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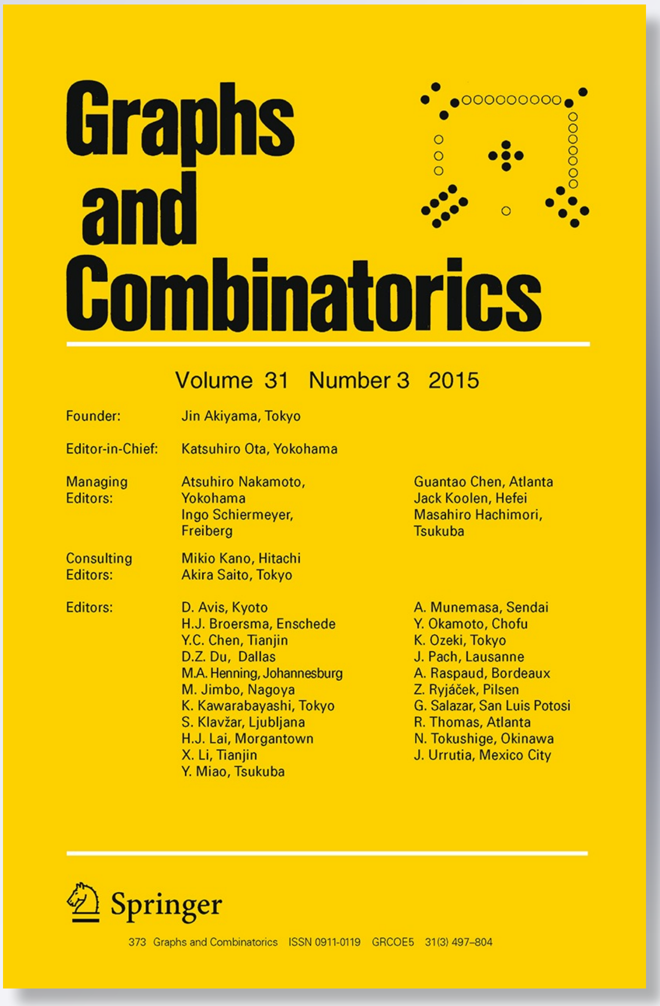

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ORIGINAL PAPER

Characterizing Heavy Subgraph Pairs for Pancyclicity

Binlong Li · Bo Ning · Hajo Broersma · Shenggui Zhang

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Abstract Earlier results originating from Bedrossian's PhD Thesis focus on characterizing pairs of forbidden subgraphs that imply hamiltonian properties. Instead of forbidding certain induced subgraphs, here we relax the requirements by imposing Ore-type degree conditions on the induced subgraphs. In particular, adopting the terminology introduced by Čada, for a graph G on n vertices and a fixed graph H, we say that G is H-o_1-heavy if every induced subgraph of G isomorphic to H contains two nonadjacent vertices with degree sum at least $n+1$ in G. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-o_1-heavy if G is H-o_1-heavy for every $H \in \mathcal{H}$. In this paper we characterize all connected graphs R and S other than P_3 (the path on three vertices) such that every 2-connected $\{R, S\}$-o_1-heavy graph is either a cycle or pancyclic, thereby extending previous results on forbidden subgraph conditions for pancyclicity and on heavy subgraph conditions for hamiltonicity.

Keywords Forbidden subgraph · o_1-Heavy subgraph · Pancyclic graph · Hamiltonian graph

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1 Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph and let G' be a subgraph of G. If G' contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G. For a given graph H, we say that G is H*-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$.

The graph $K_{1,3}$ is called a *claw*; its (only) vertex of degree 3 is called the *center* of the claw, and the other vertices are called the *end vertices* of the claw.

Following the terminology of [4], a graph G on n vertices is said to be *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle of length n passing through all the vertices of G. If G contains cycles of length k for every k with $3 \leq k \leq n$, we say that G is *pancyclic*. Note that a pancyclic graph is necessarily hamiltonian. Bedrossian [1] studied forbidden subgraph conditions for a 2-connected graph to be hamiltonian and to be pancyclic. In his PhD thesis, he proved the following nice results, characterizing all pairs of forbidden subgraphs for these properties. We note here that in Bedrossian's results and throughout this paper, all graphs are assumed to be *nontrivial*, i.e., having a nonempty edge set. As a consequence we do not consider P_2-free graphs. We also note here that a connected P_3-free graph is a *complete graph*, i.e., its vertex set is a *clique*, i.e., all its vertices are mutually adjacent, and hence it is hamiltonian and pancyclic if it has order at least 3. In fact, it is not hard to show that the statement 'every connected (nontrivial) H-free graph is hamiltonian (pancyclic)' only holds if $H = P_3$. The case with pairs of forbidden subgraphs (different from P_3) is much more interesting.

Theorem 1 (Bedrossian [1]) *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Figure 1).*

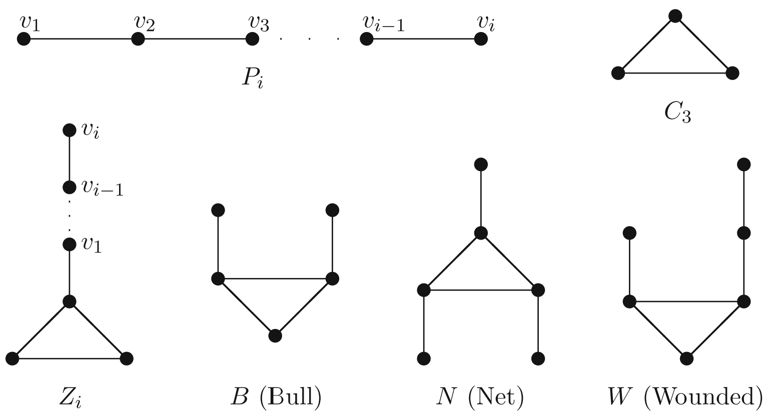

Fig. 1 The graphs P_i, C_3, Z_i, B, N and W

Theorem 2 (Bedrossian [1]) *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Forbidding pairs of graphs as induced subgraphs might impose such a strong condition on the graphs under consideration that hamiltonian properties are almost trivially obtained. As an example, one easily shows that, apart from paths and cycles, connected $\{K_{1,3}, Z_1\}$-free graphs are only a matching away from complete graphs, i.e., their complements consist of isolated vertices and isolated edges. This is one of the motivations to relax forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where additional conditions are imposed on these subgraphs if they appear. Early examples of this approach in the context of hamiltonicity and pancyclicity date back to the early 1990s [2,6]. The idea to put a minimum degree bound on one or two of the end vertices of an induced claw has been explored in [5]. Here we follow the ideas and terminology of [7] by putting an Ore-type degree sum condition on at least one pair of nonadjacent vertices in certain induced subgraphs. These degree sum conditions refer to one of the earliest papers in this area, in which Ore [9] proved that a graph G on $n \geq 3$ vertices is hamiltonian if the degree sum of any two nonadjacent vertices of G is at least n.

Let G be a graph on n vertices and let G' be an induced subgraph of G. We say that G' is *heavy* in G if there are two nonadjacent vertices in $V(G')$ with degree sum at least n in G. For a given graph H, G is called *H-heavy* if every induced subgraph of G isomorphic to H is heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-heavy* if G is H-heavy for every $H \in \mathcal{H}$. Note that an H-free graph is trivially H-heavy.

The counterpart of Theorem 1 for heavy subgraphs was studied in [8]. For hamiltonicity of 2-connected graphs the authors in [8] obtained the following result.

Theorem 3 (Li et al. [8]) *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W.*

Note that, apart from P_6, the same graphs appear in Theorems 1 and 3. Examples is [8] show that P_6 has to be excluded in Theorem 3.

Is there a natural counterpart of Theorem 2 involving heavy subgraphs? What can we say about the pancyclicity of graphs when we consider heavy subgraph conditions instead of forbidden subgraph conditions? To start with a negative observation, let us first consider the complete bipartite graph $K_{n/2,n/2}$. Note that every induced subgraph of $K_{n/2,n/2}$ (other than P_1 and P_2) is heavy, but $K_{n/2,n/2}$ is clearly not pancyclic. This implies that for any family $\mathcal{H}$ of graphs, a 2-connected graph G (not being a cycle) cannot be guaranteed to be pancyclic by imposing that G is $\mathcal{H}$-heavy. As in existing degree condition results for pancyclicity, we have to impose a slightly stronger degree condition in order to exclude the above counterexamples.

