Edge colorings of the direct product of two graphs

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1 Introduction

Let G be a finite simple undirected graph. An *edge coloring* of G is a map α from the edge set $E(G)$ of G to a finite set of colors C. The coloring α is *proper* if $\alpha(e_1) \neq \alpha(e_2)$ whenever edges e_1, e_2 are adjacent. One of the most studied graph invariants, the *chromatic index* of G, is the minimum number of colors $\chi'(G)$ in a proper edge coloring of G. By the well-known Vizing's Theorem $\chi'(G)$ is either $\Delta(G)$, the maximum degree of G (G is *Class* 1), or $\Delta(G) + 1$ (G is *Class* 2). Note that deciding whether a graph G is Class 1 is an NP-complete problem even for cubic graphs (Holyer [7]).

The color set of a vertex $u \in V(G)$ with respect to the coloring α is the set $C_\alpha(u) := \{\alpha(uv) : uv \in E(G)\}$ of colors assigned by α to edges incident to u. The coloring α is *adjacent vertex distinguishing* (avd for short) if $uv \in E(G)$ implies $S_\alpha(u) \neq S_\alpha(v)$. The *adjacent vertex distinguishing chromatic index* of the graph G is the minimum number $\chi'_a(G)$ of colors in a proper avd edge coloring of G. Since $\chi'_a(K_1) = 0$ and the graph K_2 does not admit an avd

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coloring at all, when analyzing the invariant $\chi'_a(G)$ it is sufficient to restrict our attention to connected graphs of order at least 3. This is justified by the obvious fact that if G is a disconnected graph with (non-K_2) components G_i, $1 \le i \le q$, then $\chi'_a(G) = \max(\chi'_a(G_i) : 1 \le i \le q)$.

The invariant $\chi'_a(G)$ was introduced and treated for classes of graphs with simple structure (trees, cycles, complete graphs, complete bipartite graphs) by Zhang et al. in [12]. Among other things, it is easy to see that $\chi'_a(C_5) = 5$. However, the other results led the authors of the introductory paper to formulate

Conjecture 1. *If a connected graph $G \neq C_5$ has at least 3 vertices, then $\chi'_a(G) \le \Delta(G) + 2$.*

Conjecture 1 is known to be true for

• subcubic graphs, bipartite graphs (Balister et al. [1]),

• graphs G with $\mathrm{mad}(G) < 3$ (Wang and Wang [11]), where $\mathrm{mad}(G)$ (the parameter called the *maximum average degree* of the graph G) is defined by $\mathrm{mad}(G) := \max(2|E(H)|/|V(H)| : H \subseteq G)$,

• planar graphs G with $\Delta(G) \ge 12$ (Horňák et al. [8]).

There are classes of graphs for which $\chi'_a(G)$ can be upper bounded even by $\Delta(G) + 1$:

• graphs satisfying either $\mathrm{mad}(G) < \frac{5}{2}$ and $\Delta(G) \ge 4$ or $\mathrm{mad}(G) < \frac{7}{3}$ and $\Delta(G) = 3$ [11],

• bipartite planar graphs with $\Delta(G) \ge 12$ (Edwards *et al.* [4]).

The best general bound so far is given by Hatami [6] who proved that $\chi'_a(G) \le \Delta(G) + 300$ if $\Delta(G) > 10^{20}$.

The avd chromatic index was discussed also for graphs resulting from binary graph operations. (A good information about such operations can be found in a monograph [9] by Imrich and Klavžar.) One can mention the Cartesian product (Baril *et al.* [2, 3]), the direct product (Frigerio *et al.* [5], Munarini *et al.* [10], [3]), the strong product [3] and the lexicographic product [3].

The *direct product* of graphs G and H is the graph $G \times H$ with $V(G \times H) := V(G) \times V(H)$ and $E(G \times H) := \{(u,x)(v,y) : uv \in E(G), xy \in E(H)\}$ (where $(u,x)(v,y)$ is a simplified notation for the undirected edge $\{(u,x),(v,y)\}$). This product is commutative and associative (up to isomorphisms). If at least one of the graphs G, H is bipartite, so is the graph $G \times H$. Let $N_G(u)$ be the set of all neighbors and $d_G(u) = |N_G(u)|$ the degree of a vertex $u \in V(G)$; then $N_{G\times H}(u,x) = N_G(u) \times N_H(x)$ and $d_{G\times H}(u,x) = d_G(u)d_H(x)$.

For $p, q \in \mathbb{Z}$ we denote by $[p,q]$ the (finite) *integer interval* bounded by p, q, *i.e.*, the set $\{z \in \mathbb{Z} : p \le z \le q\}$. Similarly, $[p,\infty)$ is the (infinite) integer interval lower bounded by p, *i.e.*, the set $\{z \in \mathbb{Z} : p \le z\}$. If $k \in [2,\infty)$ and $z \in \mathbb{Z}$, we use the notation $(z)_k$ for the (unique) $i \in [1,k]$ satisfying $i \equiv z \pmod{k}$.

For a finite sequence A we denote by $l(A)$ the *length* of A. The *concatenation* of finite sequences A and B is the sequence AB of length $l(A) + l(B)$, in which the terms of A are followed by the terms of B. The unique sequence of length 0, the empty sequence ( ), is both left- and right-concatenation-neutral. If $p, q \in \mathbb{Z}$

and A_i is a finite sequence for $i \in [p,q]$, then $\prod_{i=p}^{q} A_i$ denotes the sequence of length $\sum_{i=p}^{q} l(A_i)$, in which the terms of A_i are followed by the terms of A_{i+1} for each $i \in [p, q-1]$; thus, if $q < p$, then $\prod_{i=p}^{q} A_i = (\ )$. The sequence $\prod_{i=1}^{q} A$ will be for simplicity denoted by A^q. The *support* of a finite sequence $\prod_{i=1}^{k}(a_i)$ is the set $\sigma(A) := \bigcup_{i=1}^{k}\{a_i\}$. A finite sequence A is *simple* if $|\sigma(A)| = l(A)$. A finite sequence A is a *left factor* of a finite sequence B, in symbols $A \leq B$, if there is a finite sequence A' with $AA' = B$.

As usual, P_k and C_k is a path and a cycle of order k, respectively. Further, we assume that $V(P_k) = [1,k]$, $E(P_k) = \{\{i, i+1\} : i \in [1, k-1]\}$ for $k \in [1, \infty)$ and $V(C_k) = [1,k]$, $E(C_k) = \{\{i, (i+1)_k\} : i \in [1,k]\}$ for $k \in [3, \infty)$.

Consider a set $D \subseteq [1, \Delta(G)]$; the graph G is said to be D-*neighbor irregular*, if for any $d \in D$ the set $V_d(G) := \{u \in V(G) : d_G(u) = d\}$ is independent. In other words, if an edge uv in a D-neighbor irregular graph joins vertices of the same degree d, then $d \in [1, \Delta(G)] \setminus D$.

