

Edge Decompositions of Hypercubes by Paths and by Cycles

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Abstract

If H is isomorphic to a subgraph of G , we say that H *divides* G if there exist embeddings $\theta_1, \theta_2, \dots, \theta_k$ of H such that

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \dots, E(\theta_k(H))\}$$

is a partition of $E(G)$. For purposes of simplification we will often omit the embeddings, saying that we have an edge decomposition by copies of $E(H)$.

Many authors have studied this notion for various subgraphs of hypercubes. We continue such a study in this paper.

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1 Introduction and Preliminary Results

Definition 1 *If H is isomorphic to a subgraph of G , we say that H divides G if there exist embeddings $\theta_1, \theta_2, \dots, \theta_k$ of H such that*

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \dots, E(\theta_k(H))\}\}$$

is a partition of $E(G)$.

Ramras [8] has defined a more restrictive concept.

Definition 2 *A fundamental set of edges of a graph G is a subset of $E(G)$ whose translates under some subgroup of the automorphism group of G partition $E(G)$.*

Edge decompositions of graphs by subgraphs have a long history. For example, there is a Steiner triple system of order n if and only if the complete graph K_n has an edge-decomposition by K_3 . In 1847 Kirkman [5] proved that for a Steiner triple system to exist it is necessary that $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$. In 1850 he proved the converse holds also [6].

Theorem 1 *A Steiner system of order $n \geq 3$ exists if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$.*

In more modern times (1964) G. Ringel [11] stated the following conjecture, which is still open.

Conjecture 1 *If T is a fixed tree with m edges then K_{2m+1} is edge-decomposable into $2m + 1$ copies of T .*

By Q_n we mean the n -dimensional hypercube. We regard its vertex set, $V(Q_n)$, as $\mathcal{P}(\{1, 2, \dots, n\})$, the set of subsets of $\{1, 2, \dots, n\}$. Two vertices x and y are considered adjacent (so $\langle x, y \rangle \in E(Q_n)$) if $|x \Delta y| = 1$, where Δ denotes the symmetric difference of the two subsets x and y . $(V(Q_n), \Delta)$ is isomorphic as a group to $(\mathbb{Z}_2^n, +)$. Occasionally, when convenient, we shall use the vector notation for vertices; thus \vec{x} and \vec{y} are adjacent precisely when they differ in exactly one component. Note that for $k < n$, $\mathcal{P}(\{1, 2, \dots, k\}) \subset \mathcal{P}(\{1, 2, \dots, n\})$ so that $V(Q_k) \subset V(Q_n)$. In fact, from the definition of adjacency, it follows that Q_k is an induced subgraph of Q_n .

Beginning in the early 1980's, interest in hypercubes (and similar hypercube-like networks such as "cube-connected cycles" and "butterfly" networks) increased dramatically with the construction of massively parallel-processing

computers, such as the “Connection Machine” whose architecture is that of the 16-dimensional hypercube, with $2^{16} = 65,536$ processors as the vertices. Problems of routing message packets simultaneously along paths from one processor to another led to an interest in questions of edge decompositions of $E(Q_n)$ by paths. An encyclopedic discussion of this and much more can be found in [7].

In [8] we have shown that if \mathcal{G} is a subgroup of $\text{Aut}(Q_n)$ and for all $g \in \mathcal{G}$, with $g \neq id$ (where id denotes the identity element), $g(E(H)) \cap E(H) = \emptyset$, then there is a packing of these translates of $E(H)$ in Q_n , *i.e.* they are pairwise disjoint. If, in addition, $|E(H)| \cdot |\mathcal{G}| = n \cdot 2^{n-1} = |E(Q_n)|$, then the translates of $E(H)$ by the elements of \mathcal{G} yield an edge decomposition of Q_n . In [8] it is shown that every tree on n edges can be embedded in Q_n as a fundamental set. (This result for edge decompositions was obtained independently by Fink [3]). In [9] this is extended to certain trees and certain cycles on $2n$ edges. Decompositions of Q_n by k -stars are proved for all $k \leq n$ in [2]. Recently, Wagner and Wild [12] have constructed, for each value of n , a tree on 2^{n-1} edges that is a fundamental set for Q_n . The structure of $\text{Aut}(Q_n)$ is discussed in [8]. For each subset A of $\{1, 2, \dots, n\}$, the complementing automorphism σ_A is defined by $\sigma_A(x) = A\Delta\{x\}$. Another type of automorphism arises from the group of permutations \mathcal{S}_n of $\{1, 2, \dots, n\}$. For $x = \{x_1, x_2, \dots, x_m\} \subseteq \{1, 2, \dots, n\}$ and $\theta \in \mathcal{S}_n$ we denote by $\rho_{\theta(x)}$ the vertex $\{\theta(x_1), \theta(x_2), \dots, \theta(x_m)\}$. The mapping $\rho_\theta : V(Q_n) \rightarrow V(Q_n)$ defined in this way is easily seen to belong to $\text{Aut}(Q_n)$. Every automorphism in $\text{Aut}(Q_n)$ can be expressed uniquely in the form $\sigma_A \circ \rho_\theta$, where this notation means that we first apply ρ_θ . Note: $\rho_\theta \circ \sigma_A = \sigma_{\theta(A)} \circ \rho_\theta$.