Let G be a graph on n vertices and let G' be an induced subgraph of G. Following [7], we say that G' is o_1-heavy in G if there are two nonadjacent vertices in $V(G')$ with degree sum at least $n + 1$ in G (Ore's condition with $n + 1$ instead of n). For a given graph H, G is called *H-o_1-heavy* if every induced subgraph of G isomorphic to H is o_1-heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-o_1-heavy* if G is H-o_1-heavy for every $H \in \mathcal{H}$.

Note that an H-free graph is trivially H-o_1-heavy, and an H-o_1-heavy graph is also H-heavy. Moreover, if H_1 is an induced subgraph of H_2, then an H_1-free (H_1-heavy, H_1-o_1-heavy) graph is also H_2-free (H_2-heavy, H_2-o_1-heavy).

Imposing this natural, slightly stronger Ore-type degree condition, we obtain the following counterpart of Theorem 2.

Theorem 4 *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Note that exactly the same graphs appear in Theorems 2 and 4. The 'only if' part of the theorem follows almost directly from Theorem 2, since H-free graphs are H-o_1-heavy. For the 'if' part of the theorem, noting that P_4 and Z_1 are both induced subgraphs of Z_2, it is sufficient to prove the following result.

Theorem 5 *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy, then G is pancyclic.*

The proof is by induction on the order n of the graph. Using Theorem 2, we are done if G is $\{K_{1,3}, P_5\}$-free or $\{K_{1,3}, Z_2\}$-free, so we may assume there is at least one pair of vertices with degree sum at least $n + 1$. Cycles of length 3, 4 and 5 are easily obtained from the degree conditions, and cycles of length n and $n - 1$ are easily obtained by using Theorem 3 directly, and after establishing the existence of a vertex v whose removal does not affect the 2-connectedness and then using Theorem 3 on $G - v$. For the other cases, we are done if we can find a vertex or a pair of vertices whose removal does not affect the 2-connectedness and the degree sum conditions on remaining pairs in the smaller graph. Assuming that such vertices or pairs of vertices do not exist, forces a lot of structure on the vertex cuts S with $|S| = 2$ of G and on the components of $G - S$, enabling us to prove Theorem 5. We postpone the details of the proof of Theorem 5 to Sect. 3.

Let $F = P_5$ or $F = Z_2$. As we pointed out before, $K_{n/2,n/2}$ is a 2-connected $\{K_{1,3}, F\}$-heavy graph which is not pancyclic. Another graph with this property is $K_{n/2,n/2} - e$ (the graph obtained from $K_{n/2,n/2}$ by deleting an arbitrary edge). Apart from the cycles and these two types of graphs, we do not know whether there exist any other graphs with the above properties, so we raise it as an open problem.

Problem 1 Is there some graph G on n vertices other than C_n, $K_{n/2,n/2}$ and $K_{n/2,n/2}$ $-e$ such that G is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy but not pancyclic?

2 Some Preliminaries

In the next section we will prove Theorem 5. Before we do so, in this section we introduce some additional terminology, and we will prove some useful lemmas.

Let G be a graph. For a subgraph H of G, when no confusion can arise we also use H to denote the vertex set of H; and similarly, for a subset S of $V(G)$, we also use S to denote the subgraph of G induced by S.

The following useful lemma is an easy exercise that can be found in [4]. It appeared as Lemma 1 in [3]. We present it here without a proof.

Lemma 1 *Let G be a graph on $n \geq 4$ vertices, and let x be a vertex of G. If $d(x) \geq n/2$ and $G - x$ is hamiltonian, then G is pancyclic.*

Throughout this paper, instead of $K_{1,3}$-free, $K_{1,3}$-heavy and $K_{1,3}$-o_1-heavy, we use the terminology claw-free, claw-heavy and claw-o_1-heavy, respectively. Our next result is a structural lemma on claw-o_1-heavy graphs.

Lemma 2 *Let G be a claw-o_1-heavy graph on $n \geq 4$ vertices, and let x and x' be two vertices of G. Then*

1. *if $xx' \in E(G)$ and $d(x) + d(x') \geq n + 1$, then xx' is contained in a triangle;*
2. *if $d(x) \geq (n + 1)/2$, then x is contained in a triangle; and*
3. *if $xx' \notin E(G)$ and $d(x) + d(x') \geq n + 1$, then*
   (a) *x and x' have at least three common neighbors in G, and*
   (b) *x and x' are contained in a common quadrangle and a common pentagon.*

Proof (1) Since $d(x) + d(x') \geq n + 1$, x and x' have at least one common neighbor y. Then $xyx'x$ is a triangle containing xx'.

(2) Since $d(x) \geq (n + 1)/2$ and $n \geq 4$, we have $d(x) \geq 3$. Let y, y', y'' be three neighbors of x. If $yy' \in E(G)$, then $xyy'x$ is a triangle containing x. Next assume that $yy' \notin E(G)$, and similarly, $yy'', y'y'' \notin E(G)$. Then the subgraph induced by $\{x, y, y', y''\}$ is a claw. Since G is claw-o_1-heavy, there must be a vertex in $\{y, y', y''\}$ with degree at least $(n + 1)/2$. Without loss of generality, we assume that $d(y) \geq (n+1)/2$. Then we have $d(x)+d(y) \geq n+1$. By (1), xy is contained in a triangle.

(3) Here we assume $xx' \notin E(G)$ and $d(x) + d(x') \geq n + 1$. If x and x' have at most two common neighbors, then $d(x) + d(x') \leq (n - 2) + 2 = n$, a contradiction. Thus x and x' have at least three common neighbors. We may assume without loss of generality that $d(x) \geq (n + 1)/2$.

Let y, y', y'' be three common neighbors of x and x'. Then $xyx'y'x$ is a quadrangle containing x and x'. If $yy' \in E(G)$, then $xyy'x'y''x$ is a pentagon containing x and x'. Next assume that $yy' \notin E(G)$, and similarly, $yy'', y'y'' \notin E(G)$. Then the subgraph induced by $\{x, y, y', y''\}$ is a claw. Without loss of generality, we assume that $d(y) \geq (n + 1)/2$. Then we have $d(x) + d(y) \geq n + 1$. By (1), xy is contained in a triangle $xyzx$. Noting that $z \neq x', y'$, $xzyx'y'x$ is a pentagon containing x and x'. □

Let G be a graph on n vertices. In the following, we call a vertex x a *super heavy vertex* of G if $d(x) \geq (n+1)/2$, and we call a pair of vertices $\{x, y\}$ a *super heavy pair* of G if $xy \notin E(G)$ and $d(x) + d(y) \geq n + 1$. Note that a super heavy pair contains at least one super heavy vertex. The importance of the existence of super heavy vertices for pancyclicity is already demonstrated by Lemma 1. The next lemma relates the (non)existence of such vertices to the structure of the neighborhood of a vertex-cut.

Lemma 3 *Let G be a 2-connected claw-o_1-heavy graph, and suppose $\{r, s\}$ is a vertex-cut of G. Then*

1. *$G - \{r, s\}$ has exactly two components; and*

2. *for any distinct neighbors x and x' of r: x and x' are in a common component of $G - \{r, s\}$ if and only if $xx' \in E(G)$ or $\{x, x'\}$ is a super heavy pair of G.*

Proof (1) If there are at least three components of $G - \{r, s\}$, then let H, H' and H'' be three such components. Let x, x' and x'' be neighbors of r in H, H' and H'', respectively. Then the subgraph induced by $\{r, x, x', x''\}$ is a claw. Since x and x' have at most the two common neighbors r and s, by Lemma 2, $d(x)+d(x') \le n$. Similarly, $d(x) + d(x'') \le n$ and $d(x') + d(x'') \le n$, contradicting that G is claw-o_1-heavy. Thus, $G - \{r, s\}$ has exactly two components.

