

# On a covering problem in the hypercube

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## Abstract

In this paper, we address a particular variation of the Turán problem for the hypercube. Alon, Krech and Szabó (2007) asked “In an  $n$ -dimensional hypercube,  $Q_n$ , and for  $\ell < d < n$ , what is the size of a smallest set,  $S$ , of  $Q_\ell$ ’s so that every  $Q_d$  contains at least one member of  $S$ ?” Likewise, they asked a similar Ramsey type question: “What is the largest number of colors that we can use to color the copies of  $Q_\ell$  in  $Q_n$  such that each  $Q_d$  contains a  $Q_\ell$  of each color?” We give upper and lower bounds for each of these questions and provide constructions of the set  $S$  above for some specific cases.

## 1 Introduction

For graphs  $Q$  and  $P$ , let  $\text{ex}(Q, P)$  denote the *generalized Turán number*, i.e., the maximum number of edges in a  $P$ -free subgraph of  $Q$ . The  $n$ -dimensional hypercube,  $Q_n$ , is the graph whose vertex set is  $\{0, 1\}^n$  and whose edge set is the set of pairs that differ in exactly one coordinate. For a graph  $G$ , we use  $n(G)$  and  $e(G)$  to denote the number of vertices and the number of edges of  $G$ , respectively.

In 1984, Erdős [8] conjectured that

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(Q_n, C_4)}{e(Q_n)} = \frac{1}{2}.$$

Note that this limit exists, because the function above is non-increasing for  $n$  and bounded. The best upper bound  $\text{ex}(Q_n, C_4)/e(Q_n) \leq 0.62256$  was obtained by Thomason and Wagner [16] by slightly improving the bound 0.62284 given by Chung [4]. Brass, Harborth and Nienborg [3] showed that the lower bound is  $\frac{1}{2}(1 + 1/\sqrt{n})$ , when  $n = 4^r$  for integer  $r$ , and  $\frac{1}{2}(1 + 0.9/\sqrt{n})$ , when  $n \geq 9$ .

Erdős [8] also asked whether  $o(e(Q_n))$  edges in a subgraph of  $Q_n$  would be sufficient for the existence of a cycle  $C_{2k}$  for  $k > 2$ . The value of  $\text{ex}(Q_n, C_6)/e(Q_n)$  is between  $1/3$  and  $0.3941$  given by Conder [6] and Lu [11], respectively. On the other hand, nothing is known for the cycle of length 10. Except  $C_{10}$ , the question of Erdős is answered positively by showing that  $\text{ex}(Q_n, C_{2k}) = o(e(Q_n))$  for  $k \geq 4$  in [4], [7] and [10].

A generalization of Erdős' conjecture above is the problem of determining  $\text{ex}(Q_n, Q_d)$  for  $d \geq 3$ . As for  $d = 2$ , the exact value of  $\text{ex}(Q_n, Q_3)$  is still not known. The best lower bound for  $\text{ex}(Q_n, Q_3)/e(Q_n)$  has been  $1 - (5/8)^{0.25} \approx 0.11086$  due to Graham, Harary, Livingston and Stout [14] until recently Offner [12] improved it to  $0.1165$ . The best upper bound is  $\text{ex}(Q_n, Q_3)/e(Q_n) \leq 0.25$  due to Alon, Krech and Szabó [1]. They also gave the best bounds for  $\text{ex}(Q_n, Q_d)$ ,  $d \geq 4$ , as

$$\Omega\left(\frac{\log d}{d^{2d}}\right) = 1 - \frac{\text{ex}(Q_n, Q_d)}{e(Q_n)} \leq \begin{cases} \frac{4}{(d+1)^2} & \text{if } d \text{ is odd,} \\ \frac{4}{d(d+2)} & \text{if } d \text{ is even.} \end{cases} \quad (1)$$

These Turán problems are also asked when vertices are removed instead of edges and most of these problems are also still open. In a very recent paper, Bollobás, Leader and Malvenuto [2] discuss open problems on the vertex-version and their relation to Turán problems on hypergraphs.

