is a 1-factor of a graph G of order n, then $F = \frac{n}{2}K_2$ and the edges of F form a perfect matching in G.

Theorem 21 For $k \ge 2$ and n even, if F_1, F_2, \ldots, F_k are 1-factors on n vertices, then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \leq \begin{cases} \left(\frac{k-1}{k}\right)n & \text{if } k \text{ is even} \\ \left(\frac{k}{k+1}\right)n & \text{if } k \text{ is odd.} \end{cases}$$

and these bounds are sharp.

Proof. Let $G = G(F_1, \ldots, F_k)$ denote the combined graph of the factoring F_1, F_2, \ldots, F_k and let G have vertex set V. Then, $\Delta(G) \leq k$. By Brook's Coloring Theorem, $\chi(G) \leq k+1$ with equality if and only if G has a component isomorphic to K_{k+1} or a component that is an odd cycle and k = 2.

We show that every component of G has even order. Suppose to the contrary that there is a component, F, in G of odd order. For each vertex v in V(F), let v' be its neighbor in $F_1 = \frac{n}{2}K_2$ and let $S = \bigcup_{v \in V(F)} \{v, v'\}$. Then, V(F) = S. However, |S| is even, while |V(F)|is odd, a contradiction. Therefore, every component of G has even order. In particular, no component of G is an odd cycle.

If k is odd, then by Theorem 10(c), $\gamma_{sd}(F_1, F_2, \ldots, F_k) \leq kn/(k+1)$, as desired. If k is even, then no component of G is isomorphic to K_{k+1} , implying that $\chi(G) \leq k$. This in turn implies that $\alpha(G) \geq n/\chi(G) = n/k$, and so, by Observation 7 and Theorem 10(a) we have that $\gamma_{sd}(F_1, F_2, \ldots, F_k) \leq \tau(G) = n - \alpha(G) \leq (k-1)n/k$, as desired.

That these bounds are sharp may be seen as follows. For k odd, take $n \equiv 0 \pmod{k+1}$. Then the 1-factors F_1, F_2, \ldots, F_k of K_n can be chosen so that the combined graph G consists of the disjoint union of n/(k+1) copies of K_{k+1} . Let S be an SD-set in G of minimum cardinality and let F be an arbitrary copy of K_{k+1} in G. If $|S \cap V(F)| \leq k - 1$, then there would be two vertices, u and v, in F that do not belong to S. However the edge uv belongs to one of the factor of G, implying that in such a 1-factor neither u nor v is dominated by S, a contradiction. Hence, $|S \cap V(F)| \geq k$. This is true for every copy of K_{k+1} in G. Therefore, $\gamma_{\rm sd}(F_1, F_2, \ldots, F_k) = |S| \geq kn/(k+1)$. As shown earlier, $\gamma_{\rm sd}(F_1, F_2, \ldots, F_k) \leq kn/(k+1)$. Consequently, $\gamma_{\rm sd}(F_1, F_2, \ldots, F_k) = kn/(k+1)$.

For k even, we simply take $F_{k-1} = F_k$, and note that in this case $\gamma_{\rm sd}(F_1, F_2, \ldots, F_k) = \gamma_{\rm sd}(F_1, F_2, \ldots, F_{k-1})$. Since k-1 is odd, the construction in the previous paragraph shows that the 1-factors $F_1, F_2, \ldots, F_{k-1}$ of K_n can be chosen so that the combined graph G satisfies $\gamma_{\rm sd}(F_1, F_2, \ldots, F_k) = (k-1)n/k$. \Box

We remark that the bound in Theorem 21 is better than the bound of Theorem 13 always, better than the bound of Theorem 3 for $k \ge 2$, and better than the bound of Theorem 10(c) for k even.

6 Cycle Factors

In this section, we consider the case when each factor is a cycle or a disjoint union of cycles. As a consequence of Corollary 16, we have the following upper bound on the simultaneous domination number of a combined graph with two factors, both of which are cycles or paths.

Theorem 22 The following holds.