(2) If x and x' are not in a common component, then clearly $xx' \notin E(G)$, and since x and x' have at most the two common neighbors r and s, by Lemma 2, $d(x)+d(x') \le n$. Thus $\{x, x'\}$ is not a super heavy pair. On the other hand, if x and x' are in a common component, then let x'' be a neighbor of r in the component not containing x and x'. If $xx' \notin E(G)$, then the subgraph induced by $\{r, x, x', x''\}$ is a claw and $d(x)+d(x'') \le n$, $d(x') + d(x'') \le n$. Since G is claw-o_1-heavy, we have $d(x) + d(x') \ge n + 1$, so $\{x, x'\}$ is a super heavy pair, completing the proof of Lemma 3. □

In the sequel, by the concept *cut* we always refer to a vertex-cut with 2 vertices. A pair of vertices $\{x, y\}$ is called a *separable pair* of G if x and y are in distinct components of $G - \{r, s\}$ for some cut $\{r, s\}$ of G. So by Lemma 3, a separable pair cannot be a super heavy pair.

Let G be a 2-connected graph, let $\{r, s\}$ be a cut of G, and let H be a component of $G - \{r, s\}$. We call the subgraph induced by $H \cup \{r, s\}$ a *link* of G (this is called an $\{r, s\}$-component in [4]). For such a link, $\{r, s\}$ is called the *bolt* of the link, H is called the *inside*, and $H' = G - \{r, s\} - H$ is called the *outside* of the link. Let L be a link of G with bolt $\{r, s\}$ and inside H. Then if its outside $H' = G - \{r, s\} - H$ is connected, then the subgraph induced by $H' \cup \{r, s\}$ is also a link, called the *co-link* of L, and denoted by L_c.

Note that if a link L has a co-link, then its co-link is unique, and L is the co-link of its co-link. By Lemma 3, we see that if a graph G has connectivity 2 and is claw-o_1-heavy, then every link of G has a co-link. It is convenient to denote a link L with bolt $\{r, s\}$ by $L(r, s)$, and its co-link by $L_c(r, s)$.

The next series of lemmas provides some structural information on cuts and links.

Lemma 4 *Let G be a 2-connected graph, let $L = L(r, s)$ be a link of G, and let H be the inside of L. If $\{r', s'\}$ is a cut of G with $r', s' \in L$, then there is a component of $G - \{r', s'\}$ contained in H.*

Proof If $\{r', s'\} = \{r, s\}$, then the result is trivially true. So we assume that $\{r', s'\} \ne \{r, s\}$. Without loss of generality, we assume that $r \ne r', s'$. Note that r has a neighbor in every component of $G - \{r, s\}$. Since $r', s' \in L$, every component of $G - \{r, s\}$ other than H is contained in a component of $G - \{r', s'\}$ containing r (and also containing s if $s \ne r', s'$). Thus any other component of $G - \{r', s'\}$ is contained in H. □

Let L be a link of a graph. A vertex of the inside (outside) of L is called a vertex inside (outside) L.

Lemma 5 *Let G be a 2-connected graph, let $L = L(r, s)$ be a link of G, and let x be a vertex inside L. If $\{x, y\}$ is a cut of G for some vertex y outside L, then*

(1) *$rs \notin E(G)$, and r and s are in distinct components of $G - \{x, y\}$;*
(2) *x is a cut vertex of L, and r and s are in distinct components of $L - x$; and*
(3) *$rx \in E(G)$ and x is the only neighbor of r in H, or $\{r, x\}$ is a cut.*

Proof (1) Let u be an arbitrary vertex inside L other than x. By the 2-connectedness of G, there is a path from u to r or s not passing through x with all internal vertices inside L. Similarly, for an arbitrary vertex v outside L other than y, there is a path from v to r or s not passing through y with all internal vertices outside L. Thus if $rs \in E(G)$ or if r and s are in a common component of $G - \{x, y\}$, then $G - \{x, y\}$ is connected, a contradiction.

(2) Let P be an arbitrary path of L from r to s. Note that P cannot pass through y. By (1), P passes through x. This implies that x is a cut vertex of L, and that r and s are in distinct components of $L - x$.

(3) Suppose that r has a neighbor r' in L other than x. By (1), r and s are in distinct components of $G - \{x, y\}$. Clearly r and r' are in a common component. Let P be an arbitrary path of G from r' to s. Thus P either passes through x or passes through y. If P passes through y, then it also passes through r. This implies that $\{r, x\}$ is a cut. □

Lemma 6 *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a link of G, let H be the inside of L, and let x be a vertex in H. Then the following two statements are equivalent:*

(1) *$rx \in E(G)$ and x is the only neighbor of r in H, or $\{r, x\}$ is a cut.*
(2) *$sx \in E(G)$ and x is the only neighbor of s in H, or $\{s, x\}$ is a cut.*

Proof First assume that $rx \in E(G)$ and x is the only neighbor of r in H. If $sx \notin E(G)$ or s has at least two neighbors in H, then there is a neighbor $s' \neq x$ of s in H. Let P be an arbitrary path of G from s' to r. If P does not pass through s, then every internal vertex of P is in H. Noting that r has only one neighbor x in H, this implies that P then passes through x. Hence $\{s, x\}$ is a cut.

Suppose now that $\{r, x\}$ is a cut. Let H_c be the outside of L. Using Lemma 4, let H' be the component of $G - \{r, x\}$ contained in H. If $sx \notin E(G)$ or s has at least two neighbors in H, then let r' be a neighbor of r in H', let r'_c be a neighbor of r outside L, and let s' be a neighbor of s inside L other than x. Clearly $s' \notin H'$.

We claim that every neighbor of r is either in $H' \cup \{x\}$ or in $H_c \cup \{s\}$. Otherwise, let r'' be a neighbor of r in $H - x - H'$. Then the subgraph induced by $\{r, r', r'_c, r''\}$ is a claw. It is easily seen that any pair of vertices from $\{r', r'_c, r''\}$ is separable. By Lemma 3, the claw induced by $\{r, r', r'_c, r''\}$ is not o_1-heavy, a contradiction.

Recall that r' and s' are in distinct components of $G - \{r, x\}$. Let P be an arbitrary path of G from r' to s'. Then P passes through either r or x. Also recall that every neighbor of r not in $H' \cup \{x\}$ is in $H_c \cup \{s\}$. Thus if P passes through r, then it will also pass through s. This implies that $\{s, x\}$ is a cut. This completes the proof of one direction of the lemma.

The opposite direction follows by symmetry. □

A link $L = L(r, s)$ is said to be *simple* if both r and s have at least two neighbors inside L, and for every vertex x inside L, $\{r, x\}$ and $\{s, x\}$ are not cuts. By Lemma 6, we can see that if $L = L(r, s)$ is a link of a 2-connected claw-o_1-heavy graph, then L is simple if and only if r has at least two neighbors inside L, and for every vertex x inside L, $\{r, x\}$ is not a cut.

Lemma 7 *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a link of G, and let H be the inside of L. Then L is 2-connected if and only if $rs \in E(G)$ or L is simple.*

Proof First we assume that L has a cut vertex x. Clearly r and s are in distinct components of $L - x$; otherwise x is a cut vertex of G. Thus we have $rs \notin E(G)$. Moreover, if r has at least two neighbors in H, then let r' be a neighbor of r in H other than x. Let P be an arbitrary path of G from r' to s. If P does not pass through r, then every internal vertex of P is in H. Note that x is a cut vertex of L, and clearly r' and r are in a common component of $L - x$. P will pass through x. This implies that $\{r, x\}$ is a cut and L is not simple.

Suppose now that L is 2-connected. We assume that $rs \notin E(G)$. If r has only one neighbor x in H, then clearly x is a cut vertex of L. So we assume that r has at least two neighbors in H. If $\{r, x\}$ is a cut of G for some x in H, then let H' be the component of $G - \{r, x\}$ contained in H, and let H_c be the outside of L. Let P be an arbitrary path of L from r to s. Similarly as in the proof of Lemma 6, we can prove that every neighbor of r is either in $H' \cup \{x\}$ or in H_c. Note that every internal vertex of P is in H. P must pass through x. This implies x is a cut vertex of G, a contradiction. So we have that L is simple. □

Let G be a 2-connected claw-o_1-heavy graph, and let $rxsr$ be a triangle such that $d(x) = 2$, $d(r) \geq 3$, and $d(s) \geq 3$. Then by Lemma 7, we get that $G - x$ is 2-connected. Similarly, let $rxysr$ be a quadrangle such that $d(x) = d(y) = 2$, $d(r) \geq 3$, and $d(s) \geq 3$. Then $G - \{x, y\}$ is 2-connected.