Here, we present results on a similar dual version of the hypercube Turán problem that is asked by Alon, Krech and Szabó in [1]. Let  $\mathcal{H}_n^i$  denote the collection of  $Q_i$ 's in  $Q_n$  for  $1 \leq i \leq n-1$ . Call a subset of  $\mathcal{H}_n^\ell$  a  $(d, \ell)$ -covering set if each member of  $\mathcal{H}_n^d$  contains some member of this set, i.e.,  $\mathcal{H}_n^d$  is covered by this set. A smallest  $(d, \ell)$ -covering set is called *optimal*. Alon, Krech and Szabó [1] asked what the size of the optimal  $(d, \ell)$ -covering set of  $Q_n$  is for fixed  $\ell < d$ . Call this function  $f^{(\ell)}(n, d)$ . Determining this function when  $\ell = 1$  is equivalent to the determination of  $\text{ex}(Q_n, Q_d)$ , since  $\text{ex}(Q_n, Q_d) + f^{(1)}(n, d) = e(Q_n)$  and the best bounds for  $f^{(1)}(n, d)$  are given in [1] as (1). In [1], also the Ramsey version of this problem is asked as follows. A coloring of  $\mathcal{H}_n^\ell$  is  $d, \ell$ -polychromatic if all colors appear on each copy  $Q_d$ 's. Let  $pc^{(\ell)}(n, d)$  be the largest number of colors for which there exists a  $d, \ell$ -polychromatic coloring of  $\mathcal{H}_n^\ell$ .

Let  $c^{(\ell)}(n, d)$  be the ratio of  $f^{(\ell)}(n, d)$  to the size of  $\mathcal{H}_n^\ell$ , i.e.,

$$c^{(\ell)}(n, d) = \frac{f^{(\ell)}(n, d)}{2^{n-\ell} \binom{n}{\ell}}. \quad (2)$$

One can observe that

$$c^{(\ell)}(n, d) \leq \frac{1}{pc^{(\ell)}(n, d)}, \quad (3)$$

since any color class used in a  $d, \ell$ -polychromatic coloring is a  $(d, \ell)$ -covering set of  $Q_n$ . Note that the following limits exist, since  $c^{(\ell)}(n, d)$  is non-decreasing,  $pc^{(\ell)}(n, d)$  is non-increasing and both are bounded.

$$c_d^{(\ell)} = \lim_{n \rightarrow \infty} c^{(\ell)}(n, d), \quad p_d^{(\ell)} = \lim_{n \rightarrow \infty} pc^{(\ell)}(n, d).$$

In Section 2, we obtain bounds on the polychromatic number.

**Theorem 1.** *For integers  $n > d > \ell$ , let  $0 < r \leq \ell + 1$  such that  $r = d + 1 \pmod{\ell + 1}$ . Then*

$$e^{\ell+1} \left( \frac{d+1}{\ell+1} \right)^{\ell+1} \geq \binom{d+1}{\ell+1} \geq p_d^{(\ell)} \geq \left\lceil \frac{d+1}{\ell+1} \right\rceil^r \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{\ell+1-r} \approx \left( \frac{d+1}{\ell+1} \right)^{\ell+1}.$$

In Section 3, we present the following bounds on  $c_d^{(\ell)}$  and  $c^{(\ell)}(n, d)$ .

**Theorem 2.** *For integers  $n > d > \ell$  and  $r = d - \ell \pmod{\ell + 1}$ ,*

$$\left( 2^{d-\ell} \binom{d}{\ell} \right)^{-1} \leq c_d^{(\ell)} \leq \left\lceil \frac{d+1}{\ell+1} \right\rceil^{-r} \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{-(\ell+1-r)}.$$

The determination of the exact values of  $p_d^{(\ell)}$  and  $c_d^{(\ell)}$  remains open. The lower and upper bounds on  $c^{(\ell)}(n, d)$  provided in Theorem 2 and Theorem 3, respectively, are a constant factor of each other when  $d$  and  $\ell$  have a bounded difference from  $n$ .