In the remaining text we shall suppose that G is a connected graph of order at least 2 (or at least 3 if the avd chromatic index is involved) and of maximum degree Δ. When working with the avd chromatic index, there are several useful observations following directly from the definitions and from the fact that the color set of a vertex of degree d is of cardinality d.

Proposition 2. $\Delta \leq \chi'(G) \leq \chi'_a(G)$ *for any graph* G.

Proposition 3. *If a graph* G *has adjacent vertices of degree* Δ*, then* $\chi'_a(G) \geq \Delta + 1$.

Proposition 4. $\chi'_a(G) = \chi'(G)$ *for any* $[1, \Delta]$*-neighbor irregular graph* (G).

2 Chromatic index

A vertex (u,i) of a graph $G \times K_2 = G \times P_2$ is said to be of *type* i. Let the *partner* of the vertex (u,i) be the vertex $(u, 3-i)$. Clearly,

$$d_{G\times K_2}(u,i) = d_G(u) = d_{G\times K_2}(u, 3-i),\ u \in V(G),\ i = 1,2.$$

An edge coloring β of the graph $G \times K_2$ is said to be *symmetric* provided that $S_\beta(u,1) = S_\beta(u,2)$ for every $u \in V(G)$.

An edge coloring $\alpha : E(G) \to C$ induces in a natural way the edge coloring $\alpha^\times : E(G \times K_2) \to C$ defined so that

$$\alpha^\times((u,1)(v,2)) := \alpha(uv) =: \alpha^\times((u,2)(v,1)),\ uv \in E(G).$$

From the definition it immediately follows:

Proposition 5. *Let* α *be an edge coloring of a graph* G*. Then*
1. $\alpha^\times$ *is a symmetric edge coloring of the graph* $G \times K_2$*;*
2. $\alpha^\times$ *is proper if* α *is proper;*
3. $\alpha^\times$ *is avd if* α *is avd.*

Proposition 5.2 yields the inequality $\chi'(G \times K_2) \leq \chi'(G)$. However, we are able to prove more:

Theorem 6. *For any graph G there is a symmetric proper edge coloring of the graph $G \times K_2$ that uses Δ colors.*

Proof. First observe that if G is Class 1, the statement follows from Proposition 5.2. Therefore, for a proof by induction on the number of edges of G we may suppose that G is Class 2 and if G' is a graph with $|E(G')| < |E(G)|$, there is a symmetric proper edge coloring of the graph $G' \times K_2$ using $\Delta(G')$ colors.

Since G is Class 2, it has a subgraph H isomorphic to a cycle. Choose an edge $uv \in E(H)$ so that $d_G(u)$ minimizes degrees (in G) of vertices of H and $d_G(v)$ minimizes degrees of (the two) neighbors of u in H. By the induction hypothesis for the graph $G' := G - uv$ there exists a symmetric proper edge coloring $\alpha' : E(G' \times K_2) \to C$ with $|C| = \Delta(G') = \Delta$.

For a vertex $w \in \{u, v\}$ let $M(w)$ be the (nonempty) set of colors missing at both $(w, 1)$ and $(w, 2)$ with respect to α'. If $a \in M(u) \cap M(v) \neq \emptyset$, define the coloring $\alpha : E(G \times K_2) \to C$ as the extension of α' with

$$\alpha((u, 1)(v, 2)) := a =: \alpha((u, 2)(v, 1))$$

to obtain a required symmetric proper edge coloring of $G \times K_2$ with Δ colors.

In the sequel suppose that $M(u) \cap M(v) = \emptyset$. Then there are colors $a \in M(v) \setminus M(u)$ and $b \in M(u) \setminus M(v)$. Consider the subgraph of $G' \times K_2$ induced by the colors a and b. It consists of *alternating* $\{a, b\}$*-cycles* and *alternating* $\{a, b\}$*-paths*. Let $\vec{\pi}_1$ be the *oriented* alternating $\{a, b\}$-path with the first vertex $(u, 1)$; the first edge of $\vec{\pi}_1$ is colored a. Form the *non-extendable* sequence $\prod_{i=1}^{q}(\vec{\pi}_i)$ of distinct (and hence pairwise vertex disjoint) oriented alternating $\{a, b\}$-paths such that

- the first vertex of $\vec{\pi}_{i+1}$ is the partner of the last vertex of $\vec{\pi}_i$ and the first edge of $\vec{\pi}_{i+1}$ has the same color as the last edge of $\vec{\pi}_i$ for each $i \in [1, q-1]$,
- if the last vertex of $\vec{\pi}_j$ is $(v, 1)$, then $j = q$;

so, $\prod_{i=1}^{q}(\vec{\pi}_i)$ is the longest sequence having the above properties. The correctness of the definition follows from the fact that α' is a symmetric edge coloring of $G' \times K_2$ and from the finiteness of the graph $G' \times K_2$.

Interchange the colors a and b in all paths $\vec{\pi}_i$, $i \in [1, q]$, to get the proper edge coloring $\alpha'' : E(G' \times K_2) \to C$ with the following structure of color sets of vertices of affected paths: color sets of internal vertices remain unchanged and in color sets of leaves the colors a and b are interchanged. Now we are ready to color the edges $(u, 1)(v, 2)$ and $(u, 2)(v, 1)$ to create a symmetric proper edge coloring $\alpha : E(G \times K_2) \to C$ as an extension of the coloring α''.

If the last vertex of $\vec{\pi}_q$ is $(v, 1)$, then in the coloring α'' the color a is missing at both vertices $(u, 1), (v, 2)$ and the color b at both vertices $(u, 2), (v, 1)$. Thus, we can define

$$\alpha((u, 1)(v, 2)) := a, \alpha((u, 2)(v, 1)) := b;$$

the common color set of $(u,1)$ and $(u,2)$ is extended (when compared to α') by the color b and the common color set of $(v,1)$ and $(v,2)$ by the color a.

If the last vertex of $\vec{\pi}_q$ is distinct from $(v,1)$, then for each $i \in [1,q]$ the last vertex of $\vec{\pi}_i$ is distinct from $(v,1)$ (see the second part of the definition of the sequence $\prod_{i=1}^{q}(\vec{\pi}_i)$) as well as from $(v,2)$ (this follows from the fact that all edges of the paths $\vec{\pi}_i$, $i \in [1,q]$, colored b end in a vertex of type 1). Consequently, the first vertex of $\vec{\pi}_i$ is distinct from both $(v,1)$ and $(v,2)$ for every $i \in [1,q]$. Finally, since the sequence $\prod_{i=1}^{q}(\vec{\pi}_i)$ is non-extendable, the partner of the last vertex of $\vec{\pi}_q$ must be $(u,1)$, and so the last edge of $\vec{\pi}_q$ is colored a. Having all this in mind we conclude that in the coloring α'' the color a is missing at each of the vertices $(u,1),(u,2),(v,1),(v,2)$ and we can define

$$\alpha((u,1)(v,2)) := a =: \alpha((u,2)(v,1));$$

the color sets of the mentioned four vertices are changed in the same way as above.