To avoid ambiguity in what follows we make this definition:

Definition 3 *By P_k , the “ k -path”, we mean the path with k edges.*

Questions

- (1) For which k dividing $n \cdot 2^{n-1}$ does P_k divide Q_n ?
- (2) For which k dividing $n \cdot 2^{n-1}$ does C_k , the cycle on k edges, divide Q_n ?
- (3) For those k for which the answer to either (1) or (2) is “yes”, is the edge set used in the decomposition a fundamental set for Q_n ?

We begin this introductory section with some examples. In later sections we prove a variety of results relating to these questions, and in the final section we summarize our findings.

Example 1

Let T be the 2-star (= the 2-path) contained in Q_3 with center 000, and leaves 100, 010. Then $\mathcal{G} = \{id, \sigma_{123}, \sigma_1\rho_{(123)}, \sigma_{12}\rho_{(132)}, \sigma_3\rho_{(132)}, \sigma_{23}\rho_{(123)}\}$ is a (cyclic) subgroup of $\text{Aut}(Q_3)$ of order 6, and the 6 translates of T under \mathcal{G} yield an edge decomposition of Q_3 . \square

Note, however, that \mathcal{G} does not work for the 2-star T' , whose center is 000 and whose leaves are 100 and 001. The subgroup which works for this 2-star is $\mathcal{G}' = \{id, \sigma_{123}, \sigma_1\rho_{(132)}, \sigma_{13}\rho_{(123)}, \sigma_2\rho_{(123)}, \sigma_{23}\rho_{(132)}\}$.

Example 2

P_6 does not divide Q_3 . For since Q_3 has 12 edges, if P_6 *did* divide Q_3 then Q_3 would have an edge-decomposition consisting of 2 copies of P_6 . The degree sequence (in decreasing order) of each P_6 is 2, 2, 2, 2, 2, 1, 1, 0, whereas Q_3 , of course, is 3-regular. Thus the vertex of degree 0 in one P_6 would require a degree of 3 in the other, which is impossible. \square

Example 3

P_4 does not divide Q_3 . Since P_4 has 4 edges, we would need 3 copies of P_4 for an edge-decomposition of Q_3 . Call the three copies of P_4 $P^{(1)}$, $P^{(2)}$, and $P^{(3)}$. At each vertex v of Q_3 , $\sum_{1 \leq i \leq 3} \deg_{P^{(i)}}(v) = 3$. Label the vertices of Q_3 (v_1) to (v_8) such that the degree sequence of $P^{(1)}$, is decreasing. Consider the 3×8 array $\deg_{P^{(i)}}(v_j)$. The first row is thus 2 2 2 1 1 0 0 0. In the second and third rows, in order for the column sums to be 3, there must be exactly 3 1's (and 3 0's) in the first 3 columns. Similarly, in the last 3 columns there must be exactly 3 1's (and 3 0's). Thus in the second and third rows we have at least 6 1's, and so at least one of these rows must have at least 3 1's. But each row is a permutation of the first, which has only 2 1's. Contradiction. Hence P_4 does not divide Q_3 . \square

Example 4

Since Q_3 is 3-regular, the 4-star is not a subgraph. The *other* tree on 4 edges *does* divide Q_3 . Let T be the 3-star centered at 000 union the edge $\langle 001, 101 \rangle$. Let $\mathcal{G} = \langle \sigma_{23}\rho_{(123)} \rangle$, which is a cyclic subgroup of $\text{Aut}(Q_3)$ of order 3. A straight-forward calculation shows that the translates of T under \mathcal{G} form an edge decomposition of Q_3 . \square

Proposition 1 For $k \geq 3$, P_{2^k} does not divide $Q_{2^{k+1}}$.

Proof. Suppose that $k \geq 3$, and suppose that P_{2^k} divides $Q_{2^{k+1}}$. The matrix (a_{iv}) formed by the degree sequences of copies of P_{2^k} has $2^{2^{k+1}}$ columns, and

$$(2k + 1) \cdot 2^{2^k} / 2^k = (2k + 1)2^k$$

rows. Then since each row has exactly two 1's, the entire matrix has $(2k + 1)2^{k+1}$ 1's. But since each vertex of $Q_{2^{k+1}}$ has degree $2k + 1$, each column sum is $2k + 1$, and thus each column has at least one 1. Thus there must be at least $2^{2^{k+1}}$ 1's in the matrix. Therefore, $(2k + 1)2^{k+1} \geq 2^{2^{k+1}}$. This is equivalent to $2k + 1 \geq 2^k$. But for $k \geq 3$ this is clearly false. Thus for $k \geq 3$, P_{2^k} does not divide $Q_{2^{k+1}}$. \square

We will prove in Section 3 that for $k = 2$, P_{2^k} does divide $Q_{2^{k+1}}$.

The next result is Proposition 8 of [9].

Proposition 2 Let n be odd, and suppose that P_k divides Q_n . Then $k \leq n$.

Lemma 1 “Divisibility” is transitive, i.e. if G_1 divides G_2 and G_2 divides G_3 , then G_1 divides G_3 .

Proof. This follows immediately from the definition of “divides”. \square

Corollary 1 If k divides n then P_k divides Q_n .

Proof. By [8], Theorem 2.3, T divides Q_n for every tree T on n edges. In particular, then, P_n divides Q_n . Clearly, if k divides n then P_k divides P_n . Hence, by Lemma 1, P_k divides Q_n . \square

We have the following partial converse.

Proposition 3 If P_k divides Q_n and k is odd, then k divides n .

Proof. Since P_k divides Q_n , k divides $n \cdot 2^{n-1}$. But since k is odd, this means that k divides n . \square

Definition 4 If G_1 and G_2 are graphs then by $G_1 \square G_2$ we mean the graph that is the Cartesian product of G_1 and G_2 .