**Theorem 3.** *Let  $n - d$  and  $n - \ell$  be fixed finite integers, where  $d > \ell$ . Then, for sufficiently large  $n$ ,*

$$c^{(\ell)}(n, d) \leq \left\lceil \frac{r \log(n - \ell)}{\log\left(\frac{r^r}{r^r - r!}\right)} \right\rceil \frac{1 + o(1)}{2^{d-\ell} \binom{d}{\ell}},$$

where  $r = n - d$ .

Finally, we show an exact result for  $c^{(\ell)}(n, d)$  when  $d = n - 1$ .

**Theorem 4.** *For integers  $n - 1 > \ell$ ,*

$$c^{(\ell)}(n, n - 1) = \frac{\left\lceil \frac{2n}{n-\ell} \right\rceil}{2^{n-\ell} \binom{n}{\ell}}.$$

In our proofs, we make use of the following terminology. The collection of  $i$ -subsets of  $[n] = \{1, \dots, n\}$ ,  $1 \leq i \leq n$ , is denoted by  $\binom{[n]}{i}$ . For an edge  $e \in E(Q_n)$ ,  $\text{star}(e)$  denotes the coordinate that is different at endpoints of  $e$ . The set of coordinates whose values are 0 (or 1, resp.) at both endpoints of  $e$  are denoted by  $\text{zero}(e)$  (or  $\text{one}(e)$ , resp.). For a subcube  $F \subset Q_n$ ,  $\text{star}(F) := \cup_{e \subseteq E(F)} \text{star}(e)$ ,  $\text{one}(F) := \cap_{e \subseteq E(F)} \text{one}(e)$  and  $\text{zero}(F) := \cap_{e \subseteq E(F)} \text{zero}(e)$ . Note that  $E_1$  covers  $E_2$  for  $E_1 \in \mathcal{H}_n^\ell$  and  $E_2 \in \mathcal{H}_n^d$  ( $d > \ell$ ) if and only if  $\text{zero}(E_2) \subset \text{zero}(E_1)$  and  $\text{one}(E_2) \subset \text{one}(E_1)$ .

**Definition 5.** For any  $Q \in \mathcal{H}_n^\ell$  and  $\text{star}(Q)$  with coordinates  $s_1 < s_2 < \dots < s_\ell$ , we define an  $(\ell + 1)$ -tuple  $w(Q) = (w_1, w_2, \dots, w_{\ell+1})$  as

- $w_1 = |\{x \in \text{one}(Q) : x < s_1\}|$ ,
- $w_j = |\{x \in \text{one}(Q) : s_{j-1} < x < s_j\}|$ , for  $2 \leq j \leq \ell$ ,
- $w_{\ell+1} = |\{x \in \text{one}(Q) : x > s_\ell\}|$ .

## 2 Polychromatic Coloring of Subcubes

**Proof of Theorem 1.** *The lower bound:*

For any  $Q \in \mathcal{H}_n^\ell$  with  $w(Q) = (w_1, w_2, \dots, w_{\ell+1})$ , we define the color of each  $Q \in \mathcal{H}_n^\ell$  as the  $(\ell + 1)$ -tuple  $c(Q) = (c_1, \dots, c_{\ell+1})$  such that

$$\begin{aligned} c_i &= w_i \pmod{k} & \text{if } 1 \leq i \leq r \text{ and} \\ c_i &= w_i \pmod{k'} & \text{if } r + 1 \leq i \leq \ell + 1, \end{aligned} \tag{4}$$

where  $k = \lceil (d + 1)/(\ell + 1) \rceil$  and  $k' = \lfloor (d + 1)/(\ell + 1) \rfloor$ . We show that this coloring is  $d, \ell$ -polychromatic.