By help of Theorem 6 we can prove a general result.

Theorem 7. *If at least one of graphs G, H is Class* 1*, so is the graph $G \times H$.*

Proof. As the graphs $G \times H$ and $H \times G$ are isomorphic, without loss of generality we may suppose that H is Class 1 and there is a proper edge coloring $\beta : E(H) \to [1, \Delta(H)]$. Further, because of Theorem 6 we can construct a (symmetric) proper edge coloring of the graph $G \times K_2$ using Δ colors.

For every $i \in [1, \Delta(H)]$ each component of the graph H_i induced by the color class i of the coloring β is K_2. So, each component of the graph $G \times H_i$ is isomorphic to $G \times K_2$ and there is a (component-wise defined) proper edge coloring $\alpha_i : E(G \times H_i) \to [1,\Delta] \times \{i\}$. The edge coloring of the graph $G \times H$ defined as the common extension of the colorings α_i, $i \in [1, \Delta(H)]$, is evidently proper and the number of involved colors is equal to $|[1,\Delta] \times [1,\Delta(H)]| = \Delta(G)\Delta(H) = \Delta(G \times H)$.

3 Adjacent vertex distinguishing chromatic index

Consider an edge coloring $\beta : E(G \times C_k) \to C$. Clearly, for $u \in V(G)$ and $i \in [1,k]$ the set $S_\beta(u,i)$ can be expressed as $S_\beta(u,i-) \cup S_\beta(u,i+)$, the union of *color half-sets*

$$S_\beta(u,i-) := \{\beta((v,(i-1)_k)(u,i)) : v \in N_G(u)\},$$
$$S_\beta(u,i+) := \{\beta((u,i)(v,(i+1)_k)) : v \in N_G(u)\}.$$

The following auxiliary result can be viewed as a metastatement providing a method for constructing proper avd edge colorings of a graph $G \times C_k$.

Lemma 8. *Let G be a graph, $k \in [3, \infty)$ and let $\beta : E(G \times C_k) \to C$ be a proper edge coloring such that, for any $uv \in E(G)$ with $d_G(u) = d_G(v)$ and any $i \in [1, k]$, the following hold:*

A_1. $S_\beta(u, i+) = S_\beta(v, (i+1)_k-) \Leftrightarrow S_\beta(v, i+) = S_\beta(u, (i+1)_k-)$,

A_2. $S_\beta(u, i+) = S_\beta(v, (i+1)_k-) \Leftrightarrow S_\beta(u, (i-1)_k+) = S_\beta(v, i-)$,

A_3. $S_\beta(u, i+) \cap S_\beta(v, (i+1)_k+) = \emptyset$,

A_4. $S_\beta(u, (i-1)_k+) \neq S_\beta(u, (i+1)_k+)$.

Then β is an avd coloring and $\chi'_a(G \times C_k) \leq |C|$.

Proof. Suppose that β is not avd. Then there is $i \in [1, k]$ and an edge $uv \in E(G)$ joining vertices of the same degree d with $S_\beta(u, i) = S_\beta(v, (i+1)_k)$, which means that

$$S_\beta(u, i-) \cup S_\beta(u, i+) = S_\beta(v, (i+1)_k-) \cup S_\beta(v, (i+1)_k+). \qquad (1)$$

Since $|S_\beta(u, i+)| = d = |S_\beta(v, (i+1)_k-)|$, we have (using successively A_3, (1), A_2 and A_1)

$$\begin{aligned} S_\beta(u, i+) &= S_\beta(v, (i+1)_k-), \\ S_\beta(u, i-) &= S_\beta(v, (i+1)_k+), \\ S_\beta(u, (i-1)_k+) &= S_\beta(v, i-), \\ S_\beta(v, (i-1)_k+) &= S_\beta(u, i-). \end{aligned}$$

Thus, we have obtained $S_\beta(v, (i-1)_k+) = S_\beta(v, (i+1)_k+)$, which contradicts the assumption A_4.

If we analyze an edge coloring $\beta : E(G \times P_k) \to C$, color half-sets $S_\beta(u, i+)$ are defined only for $i \in [1, k-1]$ and $S_\beta(u, i-)$ only for $i \in [2, k]$. Moreover, we have $S_\beta(u, 1) = S_\beta(u, 1+)$, $S_\beta(u, i) = S_\beta(u, i-) \cup S_\beta(u, i+)$ for $i \in [2, k-1]$ and $S_\beta(u, k) = S_\beta(u, k-)$.

3.1 Graphs without adjacent vertices of maximum degree

Because of Proposition 3, if H is a cycle or a path of order at least 3, then $\chi'_a(G \times H)$ can be equal to $\Delta(G \times H) = 2\Delta$ only if $G \times H$ does not have adjacent vertices of degree 2Δ. Such a condition is fulfilled only if either $H = P_3$ or G does not have adjacent vertices of degree Δ.

Theorem 9. *$\chi'_a(G \times P_3) = 2\Delta = \Delta(G \times P_3)$.*

Proof. From Theorem 6 we know that there exists a (symmetric) proper edge coloring $\alpha : E(G \times K_2) \to [1, \Delta]$. Let the coloring $\beta : E(G \times P_3) \to [1, 2\Delta]$ be defined so that if $uv \in E(G)$, then

$$\begin{aligned} \beta((u, 1)(v, 2)) &:= \alpha((u, 1)(v, 2)), \\ \beta((u, 2)(v, 3)) &:= \alpha((u, 1)(v, 2)) + \Delta. \end{aligned}$$

Clearly, β is proper. Moreover, if vertices $(u, i), (v, i+1)$ with $i \in [1, 2]$ are adjacent in $G \times P_3$, then $S_\beta(u, i) \neq S_\beta(v, i+1)$, because exactly one of those two color sets is such that it contains elements of both subsets $[1,\Delta]$ and $[\Delta+1,2\Delta]$ of the set $[1,2\Delta]$. Thus β is also avd and the desired result comes from Proposition 2.

A finite sequence $\prod_{i=1}^{k}(p_i)\in\mathbb{Z}^k$, is said to be *$r$-distinguishing*, $r\in[1,\infty)$, if $p_{(i+2)_k}-p_i\in[-r,r]\setminus\{0\}$ for each $i\in[1,k]$.