Lemma 2 *If H divides G_1 and H divides G_2 then H divides $G_1 \square G_2$.*

Proof. This is obvious because $E(G_1 \square G_2)$ consists of $|V(G_1)|$ copies of $E(G_2)$ and $|V(G_2)|$ copies of $E(G_1)$. \square

Proposition 4 *If k divides n then Q_k divides Q_n .*

Proof. Let $n = mk$. We argue by induction on m . The statement is obvious for $m = 1$. Now let $m > 1$ and assume the statement is true for $m - 1$. The desired result follows from Lemma 2 and the fact that $Q_{(m-1)k} \square Q_k \simeq Q_{(m-1)k+k} = Q_{mk}$. \square

The converse to Proposition 4 follows easily from the next lemma.

Lemma 3 *Suppose that the subgraph H of G edge-divides G . If G is n -regular and H is k -regular, then k divides n .*

Proof. Since the copies of $E(H)$ form an edge-partition of $E(G)$, each vertex v of H must belong to exactly n/k copies of H and so k divides n . \square

Corollary 2 *If Q_k divides Q_n then k divides n .*

Proof. Since Q_k is k -regular and Q_n is n -regular, this follows immediately from Lemma 3. \square

Combining Proposition 4 and Corollary 2 we obtain

Proposition 5 *Q_k divides Q_n if and only if k divides n .*

As an immediate consequence of Lemma 1 and Proposition 4 we have

Corollary 3 *If k divides n and if P_j divides Q_k then P_j divides Q_n .*

We have a more general consequence.

Corollary 4 *If k divides n and T is any tree on k edges, then there is an embedding of T which divides Q_n .*

Proof. By [8], Theorem 2.3, by mapping any given vertex of T to \emptyset and assigning distinct labels $1, 2, \dots, k$ to the edges of T we get a subtree of Q_k isomorphic to T that divides Q_k . Hence by Lemma 1 and Proposition 4, T divides Q_n . \square

Proposition 6 *If n is even, and $j < n$ then P_{2^j} divides Q_n .*

Proof. It is proved in [1] that the cycle C_{2^n} divides Q_n . The Hamiltonian cycle C_{2^n} is divisible by any path P_q , as long as q divides 2^n and $q < 2^n$. Thus C_{2^n} is divisible by P_{2^j} provided $j < n$. The result now follows from Lemma 1. \square

Proposition 7 *If n is even, and C is the $2n$ -cycle with initial vertex \emptyset , and edge direction sequence $(1, 2, \dots, n)^2 \stackrel{\text{def}}{=} (1, 2, \dots, n, 1, 2, \dots, n)$, then Q_n is edge-decomposed by the copies of C under the action of $\mathcal{G} = \{\sigma_A \mid A \subset \{1, 2, \dots, n-1\}, |A| \text{ even}\}$. So $E(C)$ is fundamental for Q_n .*

Proof. C consists of the path P , followed by $\sigma_{\{1,2,\dots,n\}}(P)$, where P is the path with initial vertex \emptyset and edge direction sequence $1, 2, \dots, n$. Note that for any $B \subseteq \{1, 2, \dots, n\}$, for any edge e , $\sigma_B(e) = e$ implies that $B = \emptyset$ or $|B| = 1$. Now we shall show that for every subset $A \subset \{1, 2, \dots, n-1\}$ with $|A|$ even, $\sigma_A(C) \cap C = \emptyset$. It should be noted that these A 's form a subgroup of $\text{Aut}(Q_n)$ of order 2^{n-2} . So suppose that $e = \langle x, y \rangle \in C \cap \sigma_A(C)$. Let the direction of e be i . Then the direction of $\sigma_A(e)$ is i . If $A \neq \emptyset$, then since $|A|$ is even, $\sigma_A(e) \neq e$. The only other edge in C with direction i is $\sigma_{\{1,2,\dots,n\}}(e)$. So if $\sigma_A(e) \in C$, then $\sigma_A(e) = \sigma_{\{1,2,\dots,n\}}(e)$. Therefore $\sigma_A \cdot \sigma_{\{1,2,\dots,n\}}(e) = e$, i.e. $\sigma_{A\Delta\{1,2,\dots,n\}}(e) = e$. Since A and $\{1, 2, \dots, n\}$ are even, so is $A\Delta\{1, 2, \dots, n\} = \bar{A}$. Hence $A\Delta\{1, 2, \dots, n\} = \emptyset$, i.e. $A = \{1, 2, \dots, n\}$. But $n \notin A$, so we have a contradiction.

Thus we have a group \mathcal{G} of automorphisms of C of order 2^{n-2} , such that for $g \in \mathcal{G}, g \neq id$, $g(E(C)) \cap E(C) = \emptyset$. Furthermore, since $|E(C)| = 2n$, it follows that $|\mathcal{G}| \cdot |E(C)| = |E(Q_n)|$. Hence by [8], Lemma 1.1, the translates of $E(C)$ via the elements of \mathcal{G} form an edge decomposition of Q_n . \square

Corollary 5 *If n is even, $k < n$ and k divides n , then P_{2k} divides Q_n .*

Proof. Since k divides n , $2k$ divides $2n$, and thus since $2k < 2n$, P_{2k} divides the $2n$ -cycle C of Proposition 7. Hence by Proposition 7, P_{2k} divides Q_n . \square

Corollary 6 *If n and k are both even and k divides n , and C is the $2k$ -cycle with initial vertex \emptyset , and edge direction sequence $(1, 2, \dots, k)^2$, then C divides Q_n .*

Proof. By the proposition, C divides Q_k , and by Proposition 4, Q_k divides Q_n . The result now follows from Lemma 1. \square

2 P_4 divides Q_5

If k is odd then by Proposition 3 and Lemma 1 P_k divides Q_n and only if k divides n . Thus the smallest value of k for which Question (1) remains open is $k = 4$. Corollary 5 settles the matter in the affirmative when n is even and thus we now only need to consider the case of n odd. Example 3 shows that P_4 does *not* divide Q_3 .