- (a) For $n \equiv 0 \pmod{2}$ and $n \ge 4$, $\gamma_{sd}(C_n, C_n) \le n/2$ and $\gamma_{sd}(P_n, P_n) \le n/2$.
- (b) For $n \equiv 1 \pmod{2}$ and $n \ge 5$, $\gamma_{sd}(C_n, C_n) \le (n+1)/2$.

Proof. (a) For $n \equiv 0 \pmod{2}$ and $n \geq 4$, both the cycle C_n and the path P_n contains a spanning subgraph that is the vertex disjoint union of n/2 copies of K_2 , and so by Corollary 16, we have that $\gamma_{\rm sd}(C_n, C_n) \leq n/2$ and $\gamma_{\rm sd}(P_n, P_n) \leq n/2$.

(b) For $n \equiv 1 \pmod{2}$ and $n \geq 3$, let v be an arbitrary vertex in the cycle C_n . Deleting the vertex v from the cycle, we produce a path P_{n-1} , where $n-1 \equiv 0 \pmod{2}$. Applying Part (a), we have that $\gamma_{\rm sd}(P_{n-1}, P_{n-1}) \leq (n-1)/2$. Adding the deleted vertex v to a minimum SD-set in the combined graph with the two paths P_{n-1} as factors, we produce a SD-set in the original combined graph with the two cycles C_n as factors of cardinality $\gamma_{\rm sd}(P_{n-1}, P_{n-1}) + 1 \leq (n+1)/2$. \Box

For generally, we can establish the following upper bound on the simultaneous domination number of a combined graph with $k \ge 2$ factors, each of which is a cycle. For simplicity, we restrict the number of vertices to be congruent to zero modulo 6.

Theorem 23 For $k \ge 2$ and $n \equiv 0 \pmod{6}$, let F_1, F_2, \ldots, F_k be factors on n vertices, each of which is isomorphic to a cycle C_n . Then,

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(1 - \frac{1}{2} \left(\frac{2}{3}\right)^{k-2}\right) n.$$

Proof. We proceed by induction on $k \geq 2$. The base case when k = 2 follows from Theorem 22(a). Assume, then, that $k \geq 3$ and that the result holds for k' factors, each of which is isomorphic to a cycle C_n , where $2 \leq k' < k$. Let F_1, F_2, \ldots, F_k be factors on n vertices, each of which is isomorphic to a cycle C_n . First we consider the combined graph $G(F_1, F_2, \ldots, F_{k-1})$ with only $F_1, F_2, \ldots, F_{k-1}$ as factors. Let D be a $\gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$ -set in $G(F_1, F_2, \ldots, F_{k-1})$, and so $|D| = \gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$. By the inductive hypothesis,

$$|D| \le \left(1 - \frac{1}{2} \left(\frac{2}{3}\right)^{k-3}\right) n.$$

We now consider the combined graph $G(F_1, F_2, \ldots, F_k)$. Let F_k be the cycle $v_1v_2 \ldots v_nv_1$. For i = 1, 2, 3, let $D_i = \{v_j \mid j \equiv i \pmod{3}\}$. We note that for $i \in \{1, 2, 3\}$, each set D_i is a dominating set in F_k and $|D_i| = n/3$. We now extend the set D to a SD-set of $G(F_1, F_2, \ldots, F_k)$ as follows. Renaming vertices, if necessary, we may assume that

$$|D \cap D_1| = \max_{1 \le i \le 3} |D \cap D_i|$$

Thus, $|D| = \sum_{i=1}^{3} |D \cap D_i| \leq 3|D \cap D_1|$, or, equivalently, $|D \cap D_1| \geq |D|/3$. Let S be the set of vertices in D_1 that do belong to D. Then, $S = D_1 \setminus D$ and $|S| = |D_1| - |D \cap D_1| \leq n/3 - |D|/3$. Since $D_1 \subseteq D \cup S$ and D_1 is a dominating set of F_k , the set $D \cup S$ is a dominating set of F_k . Since D is a DS-set of $G(F_1, F_2, \ldots, F_{k-1})$, the set D is a dominating set in F_i for $1 \leq i \leq k-1$. Hence, $D \cup S$ is a SD-set of $G(F_1, F_2, \ldots, F_k)$, implying that