Lemma 10. *Suppose that $k\in[3,\infty)$.*
1. *If $k\equiv 0 \pmod 4$, there is a 1-distinguishing sequence of length k.*
2. *There is a 2-distinguishing sequence of length k, for any k.*

Proof. The sequence $(0,0,1,1)^{\frac{k}{4}}$ with $k\equiv 0 \pmod 4$ is 1-distinguishing (as well as 2-distinguishing), while the sequences

$$\begin{aligned}(0,0,1,1,2,2)(0,0,1,1)^{\frac{k-6}{4}},&\ k\equiv 2 \pmod 4,\\ (0,1,2,0,1,2,0)(0,1,2)^{\frac{k-7}{3}},&\ k\equiv 1 \pmod 6,\\ (0,1,2)^{\frac{k}{3}},&\ k\equiv 3 \pmod 6,\\ (0,1,1,2,0)(0,1,2)^{\frac{k-5}{3}},&\ k\equiv 5 \pmod 6\end{aligned}$$

are 2-distinguishing.

Theorem 11. *Suppose that for a graph G and $k\in[4,\infty)$ one of the following assumptions is fulfilled:*
(i) G is $\{\Delta\}$-neighbor irregular and $k\equiv 0 \pmod 4$;
(ii) $\Delta\equiv 1 \pmod 2$, G is $\{\Delta\}$-neighbor irregular and $k\equiv 2 \pmod 4$;
(iii) $\Delta\equiv 0 \pmod 2$, G is $\{\frac{\Delta}{2},\Delta\}$-neighbor irregular and $k\equiv 2 \pmod 4$.
Then $\chi'_a(G\times C_k)=2\Delta=\Delta(G\times C_k)$.

Proof. Let $r:=1$ if (i) is fulfilled and let $r:=2$ if either (ii) or (iii) is fulfilled. By Lemma 10 there is an r-distinguishing sequence $\prod_{i=1}^{k}(p_i)\in\mathbb{Z}^k$. Further, by Theorem 6 there is a symmetric proper edge coloring $\alpha: G\times K_2\to[1,\Delta]$. Let $\beta: E(G\times C_k)\to[1,2\Delta]$ be the coloring determined as follows: if $uv\in E(G)$ and $i\in[1,k]$, then

$$\begin{aligned}\beta((u,i),(v,(i+1)_k)) &:= (\alpha(uv)+p_i)_\Delta, && i\equiv 1 \pmod 2,\\ \beta((u,i),(v,(i+1)_k)) &:= (\alpha(uv)+p_i)_\Delta+\Delta, && i\equiv 0 \pmod 2.\end{aligned}$$

For $i\in[1,k]$ denote as F_i the subgraph of the graph $G\times C_k$ induced by the vertex set $V(G)\times\{i,(i+1)_k\}$ and as $\beta_i: E(F_i)\to[1,2\Delta]$ the restriction of β. From the definition it follows that β_i is proper and

$$\beta_i(E(F_i))\subseteq[1,\Delta],\qquad i\equiv 1 \pmod 2, \tag{2}$$
$$\beta_i(E(F_i))\subseteq[\Delta+1,2\Delta],\qquad i\equiv 0 \pmod 2; \tag{3}$$

as a consequence then β is proper.

Let us show now that we can use Lemma 8 to prove that $\chi'_a(G\times C_k)\le 2\Delta$. First, if $u,v\in V(G)$, then $S_\alpha(u)=S_\alpha(v)$ is equivalent to $S_\beta(u,i+)=$

$S_\beta(v,(i+1)_k-)$ as well as to $S_\beta(v,i+) = S_\beta(u,(i+1)_k-)$, which proves that the assumptions A_1 and A_2 of Lemma 8 are fulfilled. The validity of the assumption A_3 follows from (2) and (3).

To see A_4 suppose that $uv \in E(G)$, $d_G(u) = d_G(v)$ and $S_\beta(u,(i+1)_k+) = S_\beta(u,(i-1)_k+)$ for some $i \in [1,k]$. Putting $q_i := p_{(i+1)_k} - p_{(i-1)_k}$ we obtain

$$S_\beta(u,(i+1)_k+) = \{(l+q_i)_\Delta : l \in S_\beta(u,(i-1)_k+)\}, \qquad i \equiv 0 \pmod 2,$$
$$S_\beta(u,(i+1)_k+) = \{(l+q_i)_\Delta + \Delta : l \in S_\beta(u,(i-1)_k+)\}, \quad i \equiv 1 \pmod 2.$$

If i is even, the set $S_\beta(u,(i-1)_k+) \subseteq [1,\Delta]$ is invariant under the mapping $l \mapsto (l+q_i)_\Delta$. Then, however, $S_\beta(u,(i-1)_k+)$ can only be $[1,\Delta]$ (if either $q_i \in \{-2,2\}$ and Δ is odd or $q_i \in \{-1,1\}$) or one of $\{2j-1 : j \in [1,\frac{\Delta}{2}]\}$ and $\{2j : j \in [1,\frac{\Delta}{2}]\}$ (if $q_i \in \{-2,2\}$ and Δ is even, so that $k \equiv 2 \pmod 4$); in any case this contradicts the assumptions of our Theorem.

If i is odd, the set $S_\beta(u,(i-1)_k+) \subseteq [\Delta+1,2\Delta]$ is invariant under the mapping $l \mapsto (l+q_i)_\Delta + \Delta$. Then we have either $S_\beta(u,(i-1)_k+) = [\Delta+1,2\Delta]$ or $S_\beta(u,(i-1)_k+) \subseteq \{\{2j-1+\Delta : j \in [1,\frac{\Delta}{2}]\},\{2j+\Delta : j \in [1,\frac{\Delta}{2}]\}\}$ (if $q_i \in \{-2,2\}$ and Δ is even), a contradiction again.

Thus, by Lemma 8, $\chi'_a(G \times C_k) \le 2\Delta$ and we are done by Proposition 2.

Theorem 12. *If G is a $\{\Delta\}$-neighbor irregular graph and $k \in [4,\infty)$, then $\chi'_a(G \times P_k) = 2\Delta = \Delta(G \times P_k)$.*

Proof. Consider a proper avd coloring $\beta : E(G \times C_{2k}) \to [1,2\Delta]$ constructed in the proof of Theorem 11. Let $\gamma : E(G \times P_k) \to [1,2\Delta]$ be the restriction of β. Suppose that $uv \in E(G)$ and $d_{G\times P_k}(u,i) = d_{G\times P_k}(v,i+1)$ for some $i \in [1,k-1]$.

If $i = 1$, then $S_\gamma(u,1) \subseteq [1,\Delta]$ and $S_\gamma(v,2) \cap [\Delta+1,2\Delta] \ne \emptyset$ so that $S_\gamma(u,1) \ne S_\gamma(v,2)$.