Note that a simple link is not necessarily a minimal one. Now we prove the following lemma.

Lemma 8 *Let G be a 2-connected claw-o_1-heavy graph, let $L = L(r, s)$ be a simple link of G, and let H be the inside of L. Suppose that there is a link L' contained in H. Then there is a link L'' (possibly equal to L') contained in H and containing L' such that its co-link L''_c is simple.*

Proof We consider a link L'' contained in H and containing L' with the largest order. Let $\{r'', s''\}$ be the bolt, H'' the inside, and H''_c the outside of L''.

By Lemma 6, for each x in H, $\{r, x\}$ and $\{s, x\}$ are not cuts. If r'' has only one neighbor x in H''_c, then $\{s'', x\}$ is a cut and $x \in H$. Then $H'' \cup \{r''\}$ is the component of $G - \{s'', x\}$ contained in H, and the subgraph induced by $H'' \cup \{r'', s'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction. Thus we assume that r'' has at least two neighbors in H''_c, and similarly, s'' has at least two neighbors in H''_c.

If $\{r'', x\}$ is a cut of G for some $x \in H''_c$, then note that $x \neq r, s$, and by Lemma 5, $x \notin H_c \cup \{r, s\}$, where H_c is the outside of L. This implies x is inside H. By Lemma 6, $\{s'', x\}$ is a cut. Let H''' be the component of $G - \{r'', x\}$ contained in H. If H'' is contained in H''', then the subgraph induced by $H''' \cup \{r'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction. Thus we assume that H'' is not contained in H'''. Note that every neighbor of r'' is either in $H'' \cup \{s''\}$ or in $H''' \cup \{x\}$. $H'' \cup H''' \cup \{r''\}$ is the component of $G - \{s'', x\}$ contained in H, and the subgraph induced by $H'' \cup H''' \cup \{r'', s'', x\}$ is a link contained in H and containing L' with larger order than L'', a contradiction.

Thus we conclude that L''_c is simple. □

Let G be a 2-connected graph. If $G - x$ is 2-connected for a vertex x of G, then we call x a *c-removable vertex* of G (a removable vertex with respect to the connectivity condition); similarly, if $G - \{x, y\}$ is 2-connected for a pair of vertices $\{x, y\}$ of G, then we call $\{x, y\}$ a *c-removable pair* of G. Note that every vertex of a 3-connected graph is c-removable. Also note that every non-removable vertex of a 2-connected graph is contained in a cut. The existence of c-removable vertices and pairs plays a key role in our induction proof of Theorem 5 in the next section. Here we prove a preliminary lemma on c-removable pairs.

Lemma 9 *Let G be a 2-connected graph on at least 5 vertices, and let $L = L(r, s)$ and $L' = L'(r', s')$ be two 2-connected links of G that are internally disjoint. If x and x' are two c-removable vertices of G inside L and L', respectively, then $\{x, x'\}$ is a c-removable pair of G.*

Proof Let y be an arbitrary vertex of $G - \{x, x'\}$. We prove that $G' = G - \{x, x', y\}$ is connected.

If y is one of the vertices in $\{r, s, r', s'\}$, then without loss of generality, we assume that $y = r$. Then for every vertex u inside L with $u \neq x$, since x is c-removable and $\{r, x\}$ is not a cut, there is a path P of $G - \{r, x\}$ from u to s. Clearly, P does not pass through x'. This implies that u and s are connected by the path P in G'. Similarly, for every vertex v outside L with $v \neq x'$, since x' is c-removable and $\{r, x'\}$ is not a cut, there is a path Q of $G - \{r, x'\}$ from v to s that does not pass through x. This implies that v and s are connected by the path Q in G'. Thus G' is connected.

Now we assume that y is not a vertex of $\{r, s, r', s'\}$. Without loss of generality, we assume that y is outside L. Then for every vertex u inside L with $u \neq x$, since L is 2-connected, there is a path P of $L - x$ from u to r. This implies that u and r are connected by the path P in G'. In particular, r and s are connected in G'. Besides, for every vertex v outside L with $v \neq x', y$, since x' is c-removable and $\{x', y\}$ is not a cut, there is a path Q of $G - \{r, x'\}$ from v to r or s with all internal vertices outside L. This implies that v and r or s are connected by the path Q in G'. Thus G' is connected. □

Let G be a graph and let x be a vertex of G. If every super heavy pair $\{u, v\}$ of G, with $u, v \in V(G)\backslash\{x\}$, is also a super heavy pair of $G - x$ (in terms of the order of the new graph), then we call x a *d-removable vertex* of G (a removable vertex with respect to the degree condition). Let x, y be two distinct vertices of G. If every super heavy pair $\{u, v\}$ of G, with $u, v \in V(G)\backslash\{x, y\}$, is also a super heavy pair of $G - \{x, y\}$, then we call $\{x, y\}$ a *d-removable pair* of G. For an induction proof the existence of vertices or pairs of vertices that are both c-removable and d-removable is very favorable, as we will see in the next section. We finish this section with the following easy observations on d-removable vertices and pairs.

Lemma 10 *Let G be a graph, and let x, y be two distinct vertices of G. Then*

(1) *if $N(x)$ contains no super heavy pair of G, then x is a d-removable vertex of G; and*
(2) *if x and y have no common neighbors, then $\{x, y\}$ is a d-removable pair of G.*

Proof We use n to denote the order of G.

(1) Let $G' = G - x$, and let $\{u, v\}$ be an arbitrary super heavy pair of G with $u, v \in V(G)\backslash\{x\}$. If $N(x)$ contains no super heavy pairs of G, then at least one of u and v is not in $N(x)$. Without loss of generality, we assume that $u \notin N(x)$. Then $d_{G'}(u) = d(u)$ and $d_{G'}(v) \geq d(v) - 1$. Thus $d_{G'}(u) + d_{G'}(v) \geq n$. Since the order of G' is $n - 1$, $\{u, v\}$ is a super heavy pair of G'. This implies that x is a d-removable vertex of G.
(2) Let $G' = G - \{x, y\}$, and let $\{u, v\}$ be an arbitrary super heavy pair of G with $u, v \in V(G)\backslash\{x, y\}$. If x and y have no common neighbors, then at least one of ux and uy is not in $E(G)$. Then $d_{G'}(u) \geq d(u) - 1$, and similarly, $d_{G'}(v) \geq d(v) - 1$. Thus $d_{G'}(u) + d_{G'}(v) \geq n - 1$. Since the order of G' is $n - 2$, $\{u, v\}$ is a super heavy pair of G'. This implies that $\{x, y\}$ is a d-removable pair of G. □

3 Proof of Theorem 5

Let G be a 2-connected $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy graph that is not a cycle, and let $n = |V(G)|$. We are going to prove that G is a pancyclic graph by induction on n. If G contains only three vertices, then the result is trivially true. So we assume that $n \geq 4$.

If G is $\{K_{1,3}, P_5\}$-free or $\{K_{1,3}, Z_2\}$-free, then by Theorem 2, G is pancyclic. So we assume that G is neither $\{K_{1,3}, P_5\}$-free nor $\{K_{1,3}, Z_2\}$-free. This implies that G contains at least one super heavy pair.

By Lemma 2, G contains a triangle, a quadrangle and a pentagon. Next we are going to prove a number of claims. Our first claim establishes the existence of long cycles.

Claim 1 G contains a cycle of length n and a cycle of length $n - 1$.

Proof Since G is $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy, by Theorem 3, G is hamiltonian. So G contains a cycle of length n.