$$\begin{aligned} \gamma_{\rm sd}(F_1, F_2, \dots, F_k) &\leq |D| + |S| \\ &\leq |D| + \frac{n - |D|}{3} \\ &\leq \frac{n + 2|D|}{3} \\ &\leq \frac{1}{3} \left(n + 2 \left(1 - \frac{1}{2} \left(\frac{2}{3} \right)^{k - 3} \right) n \right) \\ &= \left(1 - \frac{1}{2} \left(\frac{2}{3} \right)^{k - 2} \right) n, \end{aligned}$$

completing the proof of the theorem. \Box

We remark that Theorem 23 is better than Theorem 21 when k = 3, since in this case the upper bound of Theorem 23 is 2n/3 while that of Theorem 21 is 3n/4.

6.1 C_4 -Factors

We consider here the case when every factor in the combined graph is the disjoint union of copies of a 4-cycle. As a consequence of Corollary 17, we have the following result.

Theorem 24 For $n \equiv 0 \pmod{4}$, let F_1 and F_2 be factors on n vertices, both of which are isomorphic to $\frac{n}{4}C_4$. Then, $\gamma_{sd}(F_1, F_2) = n/2$.

Proof. We observe that F_1 and F_2 are factors on n vertices both having a 1-factor. Further, each of the n/4 copies of C_4 in F_1 need two vertices to dominate that copy of C_4 , implying that $\gamma(F_1) \ge n/2$. The desired result now follows from Corollary 17. \Box

Theorem 25 For $n \equiv 0 \pmod{4}$, let F_1, F_2, F_3 be factors on n vertices, each of which is isomorphic to $\frac{n}{4}C_4$. Then, $\gamma_{sd}(F_1, F_2, F_3) \leq 3n/4$.

Proof. First we consider the combined graph $G(F_1, F_2)$ with only F_1 and F_2 as factors. Let D be a $\gamma_{sd}(F_1, F_2)$ -set in $G(F_1, F_2)$. By Theorem 24, |D| = n/2. We next consider the factor F_3 . For $0 \le i \le 4$, let n_i denote the number of copies of C_4 in F_3 that contain exactly *i* vertices in the set *D*. Counting the number of vertices not in *D*, we have that

$$\frac{n}{2} = n - |D| = \sum_{i=0}^{4} (4-i)n_i \ge 4n_0 + 3n_1,$$

implying that $2n_0 + n_1 \leq 2n_0 + 3n_1/2 \leq n/4$. We now extend the set D to a SD-set of $G(F_1, F_2, F_3)$ as follows. From each copy of C_4 in F_3 that contains exactly one vertex in D, we add to D the vertex that is not adjacent in F_3 to a vertex of D. From each copy of C_4 in F_3 that contains no vertex in D, we add any two vertices to D. The resulting set is a SD-set of $G(F_1, F_2, F_3)$, implying that $\gamma_{sd}(F_1, F_2, F_3) \leq |D| + 2n_0 + n_1 \leq n/2 + n/4 = 3n/4$. \Box

We remark that the bound in Theorem 24 is strictly better than the bounds of Theorem 3 and Theorem 10(c) when k = 2. The bound in Theorem 25, namely 3n/4, is better than the general probabilistic bound of Theorem 12, namely f(3, 2)n = 7n/9 (see Table 3).

6.2 C_5 -Factors

We consider here the case when every factor in the combined graph is the disjoint union of copies of a 5-cycle.

Theorem 26 For $n \equiv 0 \pmod{5}$ and $k \geq 2$, let F_1, F_2, \ldots, F_k be factors on n vertices, each of which is isomorphic to $\frac{n}{5}C_5$. Then, $\gamma_{sd}(F_1, F_2) \leq 3n/5$ and this bound is sharp. Further, for $k \geq 3$,