If $i \in [2,k-2]$, then $S_\gamma(u,i) = S_\beta(u,i) \ne S_\beta(v,i+1) = S_\gamma(v,i+1)$.

Finally, with $i = k-1$ we have $S_\gamma(u,k-1) \ne S_\gamma(v,k)$, since $S_\gamma(u,k-1)$ has a nonempty intersection with both $[1,\Delta]$ and $[\Delta+1,2\Delta]$, while $S_\gamma(v,k)$ is a subset of one of the sets $[1,\Delta]$ and $[\Delta+1,2\Delta]$.

Thus, γ is a proper avd coloring and $\chi'_a(G \times P_k) = 2\Delta$.

Theorem 13. *Suppose that $k \in [3,\infty)$ and G is a D-neighbor irregular bipartite graph, where either Δ is odd and $D = \{\Delta\}$ or Δ is even and $D = \{\frac{\Delta}{2},\Delta\}$. Then $\chi'_a(G \times C_k) = 2\Delta = \Delta(G \times C_k)$.*

Proof. Let $\{U,V\}$ be the bipartition of the graph G. Consider a proper coloring $\alpha : E(G) \to [1,\Delta]$ (König's Theorem) and a 2-distinguishing sequence $\prod_{i=1}^k (p_i) \in \mathbb{Z}^k$ provided by Lemma 10. Let $\beta : E(G \times C_k) \to [1,2\Delta]$ be the coloring determined as follows: if $uv \in E(G)$, $u \in U$, $v \in V$ and $i \in [1,k]$, then

$$\beta((u,i)(v,(i+1)_k) := (\alpha(uv)+p_i)_\Delta,$$
$$\beta((u,i)(v,(i-1)_k) := (\alpha(uv)+p_i)_\Delta + \Delta.$$

From the definition it immediately follows that β is proper and

$$S_\beta(u,(i+1)_k-) = \{l+\Delta : l \in S_\beta(u,i+)\},$$
$$S_\beta(v,(i-1)_k+) = \{l+\Delta : l \in S_\beta(v,i-)\}.$$

Further, for any $u \in U$ and any $v \in V$, $S_\alpha(u) = S_\alpha(v)$ is equivalent to $S_\beta(u,i+) = S_\beta(v,(i+1)_k-)$ as well as to $S_\beta(v,i+) = S_\beta(u,(i+1)_k)$. Therefore, the assumptions A_1 and A_2 of Lemma 8 are fulfilled. The assumption A_3 follows from the inclusions $S_\beta(u,i+) \subset [1,\Delta]$ and $S_\beta(v,(i+1)_k+ \subset [\Delta+1,2\Delta]$. The validity of the assumption A_4 can be checked in the same way as in the proof of Theorem 11. So, Lemma 8 can be used as before.

Theorem 14. *Suppose that G is a D-neighbor irregular bipartite graph, where either Δ is odd and $D = \{\Delta\}$ or Δ is even and $D = \{\frac{\Delta}{2},\Delta\}$. Further, let H be a regular graph having a perfect matching provided that $\Delta(H)$ is odd. Then $\chi'_a(G \times H) = \Delta(G)\Delta(H) = \Delta(G \times H)$.*

Proof. Suppose first that $\Delta(H)$ is even, say $\Delta(H) = 2h$. By Petersen's Theorem there is a 2-factorization $\{H_i : i \in [,h]\}$ of the graph H. By Theorem 13 there is a (component-wise constructed) proper avd coloring

$$\gamma_i : E(G \times H_i) \to [1,\Delta] \times [2i-1,2i],\ i \in [1,h].$$

Consider the common extension $\gamma : E(G \times H) \to [1,\Delta] \times [1,2h]$ of the colorings γ_i, $i \in [1,h]$. If $(u,y) \in V(G \times H)$, then

$$S_\gamma(u,y) = \bigcup_{i=1}^{h} S_{\gamma_i}(u,y).$$

Further, if $uv \in E(G)$, $d_G(u) = d = d_G(v)$ and $(u,y)(v,z) \in E(G \times H)$, there is $l \in [1,h]$ such that $(u,y)(v,z) \in E(G \times H_l)$, and so $S_{\gamma_l}(u,y) \neq S_{\gamma_l}(v,z)$. Both sets $S_{\gamma_l}(u,y)$ and $S_{\gamma_l}(v,z)$ are of the same cardinality $2d$, hence

$$S_{\gamma_l}(u,y) \neq S_{\gamma_l}(v,z) \Leftrightarrow S_{\gamma_l}(u,y) \cap S_{\gamma_l}(v,z) \subsetneqq S_{\gamma_l}(u,y).$$

Then we have

$$\begin{aligned} S_\gamma(u,y) \cap S_\gamma(v,z) &= \left(\bigcup_{i=1}^{h} S_{\gamma_i}(u,y)\right) \cap \left(\bigcup_{j=1}^{h} S_{\gamma_j}(v,z)\right) \\ &= \bigcup_{i=1}^{h} \bigcup_{j=1}^{h} \left(S_{\gamma_i}(u,y) \cap S_{\gamma_j}(v,z)\right) \\ &\subsetneqq \bigcup_{i=1}^{h} \bigcup_{j=1}^{h} S_{\gamma_i}(u,y) = \bigcup_{i=1}^{h} S_{\gamma_i}(u,y) = S_\gamma(u,y), \end{aligned}$$

so that γ is an avd coloring and

$$\chi'_a(G \times H) \leq |[1, \Delta] \times [1, 2h]| = \Delta(G)\Delta(H) = \Delta(G \times H).$$

Now suppose that $\Delta(H) = 2h + 1$ and the graph H has a perfect matching. Then by Petersen's Theorem there is a factorization $\{H_i : i \in [1, h + 1]\}$ of the graph H, in which H_i, $i \in [1, h]$, are 2-factors and H_{h+1} is a 1-factor. Consider proper avd colorings γ_i, $i \in [1, h]$, from the first part of the proof. By Proposition 5 and by König's Theorem there is a (component-wise constructed) proper avd coloring

$$\gamma_{h+1} : E(G \times H_{h+1}) \to [1, \Delta] \times \{2h + 1\}.$$

For the common extension $\bar{\gamma} : E(G \times H) \to [1, \Delta] \times [1, 2h + 1]$ of the colorings γ_i, $i \in [1, h + 1]$, we proceed very similarly as above to show that $\chi'_a(G \times H) \leq \Delta(G)\Delta(H)$ again.

3.2 General graphs

If a graph G has adjacent vertices of degree Δ, Proposition 3 yields $\chi'_a(G \times C_k) \geq 2\Delta + 1$. In this section we show among other things that $\chi'_a(G \times C_k) \leq 2\Delta + 1$ whenever $k \geq 2\Delta + 1$ or k is even, $k \geq 6$.