In the next two sections we show that for all odd n with $n \geq 5$, P_4 divides Q_n . We first, in this section, prove the result for $n = 5$. The strategy is to find a subgraph G of Q_5 , show that G divides Q_5 , and then show that P_4 divides G . In the next section we deduce the general case.

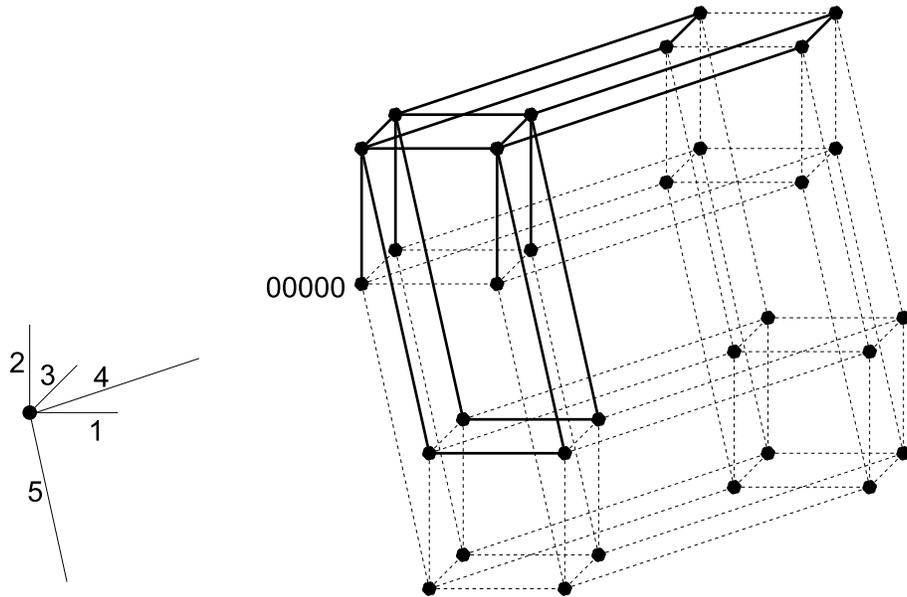


Figure 1: Q_5 and the subgraph G

We define G as follows (see figure 1). First, some notation. For $b, c \in \{0, 1\}$, $Q_5^{(**bc)}$ denotes the 3-cube induced by the vertices $x_1x_2x_3x_4x_5$ with $x_4 = b$ and $x_5 = c$. If $a \in \{0, 1\}$ $Q_5^{(**abc)}$ is the 2-cube induced by the vertices with $x_3 = a, x_4 = b$, and $x_5 = c$. We take G to be the union of (1) : $Q_5^{(**00)}$, with the edges of $Q_5^{(**00)}$ deleted; (2) : $Q_5^{(**10)}$ with all edges deleted except for $\langle 01010, 01110 \rangle$ and $\langle 11010, 11110 \rangle$; (3) : $Q_5^{(**01)}$ with all edges deleted except for $\langle 01101, 11101 \rangle$ and $\langle 01001, 11001 \rangle$; (4) : the 4 matching

edges between $Q_5^{(*1*00)}$ and $Q_5^{(*1*10)}$; and (5) the 4 matching edges between $Q_5^{(*1*00)}$ and $Q_5^{(*1*01)}$. Thus $|E(G)| = 20$. Since $|E(Q_5)| = 5 \cdot 2^4 = 80$, we must exhibit $80/20 = 4$ copies of $E(G)$ that partition $E(Q_5)$.

Lemma 4 G divides Q_5 . In fact, $E(G)$ is a fundamental set for Q_5 .

Proof. By direct inspection of figure 2 the group of translations $\mathcal{G} = \{id, \sigma_{24}, \sigma_{25}, \sigma_{45}\}$, applied to $E(G)$, partitions $E(Q_5)$. \square

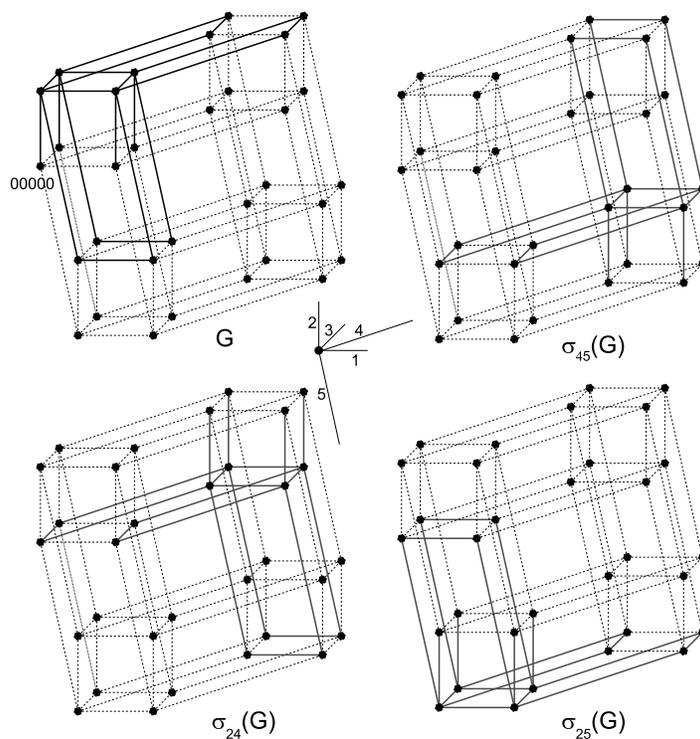


Figure 2: $E(G)$ is a fundamental set for Q_5

Lemma 5 P_4 divides G .