Let C be a Hamilton cycle of G, and let $\{r, s\}$ be a super heavy pair of G. Clearly $\{r, s\}$ divides C into two subpaths. Recall from the definition of a super heavy pair that $rs \notin E(G)$. Let $P = rx_1x_2 \cdots x_ks$ and $Q = ry_1y_2 \cdots y_\ell s$ be the two subpaths of C, where $k + \ell + 2 = n$. If $rx_2 \in E(G)$, then $C' = C - rx_1x_2 \cup rx_2$ (with the obvious meaning) is a cycle of length $n - 1$. Thus we assume that $rx_2 \notin E(G)$ and, similarly $sx_{k-1}, ry_2, sy_{\ell-1} \notin E(G)$. Let $S = \{u \in P \mid u \in N(s)\}$ and $R = \{u \in P \mid u^{++} \in N(r)\}$, where u^{++} denotes the vertex at distance 2 of u on P in the direction from r to

s. Then r and s are no elements of $S \cup R$. Clearly, $d_P(s) = |S|$ and $d_P(r) = |R| + 1$, so $d_P(s) + d_P(r) = |S| + |R| + 1 = |S \cap R| + |S \cup R| + 1$. If we assume that $S \cap R = \emptyset$, then we get that $d_P(s) + d_P(r) \leq |P| - 1$. Applying similar assumptions and counting techniques to Q, this would yield $d(r) + d(s) \leq |P| + |Q| - 2 = n$. Since $d(r) + d(s) \geq n + 1$, we conclude that without loss of generality, $S \cap R \neq \emptyset$. Hence, there is a vertex x_i, $2 \leq i \leq k - 1$, such that $rx_{i+1}, sx_{i-1} \in E(G)$. Clearly x_i is a c-removable vertex of G.

Let $G' = G - x_i$. Then G' is 2-connected. Let $\{u, v\}$ be an arbitrary super heavy pair of G. Noting that $d_{G'}(u) \geq d(u) - 1$ and $d_{G'}(v) \geq d(v) - 1$, we have $d_{G'}(u) + d_{G'}(v) \geq n - 1$. Since G' has $n - 1$ vertices, $\{u, v\}$ is a *heavy pair* of G', i.e., u, v are nonadjacent and with degree sum at least $|V(G')|$. This implies that G' is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy. Hence, by Theorem 3, G' contains a Hamilton cycle, which is a cycle of length $n - 1$. □

By Lemma 2 and Claim 1, if $n \leq 7$, then G is pancyclic. So we assume that $n \geq 8$. It suffices to prove that G contains a cycle of length k for all $k \in [6, n - 2]$.

Suppose to the contrary that G does not contain cycles of all these lengths. Our next claim shows that G has no vertices or vertex pairs that are c-removable and d-removable at the same time.

Claim 2 G contains no vertices or pairs that are both c-removable and d-removable.

Proof If G contains a vertex x that is both c-removable and d-removable, then $G' = G - x$ is 2-connected and $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy. By the induction hypothesis, G' contains a cycle of length k for all $k \in [3, n - 1]$, a contradiction. Similarly, if G contains a pair of vertices $\{x, y\}$ that is both c-removable and d-removable, then $G' = G - \{x, y\}$ is 2-connected and $\{K_{1,3}, P_5\}$-o_1-heavy or $\{K_{1,3}, Z_2\}$-o_1-heavy. By the induction hypothesis, G' contains a cycle of length k for all $k \in [3, n - 2]$, a contradiction. □

The next claim shows that super heavy vertices must be part of a cut of G.

Claim 3 Every super heavy vertex of G is contained in a cut.

Proof Let r be a super heavy vertex of G. If r is not contained in any cut, then r is c-removable and $G - r$ is 2-connected. Similarly as in the proof of Claim 1, we can prove that $G - r$ is $\{K_{1,3}, P_5\}$-heavy or $\{K_{1,3}, Z_2\}$-heavy, and hence hamiltonian. By Lemma 1, G is pancyclic, a contradiction. □

The following claim provides useful structural properties related to the links of G.

Claim 4 Let $L = L(r, s)$ be a link of G, and let H be the inside of L. Then one of the following statements holds.

1. H contains a c-removable vertex of G, or
2. L is an induced path from r to s.

Proof We use induction on $|V(H)|$. If H consists of only one vertex x, then $rx, sx \in E(G)$. If $rs \in E(G)$, then by Lemma 7, $G - x$ is 2-connected, hence x is c-removable and (1) holds. If $rs \notin E(G)$, then L is an induced path rxs and (2) holds. Thus we assume that H has at least two vertices. Suppose that both statements of the claim do not hold. We prove a number of subclaims to reach a contradiction. □

Claim 4.1 There is a vertex in H with degree at least 3.

Proof Suppose that every vertex of H has degree 2. If $rs \notin E(G)$, then L is an induced path and (2) holds. So we assume that $rs \in E(G)$. If H consists of two vertices x_1 and x_2, then by Lemma 7, $G - \{x_1, x_2\}$ is 2-connected, hence $\{x_1, x_2\}$ is a c-removable pair of G. By Lemma 10, $\{x_1, x_2\}$ is a d-removable pair of G, a contradiction to Claim 2. Thus we assume that H has $k \geq 3$ vertices.

Let $rx_1x_2 \cdots x_ks$ be the path from r to s, where $x_i \in H$, $1 \leq i \leq k$. Note that x_i cannot be in a super heavy pair of G since $d(x_i) = 2$. Let y be a neighbor of r outside L and z be a neighbor of s outside L. Then $yrx_1x_2x_3$ is an induced P_5 which is not o_1-heavy. At the same time, if $sy \in E(G)$, then the subgraph induced by $\{r, s, y, x_1, x_2\}$ is a Z_2 which is not o_1-heavy. Thus G will be neither P_5-o_1-heavy nor Z_2-o_1-heavy, a contradiction. So we assume that $sy \notin E(G)$ and similarly, $rz \notin E(G)$. Then the subgraph induced by $\{r, s, y, x_1\}$ is a claw. Thus we have that $d(s) + d(y) \geq n + 1$, and similarly, $d(r) + d(z) \geq n + 1$. This implies that $d(r) + d(y) \geq n + 1$ or $d(s) + d(z) \geq n + 1$. Without loss of generality, we assume that $d(r) + d(y) \geq n + 1$. Then by Lemma 2, ry is contained in a triangle $ryy'r$. Now the subgraph induced by $\{y, y', r, x_1, x_2\}$ is a Z_2 which is not o_1-heavy, a contradiction. □

Claim 4.2 L is simple.

Proof If r has only one neighbor x in H, then s has a neighbor in H other than x; otherwise x would be a cut vertex of G. By Lemma 6, $\{s, x\}$ is a cut. Let H' be the component of $G - \{s, x\}$ contained in H. Then $L' = H' \cup \{s, x\}$ is a link contained in L. Clearly, every vertex in L is either r or in L'. By the induction hypothesis, either H' contains a c-removable vertex of G or L' is an induced path from s to x. If L' is an induced path from s to x, then every vertex in H will have degree 2, a contradiction. Thus H' contains a c-removable vertex of G, and it is also contained in H, a contradiction.

Thus we next assume that r has at least two neighbors in H, and similarly, that s has at least two neighbors in H.

If there is a vertex x in H such that $\{r, x\}$ is a cut, then by Lemma 6, $\{s, x\}$ is a cut. Let H' be the component of $G - \{r, x\}$ contained in H, and let H'' be the component of $G - \{s, x\}$ contained in H. Then $L' = H' \cup \{r, x\}$ and $L'' = H'' \cup \{s, x\}$ are two links contained in L. Clearly, every vertex in L is either in L' or in L''. If both L' and L'' are induced paths, then every vertex in H will have degree 2, a contradiction. Thus we assume that L' or L'' is not an induced path. By the induction hypothesis, H' or H'' contains a c-removable vertex of G, and it is also contained in H, a contradiction. □

Claim 4.3 There is a link contained in H. Moreover, if H contains a super heavy vertex, then there is a link contained in H and containing a super heavy vertex.