Theorem 15. $\chi'_a(G \times K_2) \leq \min(\chi'_a(G), \Delta + 2)$ *for every graph* G.

Proof. The inequality $\chi'_a(G \times K_2) \leq \chi'_a(G)$ is known due to [5]; it follows also immediately from Proposition 5.2,3. Since $G \times K_2$ is bipartite, the inequality $\chi'_a(G \times K_2) \leq \Delta + 2$ is true because of [1].

There are graphs G such that $\chi'_a(G \times K_2)$ is smaller than $\chi'_a(G)$, *e.g.*, $\chi'_a(C_5 \times K_2) = 4 < 5 = \chi'_a(C_5)$.

Let us describe now one possibility how to construct proper edge colorings of $G \times C_k$ appropriate for using Lemma 8. By Theorem 6 there is a proper symmetric coloring $\alpha : E(G \times K_2) \to [1, \Delta]$. Consider a sequence $\prod_{i=1}^{k}(S_i)$, in which $S_i = \prod_{j=1}^{\Delta}(s_i^j)$ is a simple sequence with $\sigma(S_i) \subseteq [1, 2\Delta + 1]$ and $\sigma(S_i) \cap \sigma(S_{(i+1)_k}) = \emptyset$ for every $i \in [1, k]$. Define the coloring $\beta : E(G \times C_k) \to [1, 2\Delta + 1]$ so that for any $uv \in E(G)$ and any $i \in [1, k]$

$$\beta((u, i), (v, (i + 1)_k)) := s_i^{\alpha((u,1),(v,2))}. \tag{4}$$

From the definition it immediately follows that β is proper. Further, for any $u, v \in V(G)$ and any $i \in [1, k]$ the assumption A_3 of Lemma 8 is fulfilled and

$$S_\beta(u, i+) = S_\beta(u, (i + 1)_k-), \tag{5}$$

$$S_\beta(u, i+) = S_\beta(v, (i + 1)_k-) \Leftrightarrow S_\alpha(u, 1+) = S_\alpha(v, 2-). \tag{6}$$

The validity of the assumption A_1 (A_2, respectively) of Lemma 8 is a consequence of (5) (of (5) and (6)).

The possibility of applying Lemma 8 for the coloring β defined above depends on guaranteeing the assumption A_4 for any $uv \in E(G)$ with $d_G(u) = d_G(v)$ and any $i \in [1, k]$. To understand the idea how to do it consider simple sequences $A = \prod_{i=1}^{\Delta}(a^i), B = \prod_{i=1}^{\Delta}(b^i) \subseteq [1, 2\Delta + 1]^k$ with $|\sigma(A) \cap \sigma(B)| = \Delta - 1$ and let $G(A, B)$ be the oriented graph with $V(G(A, B)) = \sigma(A) \cup \sigma(B)$ and $E(G(A, B)) = \{(a^i, b^i) : i \in [1, \Delta]\}$. Clearly, exactly one component of $G(A, B)$ is an oriented path, which will be denoted by $P(A, B)$. (Remaining components – if any – of $G(A, B)$ are oriented cycles.) The pair (A, B) is said to be Δ-*good* if $|V(P(A, B))| \geq \Delta$. Since $G(B, A)$ results from $G(A, B)$ by changing the orientation of all the edges of $G(A, B)$, the pair (B, A) is Δ-good if and only if the pair (A, B) is.

Lemma 16. *Suppose that $\Delta \in [2, \infty)$, the pair (A, B) with simple sequences $A = \prod_{i=1}^{\Delta}(a^i), B = \prod_{i=1}^{\Delta}(b^i)$ is Δ-good and the mapping $\varphi : \sigma(A) \to \sigma(B)$ is defined by $\varphi(a^i) := b^i$ for $i \in [1, \Delta]$. Then $\varphi(X) \neq X$ for any set $X \subseteq \sigma(A)$ with $|X| \geq 2$.*

Proof. Let $P(A, B) = \prod_{i=1}^{k}(v^i)$ so that $v^k \notin X$. Since $|X| \geq 2$, $k \geq \Delta$, $|V(P(A, B)) \cap \sigma(A)| = k-1 \geq \Delta-1$ and $|\sigma(A)| = \Delta$, we have $X \cap V(P(A, B)) \neq \emptyset$. With $j := \max(i \in [1, k-1] : v^i \in X)$ then $v^{j+1} \in \varphi(X) \setminus X$ and $X \neq \varphi(X)$.

A sequence $\prod_{i=1}^{k}(S_i)$ of simple sequences S_i with $\sigma(S_i) \subseteq [1, 2\Delta + 1]$ and $l(S_i) = \Delta$, $i \in [1, k]$, is said to be Δ-*appropriate* if $\sigma(S_i) \cap \sigma(S_{(i+1)_k}) = \emptyset$ and the pair $(S_{(i-1)_k}, S_{(i+1)_k})$ is Δ-good for every $i \in [1, k]$.

Lemma 17. *If $\Delta \in [2, \infty)$, $k \in [3, \infty)$ and there is a Δ-appropriate sequence of length k, then $\chi'_a(G \times C_k) \leq 2\Delta + 1$.*

Proof. Let $\prod_{i=1}^{k}(S_i)$ be a Δ-appropriate sequence, $\alpha : E(G \times K_2) \to [1, \Delta]$ a symmetric proper coloring (Theorem 6) and let $\beta : E(G \times C_k) \to [1, 2\Delta+1]$ be a coloring defined by (4). As we have seen before Lemma 16, β is a proper coloring such that for any $u, v \in V(G)$ and any $i \in [1, k]$ the assumptions A_1, A_2 and A_3 of Lemma 8 are fulfilled. Suppose now that $i \in [1, k]$ and $d_G(u) = d = d_G(v)$ for an edge $uv \in E(G)$. From the definition of β it follows that

$$S_\beta(u, (i+1)_k+) = \beta_i(S_\beta(u, (i-1)_k+)),$$

where $\beta_i : \sigma(S_{(i-1)_k}) \to \sigma(S_{(i+1)_k})$ maps the jth term of $S_{(i-1)_k}$ to the jth term of $S_{(i+1)_k}$ for each $j \in [1, \Delta]$. The graph G of maximum degree Δ is connected, hence $|S_\beta(u, (i-1)_k+)| = d \geq 2$, and so, by Lemma 16, $S_\beta(u, (i+1)_k) \neq S_\beta(u, (i+1)_k)$. Thus, all assumptions of Lemma 8 are fulfilled, and we have $\chi'_a(G \times C_k) \leq 2\Delta + 1$.