Proof. It is easiest to describe the paths by their starting points and direction sequences (see figure 3).

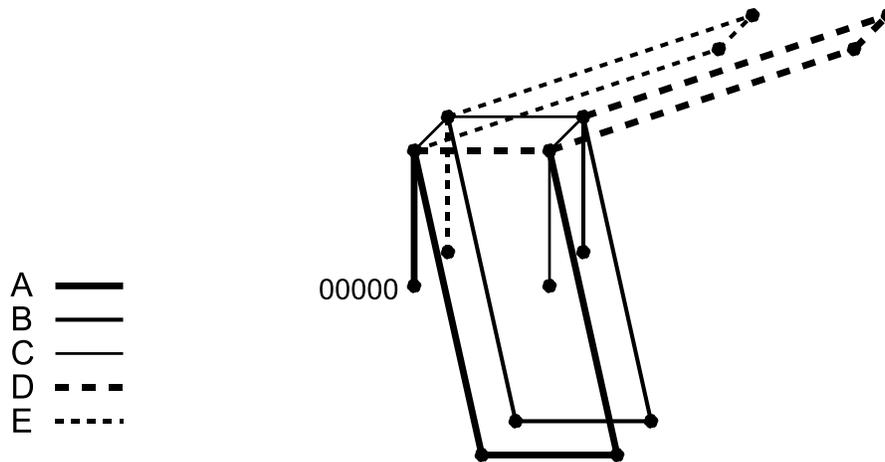


Figure 3: P_4 divides G

Path	Starting Point	Direction Sequence
A	00000	2, 5, 1, 5
B	10100	2, 5, 1, 5
C	10000	2, 3, 1, 3
D	01000	1, 4, 3, 4
E	00100	2, 4, 3, 4

□

Corollary 7 P_4 divides Q_5 .

Proof. This follows immediately from the previous two lemmas. □

3 P_4 divides Q_n , for n odd, $n \geq 5$

Let us write Q_5 as $Q_5 = Q_3 \square Q_2 = Q_3 \square C_4$. Let $G_0 = Q_5^{(***00)}$, $G_1 = Q_5^{(***10)}$, $G_2 = Q_5^{(***11)}$, $G_3 = Q_5^{(***01)}$. For $i \in \{0, 1, 2, 3\}$ let π_i be the canonical mapping from G_i to Q_3 .

* From the decomposition of Q_5 by P_4 we have a coloring $c : Q_5 \rightarrow \{1, 2, \dots, 20\}$ of the edges of Q_5 such that for any $i \in \{1, 2, \dots, 20\}$ the set of edges of Q_5 colored i induces a P_4 .

* Consider now $Q_3 \square C_{4k}$ for some $k \geq 1$. Let $G'_0, \dots, G'_{4k-1} \simeq Q_3$. Let $\pi_{i'}$ be the canonical mapping from $G'_{i'} \rightarrow Q_3$ for $i' \in \{0, 1, \dots, 4k-1\}$.

The edges of $Q_3 \square C_{4k}$ are

Case A: the edges of $G'_{i'}$ for any $i' \in \{0, 1, \dots, 4k-1\}$.

Case B: for any $i' \in \{0, 1, \dots, 4k-1\}$ the edges $\langle x', y' \rangle$ for $x' \in G'_{i'}$, $y' \in G'_{j'}$, where $|j' - i'| \equiv 1 \pmod{4k}$ and $\pi_{i'}(x') = \pi_{j'}(y')$.

* Let θ be the mapping from $Q_3 \square C_{4k} \rightarrow Q_5$ defined by: for any $x' \in G'_{i'}$, $\theta(x') = x$ where x is the element of G_i , with $i \equiv i' \pmod{4}$ such that $\pi_i(x) = \pi_{i'}(x')$. (Note that θ is not a one-to-one mapping.)

Proposition 8 *If $\langle x', y' \rangle$ is an edge of $Q_3 \square C_{4k}$ then $\langle \theta(x'), \theta(y') \rangle$ is an edge of Q_5 .*

Proof.

Case A

$\langle x', y' \rangle \in G'_{i'}$, for some i' . Then let $i \equiv i' \pmod{4}$. By the definition of θ , $\theta(x') \in G_i$, $\theta(y') \in G_i$. This implies that $\theta(x')$ and $\theta(y')$ are adjacent.