Proof Let x be an arbitrary vertex of H. If x is not c-removable, then x is contained in a cut $\{x, y\}$. By Claim 4.2, $\{r, x\}$ and $\{s, x\}$ are not cuts. Thus we have $y \neq r$ or $y \neq s$, and by Lemma 5, $y \in H$. This implies that there is a link L' contained in H (and containing x). In particular, if H contains a super heavy vertex x', then there is a link L' contained in H and containing x'. □

Here we continue the proof of Claim 4. By Lemma 8, there is a link contained in H such that its co-link is simple. Moreover, if H contains a super heavy vertex x, then by Claim 4.3 there is a link contained in H and containing x. Then by Lemma 8, there is a link contained in H and containing x such that its co-link is simple. Denote this link by L', let x be a vertex inside L' and assume that x is super heavy if H contains the super heavy vertex x. Let $\{r', s'\}$ be the bolt, H' the inside, and H'_c the outside of L'. If H' contains a c-removable vertex of G, then it is also a c-removable vertex contained in H, a contradiction. So by the induction hypothesis, we assume that L' is an induced path.

If H' consists of only one vertex x, then since $L'_c = G - x$ is simple, by Lemma 7, x is a c-removable vertex of G, and it is also contained in H, a contradiction. If H' consists of only two vertices x_1 and x_2, then since $L'_c = G - \{x_1, x_2\}$ is simple, by Lemma 7, $\{x_1, x_2\}$ is a c-removable pair of G and by Lemma 10, $\{x_1, x_2\}$ is a d-removable pair of G, a contradiction to Claim 2. Thus we assume that H' contains $k \geq 3$ vertices.

Let $r'x_1x_2 \cdots x_k s'$ be the path of L' from r' to s', where $x_i \in H'$, $1 \leq i \leq k$.

Note that x_i cannot be in a super heavy pair of G since $d(x_i) = 2$. Let y be a neighbor of r' in H'_c, and let z be a neighbor of s' in H'_c. Then $yr'x_1x_2x_3$ is an induced P_5 of G which is not o_1-heavy. At the same time, if r' is contained in a triangle, then we assume that $r'yy'r'$ is a triangle. Then the subgraph induced by $\{y, y', r', x_1, x_2\}$ is a Z_2 which is not o_1-heavy, a contradiction. Thus we assume that r' is not contained in a triangle. By Lemma 2, we have that r' is not super heavy. Similarly, we get that s' is not contained in a triangle and is not super heavy. This implies that there are no super heavy vertices in H.

Since L'_c is simple, r' has at least two neighbors in H'_c. Let y' be a neighbor of r' in H'_c other than y. Note that r' is contained in no triangles, $yy' \notin E(G)$, and the subgraph induced by $\{r', y, y', x_1\}$ is a claw. Since $d(x_1) = 2$, we have that either y or y' is a super heavy vertex of G. Without loss of generality, we assume that y is super heavy. Since H contains no super heavy vertex, we have that r' has at most one neighbor in $H - H'$, and $y = r$ or $y = s$. Without loss of generality, we assume that $y = r$. Note that r is a super heavy vertex. By Lemma 2, r is contained in a triangle $rtt'r$.

If $t \in H$, then $\{r', t\}$ is not a super heavy pair, since r' and t are not super heavy vertices. If $t = s$, then $\{r', t\}$ is not a super heavy pair, since r' has at most one neighbor in $H - H'$. If $t \in G - L$, then $\{r', t\}$ is not a super heavy pair by Lemma 3. Similarly, we have that $\{r', t'\}$ is not a super heavy pair of G. Thus the subgraph induced by $\{t, t', r, r', x_1\}$ is a Z_2 which is not o_1-heavy, a contradiction.

The next claim provides useful information on the existence of c-removable vertices in the inside of a simple link.

Claim 5 Let $L = L(r, s)$ be a simple link of G, and let H be the inside of L. Then

1. H contains a c-removable vertex of G;
2. if H contains a vertex nonadjacent to r, then H contains a c-removable vertex nonadjacent to r; and
3. if H contains a vertex nonadjacent to both r and s, then H contains a c-removable vertex nonadjacent to both r and s.

Proof By definition, a simple link cannot be an induced path. Hence, by Claim 4, H contains a c-removable vertex of G. Thus (1) holds.

In order to prove (2), we assume that H contains a vertex, but no c-removable vertices, nonadjacent to r. We first prove the following subclaim in order to reach a contradiction. □

Claim 5.1 There is a link contained in H. Moreover, if H contains a super heavy vertex, then there is a link contained in H and containing a super heavy vertex.

Proof Let r' be an arbitrary vertex of H nonadjacent to r. By our assumption, r' is contained in a cut $\{r', s'\}$. Since L is simple, $s' \neq r, s$, and by Lemma 5, s' is not outside L. Now we have that $r', s' \in H$. Let H' be the component of $G - \{r', s'\}$ contained in H. Then the subgraph induced by $H' \cup \{r', s'\}$ is a link contained in H (and containing r'). Moreover, if H contains a super heavy vertex r'', then by Claim 3, r'' is contained in a cut $\{r'', s''\}$. Similarly as in the above analysis, we get that there is a link contained in H and containing r''. □

By Lemma 8, there is a link L' contained in H such that its co-link L'_c is simple. Moreover, if H contains a super heavy vertex, L' can be chosen in such a way that it contains a super heavy vertex. Let $\{r', s'\}$ be the bolt and H' be the inside of L'. Note that every vertex in H' is nonadjacent to r. If H' contains a c-removable vertex of G, then the assertion is true. Thus, using Claim 4, we assume that L' is an induced path from r' to s'. Then similarly as in the proof of Claim 4, we get that G contains a P_5 and a Z_2 that are not o_1-heavy, a contradiction.

The third assertion can be proved similarly. We omit the details.

Let r be a super heavy vertex of G. By Claim 3, $G - r$ is *separable*, i.e., has a cut vertex, so we can consider the *blocks* of $G - r$, i.e., the maximal subgraphs of $G - r$ without a cut vertex (these blocks are either 2-connected or isomorphic to K_2). An *end-block* of $G - r$ is a block containing precisely one cut vertex of $G - r$. Note that every end-block of $G - r$ contains an inner vertex (a vertex that is not the cut vertex of $G - r$ of that end-block) adjacent to r. Using Lemma 2(3) and Lemma 3, we deduce that there are exactly two end-blocks of $G - r$. This implies that the blocks of $G - r$ can be denoted as $B_0, B_1, \ldots, B_k$ with cut vertices s_i, $1 \leq i \leq k$, common to B_{i-1} and B_i.

Our next claim shows that $G - r$ consists of two or three blocks.

Claim 6 $k = 1$ or 2.

Proof Suppose that $k \geq 3$. We prove the following subclaims in order to reach a contradiction. The first subclaim shows that all the super heavy vertices $\neq r$ are concentrated in one block. □

Claim 6.1 All the super heavy vertices of G other than r are contained in a common end-block of $G - r$.

Proof Since r is super heavy, every other super heavy vertex is either adjacent to r or forms a super heavy pair together with r.

Using Lemma 2(3) and Lemma 3, note that every neighbor of r is either in B_0 or in B_k, and every vertex in $\bigcup_{i=1}^{k-1} B_i - \{s_1, s_k\}$ has at most two neighbors in common with r. This implies that every super heavy vertex other than r is either in B_0 or in B_k.

Note that $k \geq 3$. A vertex in B_0 and a vertex in B_k have at most two common neighbors, so they cannot be super heavy at the same time. Thus we have that all the super heavy vertices of G other than r are contained in a common end-block of $G - r$. $\square$

Using Claim 6.1, without loss of generality, we assume that every super heavy vertex of G other than r is in B_0. We reach a contradiction by proving two subclaims, showing that G has an induced P_5 and an induced Z_2 that are both not o_1-heavy, respectively.

Claim 6.2 There is an induced P_5 in G that is not o_1-heavy.

Proof Note that for every vertex s' in $B_1 - s_1$, $\{r, s'\}$ cannot be a super heavy pair, and for every vertex r' in B_k, $\{s_1, r'\}$ cannot be a super heavy pair. We have that either $\{r, r'\}$ is not a super heavy pair for all $r' \in B_k$ or $\{s_1, s'\}$ is not a super heavy pair for all $s' \in B_1 - s_1$. We distinguish two cases.

Case A. $\{s_1, s'\}$ is not a super heavy pair for all $s' \in B_1 - s_1$.

In this case, let x be a neighbor of s_1 in $B_0 - s_1$, let P be a shortest path of B_1 from s_1 to s_2, let Q be a shortest path of B_2 from s_2 to s_3, and let y be a neighbor of s_3 in $B_3 - s_3$. Then $xs_1Ps_2Qs_3y$ is an induced P_ℓ with $\ell \geq 5$ that is not o_1-heavy.