Let $A = \prod_{i=1}^{d}(a^i)$, $B = \prod_{i=1}^{d}(b^i)$ be simple sequences of the same length d with $|\sigma(A) \cap \sigma(B)| = d - 1$ and let $t \in \mathbb{Z} \setminus \{0\}$. The sequence B is a t-*shift* of the sequence A provided that there is $j \in [1, d]$ such that $b^{(i+t)_d} = a^i$ for any $i \in [1, k] \setminus \{j\}$; then, clearly, $a^j \in \sigma(A) \setminus \sigma(B)$ and $b^{(j+t)_d} \in \sigma(B) \setminus \sigma(A)$. The fact that B is a t-shift of A will be denoted by $A \xrightarrow{t} B$. Evidently, $A \xrightarrow{t} B$ is equivalent to $B \xrightarrow{-t} A$.

Lemma 18. *Let A, B be simple sequences of the same length $d \in [2, \infty)$ with $|\sigma(A) \cap \sigma(B)| = d - 1$ and such that $A \xrightarrow{t} B$ for some $t \in \{-2, -1, 1, 2\}$. If either $t \in \{-2, 2\}$ and $d \equiv 1 \pmod 2$ or $t \in \{-1, 1\}$, then the pair (A, B) is d-good.*

Proof. Let $A = \prod_{i=1}^{d}(a^i)$ and $B = \prod_{i=1}^{d}(b^i)$. Suppose that there is $j \in [1, d]$ such that $b^{(i+t)_d} = a^i$ for any $i \in [1, k] \setminus \{j\}$. If $t = 1$, then $P(A, B) = \left[\prod_{i=1}^{d}(a^{(j+1-i)_d})\right](b^{j+1})$. Further, if $t = 2$ and d is odd, we have $P(A, B) = \left[\prod_{i=1}^{d}(a^{(j+2-2i)_d})\right](b^{j+2})$. In both cases $|V(P(A, B))| = d + 1$ and the pair (A, B) is d-good. If either $t = -2$ and d is odd or $t = -1$, then $B \xrightarrow{-t} A$, the pair (B, A) is d-good (by what we have just proved), hence so is the pair (A, B).

For the proof of the next theorem we will need the following obvious auxiliary result.

Lemma 19. *If $d, k, l \in [3, \infty)$ and $\mathcal{A} = \prod_{i=1}^{k}(A_i)$, $\mathcal{B} = \prod_{i=1}^{l}(B_i)$ are d-appropriate sequences with $A_i = B_i$, $i = 1, 2$, then $\mathcal{AB}$ is a d-appropriate sequence (of length $k + l$).*

Theorem 20. *Let $d \in [3, \infty)$. If $k \in [6, \infty)$ and either k is even or $k \geq 2d + 1$, there is a d-appropriate sequence of length k.*

Proof. The following sequences are important for our constructions:

$$T_{2j+1} := \left[\prod_{i=1}^{j}(2d + 1 - j + i)\right]\prod_{i=j+1}^{d}(-j + i), \quad j \in [0, d],$$
$$T_{2j+2} := \prod_{i=1}^{d}(d - j + i), \quad j \in [0, d - 1].$$

Let $\mathcal{T}^j := \prod_{i=1}^{j}(T_i)$ for $j \in [1, 2d + 1]$.

We shall in fact prove a stronger statement, namely the existence of a special d-appropriate sequence $\mathcal{S}_d^k = \prod_{i=1}^{k}(S_i^k)$ – one satisfying $\mathcal{T}^4 \leq \mathcal{S}_d^k$ if k is even and $\mathcal{T}^{2d} \leq \mathcal{S}_d^k$ if k is odd. For some k's the sequence $\mathcal{S}_d^k$ can be defined independently of the parity of d; since it can be applied for both parities of d, it will be denoted $\mathcal{B}_d^k = \prod_{i=1}^{k}(B_i^k)$. For remaining k's we will have in the role of $\mathcal{S}_d^k$ either a sequence $\mathcal{E}_d^k = \prod_{i=1}^{k}(E_i^k)$ (if d is even, in which case we shall suppose $d = 2l$) or $\mathcal{O}_d^k = \prod_{i=1}^{k}(O_i^k)$ (if d is odd).

Suppose first that k is even, $k \geq 6$, and proceed by induction on k. We start with defining $L_i^k := S_i$ for each $L \in \{B, E, O\}$ and $i \in [1, 4]$. As $T_{i-1} \xrightarrow{1} T_{i+1}$, $i = 1, 2$, by Lemma 18 we see that (S_{i-1}^k, S_{i+1}^k) is a d-good pair, $i = 1, 2$, and it only remains to be proved that $(S_{i-1}^k, S_{(i+1)_k}^k)$ is a d-good pair for each $i \in [3, k]$.

With

$$O^6_5 := \left[\prod_{i=1}^{d-2}(1+i)\right](2d,1),$$
$$O^6_6 := \left[\prod_{i=1}^{d-2}(d+1+i)\right](2d+1,d+1),$$

the sequence $\mathcal{O}^6_d$ is d-appropriate, since $O^6_3 \xrightarrow{-2} O^6_5 \xrightarrow{1} O^6_1$ and $O^6_4 \xrightarrow{-2} O^6_6 \xrightarrow{1} O^6_2$. Further, if

$$E^6_5 := (1,d-1,2d)\prod_{i=4}^{d}(-2+i),\ E^6_6 := (d+1,2d-1,2d+1)\prod_{i=4}^{d}(d-2+i),$$

the sequence $\mathcal{E}^6_d$ is d-appropriate, since

$$P(E^6_3,E^6_5) = (2d+1,1)\left[\prod_{i=3}^{d}(d+2-i)\right](2d),$$
$$P(E^6_4,E^6_6) = (d,d+1)\left[\prod_{i=3}^{d}(2d+2-i)\right](2d+1),$$
$$P(E^6_5,E^6_1) = (2d)\left[\prod_{i=2}^{l}(-1+2i)\right]\prod_{i=l+1}^{d}(-d+2i),$$
$$P(E^6_6,E^6_2) = (2d+1)\left[\prod_{i=2}^{l}(d-1+2i)\right]\prod_{i=l+1}^{d}(2i).$$

We define

$$B^8_5 := (2d,2d+1)\prod_{i=3}^{d}(-2+i),\quad B^8_6 := \left[\prod_{i=1}^{d-1}(d+i)\right](d-1),$$
$$B^8_7 := (d)\left[\prod_{i=2}^{d-1}(-1+i)\right](2d),\quad B^8_8 := (2d+1)\left[\prod_{i=2}^{d}(d-1+i)\right].$$

The sequence $\mathcal{B}^8_d$ is d-appropriate, since $B^8_3 \xrightarrow{1} B^8_5 \xrightarrow{-1} B^8_7 \xrightarrow{-1} B^8_1$ and $B^8_4 \xrightarrow{1} B^8_6 \xrightarrow{-1} B^8_8 \xrightarrow{-1} B^8_2$.