Case B

Assume $x' \in G'_{i'}$, $y' \in G'_{j'}$, with $|j' - i'| \equiv 1 \pmod{4k}$. We have $\pi_{i'}(x') = \pi_{j'}(y')$. Then $\theta(x') \in G_i$ and $\theta(y') \in G_j$ where $|j - i| \equiv 1 \pmod{4}$ since $|j' - i'| \equiv 1 \pmod{4}$ implies that $|j - i| \equiv 1 \pmod{4}$. Furthermore

$$\pi_i(\theta(x')) \stackrel{\text{def of } \theta}{=} \pi_{i'}(x') \stackrel{\text{edge}}{=} \pi_{j'}(y') \stackrel{\text{def of } \theta}{=} \pi_j(\theta(y')).$$

Thus there exists an edge between $\theta(x')$ and $\theta(y')$ □

Definition 5 *Consider the coloring $E(Q_3 \square C_{4k}) \xrightarrow{c'} \{1, 2, \dots, 20\}$ of the edges of $Q_3 \square C_{4k}$ defined by $c'(\langle x', y' \rangle) = c(\langle \theta(x'), \theta(y') \rangle)$.*

Lemma 6 *For any $i \in \{1, 2, \dots, 20\}$ the set of edges of $Q_3 \square C_{4k}$ such that $c'(x', y') = i$ is a set of disjoint paths of length 4. Therefore P_4 divides $Q_3 \square C_{4m}$ for all $m \geq 1$.*

Proof. By definition of c' , for any vertex x' of $Q_3 \square C_{4k}$ the number of edges incident to x' colored i by c' is the number of edges incident to $\theta(x')$ colored i by c . Therefore this number is ≤ 2 . Furthermore, there is no cycle colored

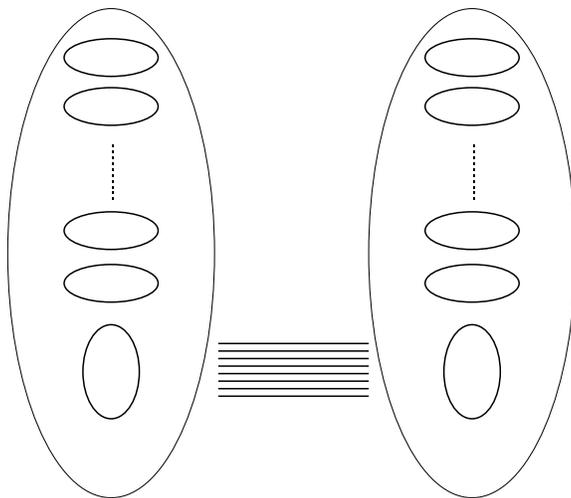


Figure 4: Decomposition of Q_{2k+1}

i in $Q_3 \square C_{4k}$ because the image by θ of this cycle would be a cycle of Q_5 colored i with c . Therefore the set of edges colored i by c' is a forest and more precisely, because of the degree, a set of disjoint paths.

Notice that the image by θ of a path colored i is a path of Q_5 of the same length (because of the degree of the endpoints of the paths). Therefore all the paths are of length 4. \square

Theorem 2 For $n \geq 4$, P_4 divides Q_n .

Proof. If n is even, the result is true by Corollary 5. If $n = 5$ then we are done by Corollary 7. Consider Q_{2k+3} , for $k \geq 2$. $Q_{2k+3} = Q_{2k+1} \square Q_2$. $E(Q_{2k})$ can be decomposed into k cycles of length 2^{2k} (Hamiltonian cycles) by Aubert and Schneider [1]. Let D be one of these cycles. The edges of Q_{2k+1} are the edges of the two copies of Q_{2k} and a matching. But every vertex of Q_{2k} appears exactly once in D so $E(Q_{2k+1})$ can be decomposed into $2(k-1)$ cycles of length 2^{2k} and $D \square Q_1 \simeq C_{2^{2k}} \square Q_1$ (see figure 4).

Every vertex of Q_{2k+1} appears once in $D \square Q_1$, thus, for the same reason, $E(Q_{2k+3})$ can be decomposed into $8(k-1)$ cycles of length 2^{2k} and $D \square Q_1 \square Q_2 \simeq C_{2^{2k}} \square Q_1 \square Q_2 \simeq C_{2^{2k}} \square Q_3$ (see figure 5).

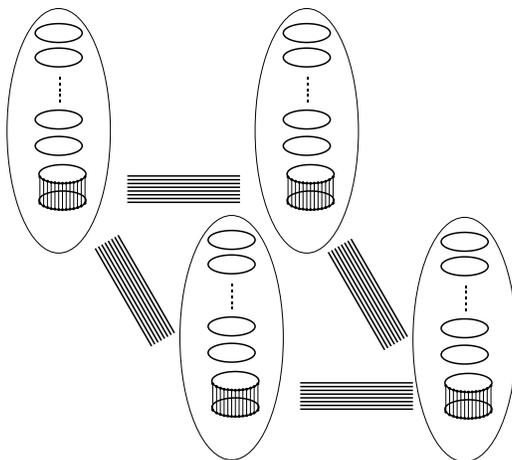


Figure 5: Decomposition of Q_{2k+3}

Since $k \geq 2$, $\frac{2^{2k}}{4}$ is an integer strictly greater than 1 so the cycles of length 2^{2k} are divisible by P_4 . By Lemma 6, P_4 divides $C_{2^{2k}} \square Q_3$, and P_4 divides $E(Q_n)$ for any odd $n \geq 5$. \square

4 Q_{2^k} has a fundamental Hamiltonian cycle.

We shall describe walks in the hypercube by specifying the starting vertex (generally \emptyset) and the sequence of edge directions.