Case B. There is a vertex $s' \in B_1 - s_1$ such that $\{s_1, s'\}$ is a super heavy pair.

In this case, $B_1 - \{s_1, s_2\} \neq \emptyset$ and $\{r, r'\}$ is not a super heavy pair for all $r' \in B_k$. Let x be a neighbor of r in $B_0 - s_1$, let P be a shortest path of $B_k \cup \{r\}$ from r to s_k, let Q be a shortest path of B_{k-1} from s_k to s_{k-1}, and let y be a neighbor of s_{k-1} in B_{k-2} such that $y \neq s_1$. Then $xrPs_kQs_{k-1}y$ is an induced P_ℓ with $\ell \geq 5$ that is not o_1-heavy. $\square$

Claim 6.3 There is an induced Z_2 in G that is not o_1-heavy.

Proof Recalling that $n \geq 8$, we have $d(r) \geq 5$. This implies that r has at least two neighbors in $B_0 - s_1$ or in $B_k - s_k$. We again distinguish two cases.

Case A. r has at least two neighbors in $B_k - s_k$.

If s_k has only one neighbor x in $B_k - s_k$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we have that s_k has at least two neighbors in $B_k - s_k$. Let x, x' be two neighbors of s_k in $B_k - s_k$. Recall that B_k contains no super heavy vertices. By Lemma 3, $xx' \in E(G)$. Let P be a shortest path of B_{k-1} from s_k to s_{k-1}, and let y be a neighbor of s_{k-1} in B_{k-2}. Then the subgraph induced by $\{x, x'\} \cup V(P) \cup \{y\}$ is a Z_ℓ with $\ell \geq 2$ that is not o_1-heavy.

Case B. r has only one neighbor in $B_k - s_k$.

We claim that r is contained in a triangle such that the two other vertices of the triangle are in $B_0 - s_0$. Note that r has at least two neighbors in $B_0 - s_1$. Let x, x' be two neighbors of r in $B_0 - s_1$. If $xx' \in E(G)$, then $rxx'r$ is the required triangle. So we assume that $xx' \notin E(G)$. By Lemma 3, $\{x, x'\}$ is a super heavy pair. Without loss of generality, we assume that x is super heavy. Thus $d(r) + d(x) \geq n + 1$. Note that

s_2 is nonadjacent to both r and x. So r and x have at least two common neighbors. Let x'' be a common neighbor of r, x other than s_1. Then $rxx''r$ is the required triangle.

Now let $rxx'r$ be a triangle such that $x, x' \in B_0 - s_1$. Let P be a shortest path of $B_k \cup \{r\}$ from r to s_k, and let y be a neighbor of s_k in B_{k-1}. Note that r has only one neighbor in B_k. No vertex in P can form a super heavy pair together with r. Thus the subgraph induced by $\{x, x'\} \cup V(P) \cup \{y\}$ is a Z_ℓ with $\ell \geq 2$ that is not o_1-heavy. □

By Claims 6.2 and 6.3 , G is neither $\{K_{1,3}, P_5\}$-o_1-heavy nor $\{K_{1,3}, Z_2\}$-o_1-heavy, a contradiction. This completes the proof of Claim 6.

By Claim 6, $G - r$ has either two or three blocks. Recalling that $d(r) \geq 5$, without loss of generality, we may assume that r has at least two neighbors in $B_0 - s_1$. Note that $\{r, x\}$ is not a cut for all $x \in B_0 - s_1$. Thus we have that $L(r, s_1) = B_0 \cup \{r\}$ is a simple link. We distinguish two cases: $k = 2$ and $k = 1$.

Case 1 $k = 2$.

In this case, $G - r$ has three blocks B_0, B_1 and B_2. We distinguish three subcases, depending on the order of B_1 and the number of neighbors of r in B_2.

Case 1.1 $B_1 - \{s_1, s_2\} \neq \emptyset$.

We first claim that $L'(s_1, s_2) = B_1$ is a simple link. If $\{s_1, x\}$ is a cut for some $x \in B_1 - \{s_1, s_2\}$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $\{s_1, x\}$ is not a cut for all $x \in B_1 - \{s_1, s_2\}$, and similarly, $\{s_2, x\}$ is not a cut for all $x \in B_1 - \{s_1, s_2\}$. If s_1 has only one neighbor x in $B_1 - \{s_1, s_2\}$, then by Lemma 6, s_2 has only one neighbor x in $B_1 - \{s_1, s_2\}$. This implies that $B_1 - \{s_1, s_2\}$ consists of only one vertex x; otherwise x is a cut vertex of G. If $s_1 s_2 \notin E(G)$, then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $s_1 s_2 \in E(G)$. By Lemma 7, x is a c-removable vertex, and by Lemma 10, x is a d-removable vertex, a contradiction to Claim 2. Thus as we claimed, $L'(s_1, s_2) = B_1$ is a simple link.

Secondly, we claim that r has at least two neighbors in $B_2 - s_2$. Suppose to the contrary that r has only one neighbor r' in $B_2 - s_2$. Suppose first that $rs_2 \in E(G)$. If s_2 has only one neighbor r' in $B_2 - s_2$, then $B_2 - s_2$ consists of only one vertex r', and r' is a c-removable and d-removable vertex, a contradiction. If $\{s_2, r'\}$ is a cut, then let H be the component of $G - \{s_2, r'\}$ contained in $B_2 - s_2$, let x be a neighbor of s_2 in $B_1 - \{s_1, s_2\}$, and let y be a neighbor of s_2 in H. Then the subgraph induced by $\{s_2, r, x, y\}$ is a claw that is not o_1-heavy, a contradiction. Thus we assume that $rs_2 \notin E(G)$. Note that $\{r, x\}$ is not a super heavy pair for every $x \in B_2$. Let x be a neighbor of r in $B_0 - s_1$, let P be a shortest path of B_2 from r' to s_2, and let y be a neighbor of s_2 in $B_1 - \{s_1, s_2\}$. Then $xrr'Ps_2y$ is an induced P_ℓ for $\ell \geq 5$ that is not o_1-heavy. At the same time, similarly as in Case B of Claim 6.3, we can prove that r is contained in a triangle $rxx'r$ with $x, x' \in B_0 - s_1$. Let y be a neighbor of r' in $B_2 - r'$. Then the subgraph induced by $\{x, x', r, r', y\}$ is a Z_2 that is not o_1-heavy. Thus G is neither $\{K_{1,3}, P_5\}$-o_1-heavy nor $\{K_{1,3}, Z_2\}$-o_1-heavy, a contradiction. So as we claimed, r has at least two neighbors in $B_2 - s_2$. Note that $\{r, x\}$ is not a cut for all $x \in B_2 - s_2$. We have that $L''(s_2, r) = B_2 \cup \{r\}$ is a simple link.

We conclude that G consists of three simple links $L = L(r, s_1)$, $L' = L'(s_1, s_2)$ and $L'' = L''(s_2, r)$.

Suppose that there is a vertex inside L nonadjacent to r. Using Claim 5, let x be a c-removable vertex inside L nonadjacent to r, and let y be a c-removable vertex in L''. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction. Thus we deduce that r is adjacent to every vertex inside L. Similarly, we can prove that s_1 is adjacent to every vertex inside L', and s_2 is adjacent to every vertex inside L''.

We claim that L contains a path from r to s_1 of length k for all $k \in [2, |V(L)| - 1]$. Recall that G is hamiltonian and that $\{r, s_1\}$ is a cut of G. There is a Hamilton path of L from r to s_1. Let $P = rx_1x_2 \cdots x_js_1$ be a Hamilton path of L, where $j = |V(L)| - 2$. Then $rx_{j-k+2} \cdots x_js_1$ is a path of L from r to s_1 of length k.

Thus as we claimed, L contains a path from r to s_1 of length k for all $k \in [2, |V(L)| - 1]$. Similarly, L' contains a path from s_1 to s_2 of length k for all $k \in [2, |V(L')| - 1]$, and L'' contains a path from s_2 to r of length k for all $k \in [2, |V(L'')| - 1]$. Thus G contains a cycle of length k for all $k \in [6, n]$.