Let  $C \in \mathcal{H}_n^d$ , where  $\text{star}(C)$  consists of the coordinates  $a_1 < a_2 < \dots < a_d$ . We choose a color  $(c_1, \dots, c_{\ell+1})$  arbitrarily and show that  $C$  contains a copy of  $Q_\ell$ , call it  $Q$ , with this color.

Since  $Q$  must be a subgraph of  $C$ ,  $\text{zero}(C) \subset \text{zero}(Q)$  and  $\text{one}(C) \subset \text{one}(Q)$ . We define  $\text{star}(Q) = \{s_1, \dots, s_\ell\}$  such that

$$s_i = \begin{cases} a_{ik} & \text{if } 1 \leq i \leq r, \\ a_{rk+(i-r)k'} & \text{if } r + 1 \leq i \leq \ell. \end{cases}$$

We include the remaining  $d - \ell$  positions of  $\text{star}(C)$  to  $\text{one}(Q)$  or  $\text{zero}(Q)$  such that  $w(Q) = (w_1, w_2, \dots, w_{\ell+1})$  satisfies (4). This is possible since by the definition of  $r$ , we have  $d - \ell = r(k - 1) + (\ell + 1 - r)(k' - 1)$ .

*The upper bound:*

Since  $pc^{(\ell)}(n, d)$  is a non-increasing function of  $n$ , we provide an upper bound for this function when  $n$  is sufficiently large which is also an upper bound for  $p_d^{(\ell)}$ .

For a subset  $S$  of  $[n]$ , we define  $\text{cube}(S)$  as the subcube  $Q$  of  $Q_n$  such that  $\text{star}(Q) = S$  and  $\text{zero}(Q) = [n] \setminus S$ . Let  $\mathcal{G}$  be a subfamily of  $\mathcal{H}_n^d$  such that  $\mathcal{G} = \{\text{cube}(S) : S \in \binom{[n]}{d}\}$ . We define a coloring of the members of  $\mathcal{G}$  as follows.

Consider a  $d, \ell$ -polychromatic coloring of  $\mathcal{H}_n^\ell$  using  $p$  colors, call this coloring  $P$ . Fix an arbitrary ordering of the copies of  $Q_\ell$ 's in  $Q_d$ . We define a coloring of  $\mathcal{H}_n^d$  such that the color of a copy of  $Q_d$  is the list of colors of each  $Q_\ell$  under  $P$  in this fixed order. By using this coloring on the members of  $\mathcal{G}$ , we obtain a coloring of  $\mathcal{G}$  using  $p \binom{d}{\ell} 2^{d-\ell}$  colors.

Now, consider the auxiliary  $d$ -uniform hypergraph  $\mathcal{G}'$  whose vertex set is the set of coordinates  $[n]$  and whose edge set is defined as the collection of  $\text{star}(E)$ 's for each  $E$  in  $\mathcal{G}$ , i.e.,  $\mathcal{G}'$  is a complete  $d$ -uniform hypergraph on the vertex set  $[n]$ . Also we define a coloring of the edges of  $\mathcal{G}'$  by using the colors on the corresponding members of  $\mathcal{G}$  as described above. Ramsey's theorem on hypergraphs implies that there is a sufficiently large value of  $n$  such that there exists a complete monochromatic subgraph on  $d^2 + d - 1$  vertices in any edge coloring of  $\mathcal{G}'$  with  $p^{\binom{d}{\ell} 2^{d-\ell}}$  colors. Let  $K \subset [n]$  be the vertex set of a monochromatic complete subgraph of  $\mathcal{G}'$  on  $d^2 + d - 1$  vertices. We define  $S$  as the collection of  $i$ <sup>th</sup> coordinates in  $K$ ,  $1 \leq i \leq d$ , so that there are at least  $d - 1$  coordinates between elements of  $S$ .