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-2}\right)\right) n.$$

Proof. We proceed by induction on $k \ge 2$. Let F_1 and F_2 be factors on n vertices, where both F_1 and F_2 consist of the vertex-disjoint union of n/5 copies of C_5 . Since the 5-cycle C_5 is well-dominated, we have by Theorem 14 that $\gamma_{sd}(F_1, F_2) \le \frac{1}{5}(2\gamma(C_5) - 1)n = 3n/5$. This establishes the base case when k = 2. Assume, then, that $k \ge 3$ and that the result holds for k' factors, each of which consist of the vertex disjoint union of n/5 copies of C_5 , where $2 \le k' < k$. Let F_1, F_2, \ldots, F_k be factors on n vertices, each of which is isomorphic to $\frac{n}{5}C_5$. First we consider the combined graph $G(F_1, F_2, \ldots, F_{k-1})$ with only $F_1, F_2, \ldots, F_{k-1}$ as factors. Let D' be a $\gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$ -set in $G(F_1, F_2, \ldots, F_{k-1})$, and so $|D'| = \gamma_{sd}(F_1, F_2, \ldots, F_{k-1})$. By the inductive hypothesis, $|D'| \le 3n/5$ if k = 3, while for $k \ge 4$, we have

$$|D'| \le \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-3}\right)\right) n.$$

We add vertices to D', if necessary, until the cardinality of the resulting superset D is either 3n/5 if k = 3 or is precisely the expression on the right-hand side of the above inequality if $k \ge 4$. Since D' is a SD-set of $G(F_1, F_2, \ldots, F_{k-1})$, so too is the set D. We now consider the combined graph $G(F_1, F_2, \ldots, F_k)$. For $0 \le i \le 5$, let n_i denote the number of copies of C_5 in F_k that contain exactly i vertices in the set D. Counting the number of vertices not in D, we have that

$$\frac{2}{5} \left(\frac{3}{5}\right)^{k-3} n = n - |D| = \sum_{i=0}^{5} (5-i)n_i \ge 5n_0 + 4n_1 + 3n_2 \ge 5n_0 + 5(n_1 + n_2)/2,$$

implying that

$$2n_0 + n_1 + n_2 \le \frac{4}{25} \left(\frac{3}{5}\right)^{k-3} n.$$

We now extend the set D to a SD-set of $G(F_1, F_2, \ldots, F_k)$ as follows. From each copy of C_5 in F_k that contains no vertex of D, we add two vertices that dominate that copy of C_5 . From each copy of C_5 in F_k that contains one or two vertices of D, we select one such vertex of D and we add to D a vertex from that copy of C_5 that is not adjacent in F_k to that selected vertex. The resulting set is a SD-set of $G(F_1, F_2, \ldots, F_k)$, implying that

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le |D| + 2n_0 + n_1 + n_2.$$

If k = 3, then

$$\gamma_{\rm sd}(F_1, F_2, \dots, F_k) \le \frac{3n}{5} + \frac{4n}{25} = \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-2}\right)\right) n.$$

If $k \geq 4$, then

$$\begin{split} \gamma_{\rm sd}(F_1, F_2, \dots, F_k) &\leq \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-3}\right)\right) \, n + \frac{4}{25}\left(\frac{3}{5}\right)^{k-3} n \\ &= \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-3} + \frac{2}{5}\left(\frac{3}{5}\right)^{k-3}\right)\right) \, n \\ &= \left(\frac{3}{5} + \frac{2}{5}\left(1 - \frac{3}{5}\left(\frac{3}{5}\right)^{k-3}\right)\right) \, n \\ &= \left(\frac{3}{5} + \frac{2}{5}\left(1 - \left(\frac{3}{5}\right)^{k-2}\right)\right) \, n. \end{split}$$

completing the proof of the upper bound of the theorem. That the bound is sharp when $k \geq 2$, may be seen as follows. For $r \geq 1$, let $G = rK_5$ be the disjoint union of r copies of K_5 and let G have order n. Then there exists two edge-disjoint spanning subgraphs, F_1 and F_2 , of G both of which are isomorphic to the disjoint union of r copies of C_5 . In order to simultaneously dominate the copies of C_5 in F_1 and F_2 corresponding to a copy of K_5 in G

at least three vertices are needed, implying that $\gamma_{\rm sd}(F_1, F_2) \ge 3r = 3n/5$. By Theorem 26, $\gamma_{\rm sd}(F_1, F_2) \le 3n/5$. Consequently, $\gamma_{\rm sd}(F_1, F_2) = 3n/5$ in this case. \Box

We remark that the bound in Theorem 26 is strictly better than the bounds of Theorem 3 and Theorem 10(c) when k = 2. Theorem 26 (when k = 2) implies the following result.