Case 1.2 $B_1 - \{s_1, s_2\} = \emptyset$ and r has at least two neighbors in $B_2 - s_2$.

Note that $\{r, x\}$ is not a cut for all $x \in B_2 - s_2$. We have that $L'(r, s_2) = B_2 \cup \{r\}$ is a simple link. So G consists of two simple links $L = L(r, s_1)$ and $L' = L'(r, s_2)$, and an edge s_1s_2.

Similarly as in the proof of Case 1.1, we get that r is adjacent to every vertex inside L, and L contains a path from r to s_1 of length k for all $k \in [2, |V(L)| - 1]$. Similarly, r is adjacent to every vertex inside L', and L' contains a path from r to s_2 of length k for all $k \in [2, |V(L')| - 1]$. Thus G contains a cycle of length k for all $k \in [5, n]$.

Case 1.3 $B_1 - \{s_1, s_2\} = \emptyset$ and r has only one neighbor in $B_2 - s_2$.

Let r' be the neighbor of r in $B_2 - s_2$. Suppose first that $B_2 - s_2$ consists of only one vertex r'. If $rs_2 \in E(G)$, then r' is a c-removable vertex and a d-removable vertex, a contradiction. If $rs_2 \notin E(G)$, then $\{r', s_2\}$ is a c-removable pair and a d-removable pair, also a contradiction. Thus we assume that $B_2 - s_2$ has at least two vertices. By Lemma 6, $\{r', s_2\}$ is a cut and $L'(r', s_2) = B_2$ is a link. Now we get that G consists of two links $L = L(r, s_1)$ and $L' = L'(r', s_2)$, and two edges rr' and s_1s_2 (and maybe an additional edge rs_2).

We claim that L' is 2-connected. If $r's_2 \in E(G)$, then by Lemma 7, L' is 2-connected. Thus we assume that $r's \notin E(G)$. If s_2 has only one neighbor x inside L' or $\{s_2, x\}$ is a cut for some x inside L', then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we have that L' is simple, and by Lemma 7, L' is 2-connected.

Note that L' is not a path. Using Claim 4, let x be a c-removable vertex inside L, and let y be a c-removable vertex inside L'. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction.

This completes the proof of Case 1.

Case 2 $k = 1$.

In this case, $G - r$ has only two blocks B_0 and B_1. We again distinguish three subcases according to the order and the number of neighbors of r in B_1.

Case 2.1 r has at least two neighbors in $B_1 - s_1$.

Note that $\{r, x\}$ is not a cut for all $x \in B_1 - s_1$. We have that $L'(r, s_1) = B_1 \cup \{r\}$ is a simple link. So G consists of two simple links $L = L(r, s_1)$ and $L' = L'(r, s_1)$.

If there is a vertex inside L nonadjacent to both r and s_1, then using Claim 5, let x be a c-removable vertex inside L nonadjacent to both r and s_1, and let y be a c-removable vertex inside L'. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction. Thus we assume that every vertex inside L is either adjacent to r or to s_1, and similarly, every vertex inside L' is either adjacent to r or to s_1.

If there is a super heavy vertex r' inside L, then by Claim 3, r' is contained in a cut $\{r', s'\}$. Since L is simple, $s' \neq r, s$, and by Lemma 5, s' is inside L. Using Lemma 4, let H be the component of $G - \{r', s'\}$ contained in $B_0 - s_1$. Then every vertex in H is nonadjacent to both r and s_1, a contradiction. Thus we assume that there are no super heavy vertices inside L.

Note that there are at least two vertices inside L. We can divide the inside of L into two nonempty subsets H and H' such that every vertex of H is adjacent to r and every vertex of H' is adjacent to s_1. Let xy be an edge connecting H and H', where $x \in H$ and $y \in H'$. Note that there are no super heavy vertices in H. By Lemma 3, H is a clique. Thus $H \cup \{r\}$ contains a path from r to x of length k for all $k \in [1, |V(H)|]$, and similarly, $H' \cup \{s_1\}$ contains a path from y to s_1 of length k for all $k \in [1, |V(H')|]$. Hence L contains a path from r to s_1 of length k for all $k \in [3, |V(L)| - 1]$, and similarly, L' contains a path from r to s_1 of length k for all $k \in [3, |V(L')| - 1]$. So G contains a cycle of length k for all $k \in [6, n]$.

Case 2.2 $B_1 - s_1$ has at least two vertices and r has only one neighbor in $B_1 - s_1$.

Let r' be the neighbor of r in $B_1 - s_1$. By Lemma 6, $\{r', s_1\}$ is a cut and $L' = L'(r', s_1) = B_1$ is a link. If L' is simple, then G consists of two simple links $L = L(r, s_1)$ and $L'(r', s_1)$, and an edge rr'. Then as in Case 1.2, we can prove that G contains a cycle of length k for all $k \in [5, n]$. Thus we assume that L' is not simple.

If $\{s_1, x\}$ is a cut for some x inside L', then by Lemma 6, $\{r, x\}$ is a cut, a contradiction. Thus we assume that $\{s_1, x\}$ is not a cut for all x inside L'. Note that L' is not simple. Now s_1 has only one neighbor s' inside L. If $r's_1 \notin E(G)$, then $\{r, s'\}$ is a cut, a contradiction. Thus we assume that $r's_1 \in E(G)$. If there is only one vertex r' inside L', then by Lemma 7, r' is a c-removable vertex, and by Lemma 10, r' is a d-removable vertex, a contradiction. Thus we assume that there are at least two vertices inside L'. By Lemma 6, we have that $\{r', s'\}$ is a cut and $L''(r', s') = B_1 - s_1$ is a link. Thus G consists of two links $L(r, s_1)$ and $L'' = L''(r', s')$, and three edges rr', s_1s' and $r's_1$. Similarly as in Case 1.3, we can prove that L'' is 2-connected. Using Claim 4, let x be a c-removable vertex inside L, and let y be a c-removable vertex inside L''. Then by Lemma 9, $\{x, y\}$ is a c-removable pair, and by Lemma 10, $\{x, y\}$ is a d-removable pair, a contradiction.

Case 2.3 $B_1 - s_1$ has only one vertex.

Let y be the vertex of $B_1 - s_1$. Recall that L is simple. Now y is a c-removable vertex. Using Claim 3, we get that y is not a d-removable vertex. By Lemma 10, we have that $\{r, s_1\}$ is a super heavy pair.

First we assume that there is a vertex inside L nonadjacent to both r and s_1. By Claim 5, let x be a c-removable vertex inside L nonadjacent to both r and s_1. Then by Lemma 10, $\{x, y\}$ is a d-removable pair.

We claim that $\{x, y\}$ is a c-removable pair. Let z be an arbitrary vertex of $G - \{x, y\}$. We prove that $G' = G - \{x, y, z\}$ is connected. If $z = r$ or s_1, then without loss of generality, we assume that $z = r$. Then for every vertex v inside L with $v \neq x$, since x is c-removable and $\{r, x\}$ is not a cut, there is a path P of $G - \{r, x\}$ from v to s_1. Clearly, P does not pass through y. This implies that v and s_1 are connected by the path P in G'. Thus G' is connected. So we assume that $z \neq r, s_1$ and then z is inside L. Then for every vertex v inside L with $v \neq x, z$, since x is c-removable and $\{x, z\}$ is not a cut, there is a path P of $G - \{x, z\}$ from v to r or s with all internal vertices inside L. This implies that v and r or s are connected by the path P in G'. Recall that $\{r, s_1\}$ is a super heavy pair. By Lemma 3, r and s_1 have a common neighbor t other than y and z. Thus r and s_1 are connected by the path rts_1 in G'. This implies that G' is connected. Thus as we claimed, $\{x, y\}$ is a c-removable pair, and recalling that $\{x, y\}$ is a d-removable pair too, we obtain a contradiction.

In the remaining case, we assume that every vertex inside L is either adjacent to r or to s_1. Similarly as in Case 2.1, we can prove that there are no super heavy vertices inside L, and that L contains a path from r to s_1 of length k for all $k \in [3, |V(L)| - 1]$. Thus G contains a cycle of length k for all $k \in [5, n]$.

This completes the proof of Theorem 5.

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