**Claim 6.** *If  $Q$  is a copy of  $Q_\ell$  in  $\text{cube}(S)$ , then the color of  $Q$  under  $P$  depends only on  $w(Q)$ .*

*Proof.* Let  $E_1$  and  $E_2$  be two different copies of  $Q_\ell$  in  $\text{cube}(S)$  such that  $w(E_1) = w(E_2)$  according to Definition 5. There exists a subset  $S' \subset K$  with  $|S'| = d$  such that

- $(\text{one}(E_2) \cup \text{star}(E_2)) \subset S'$ , i.e.,  $E_2$  is contained in  $\text{cube}(S')$  and
- the restriction of  $E_2$  on  $S'$  gives the same vector as the restriction of  $E_1$  on  $S$ .

Clearly, one can find  $S'$  that satisfies the first condition. It is also possible that  $S'$  fulfills the second condition, since we can remove or add up to  $d - 1$  coordinates from  $K$  between consecutive coordinates of ones and stars in  $E_2$  to define  $S'$ . This implies that the colors of  $E_1$  and  $E_2$  are the same under  $P$ , since  $\text{cube}(S)$  and  $\text{cube}(S')$  have the same colors.  $\square$

Hence, the number of colors used in any  $d, \ell$ -polychromatic coloring of  $\mathcal{H}_n^\ell$  is at most the number of possible vectors  $w(Q)$  for any  $Q \in \mathcal{H}_n^\ell$ . The number of possible  $(\ell + 1)$ -tuples  $w(Q)$  for any  $Q \in \mathcal{G}$  is given by the number of partitions of at most  $d - \ell$  ones into  $\ell + 1$  parts and therefore it is at most  $\binom{d+1}{\ell+1}$ .  $\square$

### 3 The Covering Problem

**Proof of Theorem 2.** Note that a trivial lower bound on  $f^{(\ell)}(n, d)$  is given by the ratio of  $|\mathcal{H}_n^d|$  to the exact number of  $Q_d$ 's that a single  $Q_\ell$  covers in  $Q_n$ . Thus, by (2), for all  $n$ ,

$$c^{(\ell)}(n, d) \geq \left\lceil \frac{2^{n-d} \binom{n}{d}}{\binom{n-\ell}{n-d}} \right\rceil \cdot \frac{1}{2^{n-\ell} \binom{n}{\ell}}. \quad (5)$$

By using the equality  $\binom{n}{d} \binom{d}{d-\ell} = \binom{n}{\ell} \binom{n-\ell}{d-\ell}$ , we are done.

The upper bound is implied together by (3) and Theorem 1.  $\square$

We define a  $(0, 1)$ -labelling of a set as an assignment of labels 0 or 1 to its elements.

**Observation 7.** *Since any subcube  $Q \subset Q_n$  is defined by  $\text{zero}(Q)$  and  $\text{one}(Q)$ , a  $(d, \ell)$ -covering set of  $Q_n$  can be defined as a collection of  $(0, 1)$ -labellings of sets chosen from  $\binom{[n]}{n-\ell}$  such that any  $(0, 1)$ -labelling of sets in  $\binom{[n]}{n-d}$  is contained in at least one of the labelled  $(n - \ell)$ -sets.*

When providing constructions for the upper bounds in Theorems 3 and 4, we provide constructions for the equivalent covering problem in Observation 7.

**Proof of Theorem 3.**

We construct a  $(d, \ell)$ -covering of  $Q_n$  by providing a construction for the equivalent problem as stated in Observation 7. In the following, we describe this construction in two steps. First, we choose the  $(n - \ell)$ -subsets of  $[n]$  to label and then, we describe an efficient way to  $(0, 1)$ -label these sets.