Theorem 27 $\gamma_{sd}(2,2,n) \ge 3n/5.$

7 Open Questions and Conjectures

Recall that in Theorem 27, we established that $\gamma_{sd}(2,2,n) \ge 3n/5$. The following conjecture was posed by Dankelmann and Laskar [8], albeit using different notation.

Conjecture 1 $\gamma_{sd}(2,2,n) = 3n/5.$

By Theorem 26, if Conjecture 1 is true, then it suffices to prove the following statement: If F_1 and F_2 are factors on n vertices both having minimum degree at least 2, then $\gamma_{\rm sd}(F_1, F_2) \leq 3n/5$.

Recall that in Theorem 22, for $n \equiv 0 \pmod{2}$ and $n \geq 4$, we show that $\gamma_{sd}(C_n, C_n) \leq n/2$ and $\gamma_{sd}(P_n, P_n) \leq n/2$. Further for $n \equiv 1 \pmod{2}$ and $n \geq 5$, $\gamma_{sd}(C_n, C_n) \leq (n+1)/2$. We pose the following problem.

Problem 1 For all $n \ge 4$, determine the exact value of $\gamma_{sd}(C_n, C_n)$ and $\gamma_{sd}(P_n, P_n)$.

Recall by Corollary 17 that if F_1 and F_2 are factors on n vertices both having a 1-factor, then $\gamma_{\rm sd}(F_1, F_2) \leq n/2$. Further, if $\max\{\gamma(F_1), \gamma(F_2)\} = n/2$, then $\gamma_{\rm sd}(F_1, F_2) = n/2$. We close with the following problem that we have yet to settle.

Problem 2 Characterize the connected factors F_1 and F_2 on n vertices that have a 1-factor and satisfy $\gamma_{sd}(F_1, F_2) = n/2$.

For *n* even, let \mathcal{G} be the family of graphs *G* whose vertex set can be partitioned into two sets *X* and *Y* such that |X| = |Y| = n/2, the set [X, Y] of edges that join a vertex of *X* and a vertex of *Y* is a 1-factor in *G*, the set *X* is independent, and the subgraph *G*[*Y*] is connected. By construction, every graph in the family \mathcal{G} is connected, has a 1-factor and has domination number one-half its order. Therefore by Corollary 17, we observe that if F_1 and F_2 are factors on *n* vertices that belong to the family \mathcal{G} , then $\gamma_{\rm sd}(F_1, F_2) = n/2$. However we have yet to provide a characterization of all factors F_1 and F_2 that meet the requirements of Problem 2.

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APPENDIX:

k	2	3	4	5	6	7
$\gamma_{\rm sd}(k,n)$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{7}{8}$	$\frac{9}{10}$	$\frac{11}{12}$

Table 1. Upper bounds on $\gamma_{\rm sd}(k,n)$ in Theorem 3 for small k.

k	2	3	4	5	6	7
$\gamma_{\rm sd}(k,n)$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

Table 2. Upper bounds on $\gamma_{\rm sd}(k,n)$ in Theorem 10(c) for small k.

			k		
		2	3	4	5
	1	0.8750	0.9167	0.9375	0.9500
	2	0.7278	0.7777	0.8075	0.8278
r	3	0.6250	0.6724	0.7023	0.7237
	4	0.5501	0.5935	0.6217	0.6432
	5	0.4930	0.5325	0.5586	0.5779

Table 3. Approximate values of $f(k, \delta)$ in Theorem 12 for small k and δ .

	i	l le				
		2	3	4	5	
	2	0.7500	0.8333	0.8750	0.9000	
r	3	0.5286	0.6151	0.6666	0.7018	
	4	0.4047	0.4800	0.5275	0.5614	
	5	0.3272	0.3921	0.4343	0.4650	

Table 4. Approximate values of $g(k, \delta)$ in Theorem 13 for small k and δ .