It is well-known that the n -dimensional hypercube Q_n is Hamiltonian, and in fact has many Hamiltonian cycles. Aubert and Schneider [1] proved that for n even, Q_n has an edge decomposition into Hamiltonian cycles. However, their construction is technical. In contrast, in this last section we shall prove that for $n = 2^k$, there is a single Hamiltonian cycle C such that $E(C)$ is a fundamental set for Q_n .

By $G_1 \square G_2$ we denote the Cartesian product of the graphs G_1 and G_2 . We will start with two easy results about Cartesian product of graphs.

Lemma 7 *Assume that $\{C^1, C^2, \dots, C^p\}$ is an edge decomposition in Hamiltonian cycles of a graph G . Then $\{C^1 \square C^1, C^2 \square C^2, \dots, C^p \square C^p\}$ is an edge decomposition of $G \square G$.*

Proof. Let (x_1, x_2) and (y_1, y_2) be adjacent in $G \square G$. Then either x_1 and

y_1 are adjacent in G and $x_2 = y_2$ or $x_1 = y_1$ and x_2 and y_2 are adjacent in G . By symmetry, it is sufficient to consider the first case. Let i be such that $\langle x_1, y_1 \rangle \in E(C^i)$. Then since C^i is Hamiltonian $x_2 = y_2 \in V(C^i)$; thus $\langle (x_1, x_2), (y_1, y_2) \rangle \in E(C^i \square C^i)$. Conversely $\langle (x_1, x_2), (y_1, y_2) \rangle \in E(C^j \square C^j)$ implies $\langle x_1, y_1 \rangle \in E(C^j)$ since $x_2 = y_2$; thus $j = i$. Therefore the $C^j \square C^j$'s are disjoint and the conclusion follows. \square

Lemma 8 *Let G_1 and G_2 be any two graphs, and for $i = 1, 2$ let $\phi_i \in \text{Aut}(G_i)$. Define $(\phi_1, \phi_2) : G_1 \square G_2 \rightarrow G_1 \square G_2$ by $(\phi_1, \phi_2)((x, y)) = (\phi_1(x), \phi_2(y))$. Then $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$.*

Proof. Let (x_1, x_2) and (y_1, y_2) be adjacent in $G_1 \square G_2$. Then either (1) x_1 and y_1 are adjacent in G_1 and $x_2 = y_2$ or (2) $x_1 = y_1$ and x_2 and y_2 are adjacent in G_2 . We must show that $(\phi_1, \phi_2)(x_1, x_2)$ and $(\phi_1, \phi_2)(y_1, y_2)$ are adjacent in $G_1 \square G_2$. By symmetry, it is sufficient to prove this for case (1). But then since $\phi_1 \in \text{Aut}(G_1)$, $\phi_1(x_1)$ and $\phi_1(y_1)$ are adjacent in G_1 , and since $x_2 = y_2$, $\phi_2(x_2) = \phi_2(y_2)$. Therefore $(\phi_1, \phi_2)(x_1, x_2)$ and $(\phi_1, \phi_2)(y_1, y_2)$ are adjacent in $G_1 \square G_2$. Conversely if $(\phi_1, \phi_2)(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ and $(\phi_1, \phi_2)(y_1, y_2) = (\phi_1(y_1), \phi_2(y_2))$ are adjacent in $G_1 \square G_2$ then $\phi_1(x_1) = \phi_1(y_1)$ or $\phi_2(x_2) = \phi_2(y_2)$. We can assume the first case by symmetry then $x_1 = y_1$ and x_2 is adjacent to y_2 in G_2 . Thus (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \square G_2$ and $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$. \square

The starting point of the theorem of Aubert and Schneider is an earlier result of G. Ringel [10] who proved that for $n = 2^k$, Q_n has an edge decomposition into Hamiltonian cycles. His proof is by induction on k . Let us recall the induction step. Let $m = 2^n$. Let θ be the mapping from $\{1, \dots, n\}$ to $\{n+1, \dots, 2n\}$ defined by $\theta(i) = i + n$. Let C be a Hamiltonian cycle of Q_n then we can construct $\Phi(C)$ and $\Gamma(C)$ two disjoint Hamiltonian cycles of $Q_{2n} = Q_n \square Q_n$ such that $E(C \square C) = E(\Phi(C)) \cup E(\Gamma(C))$. Indeed fix an arbitrary vertex (say 0) and represent C by the sequence of directions $C = (c_1, \dots, c_m)$ then consider

$$\Phi(C) = (\begin{array}{ll} c_1, \dots & \dots, c_{m-1}, c_{\theta(c_1)}, \\ c_m, c_1, \dots & \dots, c_{m-2}, c_{\theta(c_2)}, \\ c_{m-1}, c_m, c_1, \dots & \dots, c_{m-3}, c_{\theta(c_3)}, \\ \dots & \dots \\ c_2, \dots & \dots, c_m, c_{\theta(c_m)}, \end{array})$$

and

$$\Gamma(C) = \left(\begin{array}{cccc} c_{\theta(1)}, \dots & & \dots, c_{\theta(m-1)}, c_1, & \\ c_{\theta(m)}, c_{\theta(1)}, \dots & & \dots, c_{\theta(m-2)}, c_2, & \\ c_{\theta(m-1)}, c_{\theta(m)}, c_{\theta(1)}, \dots & & \dots, c_{\theta(m-3)}, c_3, & \\ \dots & & \dots & \\ c_{\theta(2)}, \dots & & \dots, c_{\theta(m)}, c_m, & \end{array} \right)$$