*Step 1:* We make use of the following well-known result on the general covering problem. An  $(n, k, t)$ -covering is defined as a collection of  $k$ -subsets of  $n$  elements such that every  $t$ -set is contained in at least one  $k$ -set. Let  $C(n, k, t)$  be the minimum number of  $k$ -sets in an  $(n, k, t)$ -covering. Rödl proved the following result by also settling a long-standing conjecture of Erdős and Hanani [9]. For any fixed integers  $k$  and  $t$  with  $2 \leq t < k < n$ ,

$$\lim_{n \rightarrow \infty} \frac{C(n, k, t)}{\binom{n}{t} / \binom{k}{t}} = 1. \tag{6}$$

By our assumption,  $n - d$  and  $n - \ell$  are fixed integers where  $n - d < n - \ell$ . By (6), there exists a  $(n, n - \ell, n - d)$ -covering  $\mathcal{F}$  for sufficiently large  $n$  such that  $|\mathcal{F}| = (1 + o(1)) \binom{n}{n-d} / \binom{n-\ell}{n-d}$ .

*Step 2:* We obtain a collection of  $(0, 1)$ -labellings for each edge  $e \in \mathcal{F}$  so that all  $(0, 1)$ -labellings of  $(n - d)$ -subsets of  $e$  are covered. The union of these  $(0, 1)$ -labellings is a covering set.

An  $r$ -cut of an  $r$ -uniform hypergraph is obtained by partitioning its vertex set into  $r$  parts and taking all edges that meet every part in exactly one vertex. An  $r$ -cut cover of a hypergraph is a collection of  $r$ -cuts such that each edge is in at least one of the cuts. An upper bound on the minimum size of an  $r$ -cut cover is shown by Cioabă, Kündgen, Timmons and Vysotsky in [5] using a probabilistic proof.

**Theorem 8** ([5]). *For every  $r$ , an  $r$ -uniform complete hypergraph on  $n$  vertices can be covered with  $\lceil c \log n \rceil$   $r$ -cuts if*

$$c > \frac{-r}{\log \left( \frac{r^r - r!}{r^r} \right)}.$$

For a fixed edge  $e$  of  $\mathcal{F}$ , let  $\mathcal{G}_e$  be the complete  $(n - d)$ -uniform hypergraph on the vertex set of  $e$ . Let  $C = \lceil c \log(n - \ell) \rceil$  be the size of a minimum  $(n - d)$ -cut cover of  $\mathcal{G}_e$  as given by Theorem 8. We obtain a collection of  $(0, 1)$ -labellings of  $e$  by labelling each cut in this cover such that the vertices in each part are labelled identically with 0 or 1. Thus, the total number of  $(0, 1)$ -labellings of  $e$  is  $2^{n-d} C$ . (If some labelling of an edge is used more than once, then we count this labelling only once.) Finally, we use similarly labellings for each edge of  $\mathcal{F}$  in the covering set. This yields that

$$c^{(\ell)}(n, d) \leq \frac{1}{2^{n-\ell} \binom{n}{\ell}} (C(1 + o(1)) 2^{n-d} \frac{\binom{n}{n-d}}{\binom{n-\ell}{n-d}}) = C(1 + o(1)) \frac{1}{2^{d-\ell} \binom{d}{\ell}},$$

where the last equality is obtained by using the relation  $\binom{n}{d}\binom{d}{d-\ell} = \binom{n}{\ell}\binom{n-\ell}{d-\ell}$ .

□

#### Proof of Theorem 4.

The lower bound follows from (5).

For the upper bound, we construct a collection of  $(0, 1)$ -labellings of sets chosen from  $\binom{[n]}{n-\ell}$ , where singletons in  $[n]$  have both 0 and 1 in some labelling. Let  $k = \lceil n/(n-\ell) \rceil$ . We choose a partition  $[n] = (P_1, \dots, P_k)$  such that  $|P_i| = n-\ell$  for  $i < k$ . Let  $P \in \binom{[n]}{n-\ell}$  such that  $P_k \subset P$ . In the covering set, we include two labellings of each of  $P_1, \dots, P_{k-1}, P$ , where all labels are the same, either 0 or 1.

□

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