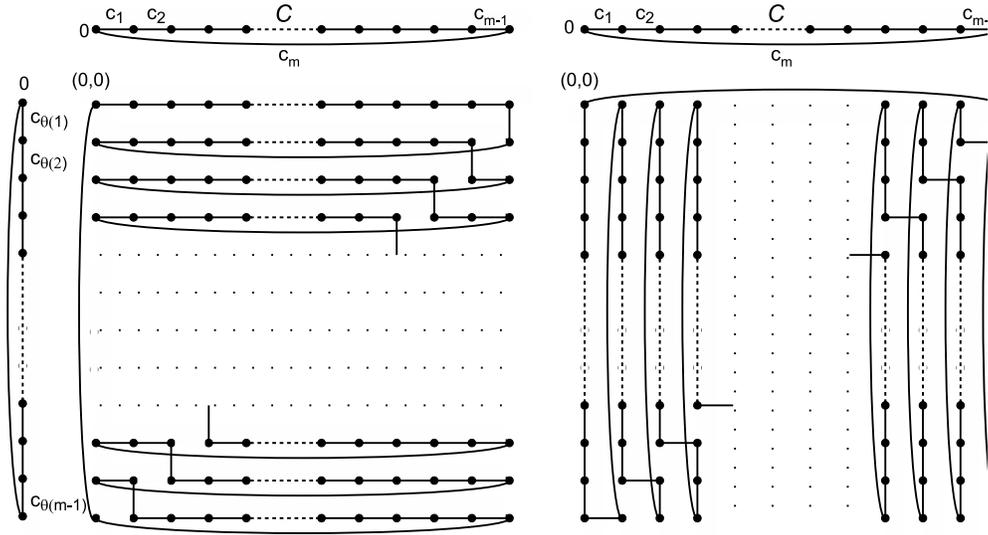


Figure 6: Construction of $\Phi(C)$ and $\Gamma(C)$ from C

It is immediate to check (see figure 6) that $\Phi(C)$ and $\Gamma(C)$ are disjoint and define a partition of the edges of $C \square C$. For n even let $p = n/2$ and assume that $\{C^1, C^2, \dots, C^p\}$ is an edge decomposition of Q_n in Hamiltonian cycles then as a consequence of Lemma 7, $\{\Phi(C^1), \Phi(C^2), \dots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \dots, \Gamma(C^p)\}$ is an edge decomposition of Q_{2n} in Hamiltonian cycles.

Theorem 3 For any $k \geq 1$, Q_{2^k} has a Hamiltonian cycle that is a fundamental set.

Proof. This is trivial for $k = 1$ since $Q_2 = C_4$. The desired result follows by induction from Ringel's construction. Indeed let $n = 2^k, k \geq 1$ and assume that there exists an edge decomposition $\{C^1, C^2, \dots, C^p\}$ of Q_n obtained as the translate of an Hamiltonian cycle C^1 under some subgroup \mathcal{E} of $\text{Aut}(Q_n)$. For any automorphism $\phi \in \text{Aut}(Q_n)$, $(\phi, \phi) \in \text{Aut}(Q_{2n})$ by Lemma 8. Furthermore if $\phi(C^1) = C^i$ then $(\phi, \phi)(\Phi(C^1)) = \Phi(C^i)$ and $(\phi, \phi)(\Gamma(C^1)) = \Gamma(C^i)$. If we consider now the permutation θ on $\{1, \dots, 2n\}$ defined by $\theta(i) = i + n \pmod{2n}$ then $\rho_\theta(\Phi(C^i)) = \Gamma(C^i)$. The conclusion follows since the subgroup of $\text{Aut}(Q_{2n})$, isomorphic to $\mathcal{E} \times S_2$, defined by $\mathcal{H} = \{(\phi, \phi); \phi \in \mathcal{E}\} \cup \{\rho_\theta \circ (\phi, \phi); \phi \in \mathcal{E}\}$ is such that $\{\Phi(C^1), \Phi(C^2), \dots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \dots, \Gamma(C^p)\}$ are the translates of $\Phi(C^1)$ under \mathcal{H} . \square

Corollary 8 *For n and m each a power of 2, with $m \leq n$, there is an m -cycle that divides Q_n .*

Proof. Let $m = 2^p$. By Theorem 3 Q_m has a fundamental 2^p -cycle, which therefore divides $Q_m = Q_{2^p}$. Since m and n are each powers of two, m divides n . Hence by Proposition 4 and Lemma 1, this cycle divides Q_n . \square

5 Summary of Results

1. For k odd, if P_k is a path on k edges that divides Q_n , then k divides n . (Proposition 3)
2. If k divides n , any tree on k edges divides Q_n . (Corollary 4)
3. If k divides n and $k < n$ then P_{2k} divides Q_n . (Corollary 5)
4. If n is even and $j < n$ then P_{2j} divides Q_n . (Proposition 6)
5. For $k = 2n$ there is a k -cycle which is a fundamental set for Q_n when n is even. (Proposition 7)
6. For $n =$ a power of 2, there is a Hamiltonian cycle which is a fundamental set for Q_n . (Theorem 3)
7. For $n =$ a power of 2 and $m =$ a power of 2, with $m \leq n$, there is an m -cycle that divides Q_n . (Corollary 8)
8. For $n \geq 4$, P_4 divides Q_n . (Theorem 2)
9. Q_k is a fundamental set for Q_n if and only if k divides n . (Proposition 5)
10. For $k \geq 3$, P_{2j} does not divide Q_{2k+1} . (Proposition 1)

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