We define

$$O^{10}_5 := \left[\prod_{i=1}^{d-1}(i)\right](2d),\qquad O^{10}_6 := (2d+1)\left[\prod_{i=2}^{d-1}(d+i)\right](d),$$
$$O^{10}_7 := \left[\prod_{i=1}^{d-2}(1+i)\right](2d,d+1),\quad O^{10}_8 := \left[\prod_{i=1}^{d-2}(d+1+i)\right](1,2d+1),$$
$$O^{10}_9 := \left[\prod_{i=1}^{d-1}(2+i)\right](2),\qquad O^{10}_{10} := (2d,2d+1)\prod_{i=3}^{d}(d-1+i)$$

to obtain a d-appropriate sequence $\mathcal{O}^{10}_d$: $O^{10}_3 \xrightarrow{-1} O^{10}_5 \xrightarrow{-1} O^{10}_7 \xrightarrow{-1} O^{10}_9 \xrightarrow{2} O^{10}_1$ and $O^{10}_4 \xrightarrow{-1} O^{10}_6 \xrightarrow{-1} O^{10}_8 \xrightarrow{2} O^{10}_{10} \xrightarrow{-1} O^{10}_2$. Further, with $E^{10}_i := O^{10}_i$, $i \in [5,8]$, and

$$E^{10}_9 := \left[\prod_{i=1}^{d-3}(2+i)\right](d+1,2,d),$$
$$E^{10}_{10} := (2d-1,2d+1)\left[\prod_{i=3}^{d-1}(d-1+i)\right](2d),$$

the sequence $\mathcal{E}_d^{10}$ is d-appropriate, because $(E_{i-1}^{10}, E_{i+1}^{10}) = (O_{i-1}^{10}, O_{i+1}^{10})$ is a d-good pair, $i \in [4,7]$, and

$$P(E_7^{10}, E_9^{10}) = (2d)\left[\prod_{i=2}^{d-1}(i)\right](d+1, d),$$
$$P(E_8^{10}, E_{10}^{10}) = (1)\left[\prod_{i=2}^{l}(2d+2-2i)\right]\left[\prod_{i=l+1}^{d-1}(3d+1-2i)\right](2d+1, 2d),$$
$$P(E_9^{10}, E_1^{10}) = (d+1)\left[\prod_{i=2}^{l}(d+2-2i)\right]\prod_{i=l+1}^{d}(2d+1-2i),$$
$$P(E_{10}^{10}, E_2^{10}) = (2d+1)\left[\prod_{i=2}^{d-1}(d+i)\right](d+1).$$

Now suppose that $k \geq 12$ and for every even $p \in [6, k-2]$ there is a d-appropriate sequence $\mathcal{S}_d^p$ of length p with $\mathcal{T}^4 \leq \mathcal{S}_d^p$. Then, by Lemma 19, the sequence $\mathcal{S}_d^k := \mathcal{S}_d^{k-6}\mathcal{S}_d^6$ of length k is d-appropriate and satisfies $\mathcal{T}^4 \leq \mathcal{S}_d^k$.

For the rest of the proof $k \geq 2d+1$ will be odd. We start with setting $L_i^k := S_i$ for each $L \in \{B, E, O\}$ and $i \in [1, 2d]$. Since $T_{i-1} \xrightarrow{1} T_{i+1}$ for every $i \in [2, 2d-1]$, it suffices to show that $(S_{i-1}^k, S_{(i+1)_k}^k)$ is a d-good pair whenever $i \in [2d-1, k]$.

If $k = 2d+1$, taking $B_{2d+1}^{2d+1} := S_{2d+1}$ leads to a d-appropriate sequence $\mathcal{B}_d^{2d+1}$; indeed, we have $B_{2d}^{2d+1} \xrightarrow{1} B_1^{2d+1}$ and $B_{2d+1}^{2d+1} \xrightarrow{1} B_2^{2d+1}$.

We define

$$O_{2d+1}^{2d+3} := (1)\left[\prod_{i=2}^{d-2}(d+1+i)\right](d+2, 2d+1),$$
$$O_{2d+2}^{2d+3} := \left[\prod_{i=1}^{d-2}(2+i)\right](2d, 2),$$
$$O_{2d+3}^{2d+3} := \left[\prod_{i=1}^{d-3}(d+2+i)\right](2d+1, d+1, d+2);$$

then $\mathcal{O}_d^{2d+3}$ is a d-appropriate sequence, because $O_{2d-1}^{2d+3} \xrightarrow{1} O_{2d+1}^{2d+3}$,

$$P(O_{2d+1}^{2d+3}, O_{2d+3}^{2d+3}) = (1)\left[\prod_{i=2}^{d-2}(d+1+i)\right](2d+1, d+2, d+1),$$

$O_{2d+3}^{2d+3} \xrightarrow{2} O_2^{2d+3}$ and $O_{2d}^{2d+3} \xrightarrow{-1} O_{2d+2}^{2d+3} \xrightarrow{2} O_1^{2d+3}$.

By defining

$$E_{2d+1}^{2d+3} := \left[\prod_{i=1}^{d-4}(d+3+i)\right](d+2, 2d+1, 1, d+3),$$
$$E_{2d+2}^{2d+3} := \left[\prod_{i=1}^{d-3}(2+i)\right](2d, 2, d),$$
$$E_{2d+3}^{2d+3} := (d+1)\left[\prod_{i=2}^{d-2}(d+1+i)\right](2d+1, d+2)$$

we obtain a d-appropriate sequence $\mathcal{E}_d^{2d+3}$, since $E_{2d-1}^{2d+3} \xrightarrow{-1} E_{2d+1}^{2d+3}$,

$$P(E_{2d}^{2d+3}, E_{2d+2}^{2d+3}) = (d+1, d) \left[\prod_{i=3}^{d} (-1+i)\right] (2d),$$

$$P(E_{2d+2}^{2d+3}, E_{1}^{2d+3}) = (2d) \left[\prod_{i=2}^{l} (d+2-2i)\right] \prod_{i=l+1}^{d} (2d+1-2i),$$

$$P(E_{2d+3}^{2d+3}, E_{2}^{2d+3}) = (2d+1) \left[\prod_{i=2}^{d-1} (2d+1-i)\right] (2d)$$

and $P(E_{2d+1}^{2d+3}, E_{2d+3}^{2d+3})$ is equal to

$$(1) \left[\prod_{i=2}^{l+1} (2d+5-2i)\right] (d+2) \left[\prod_{i=l+3}^{d} (3d+4-2i)\right] (d+1).$$

In the case $k = 2d+5$ let

$$B_{2d+1}^{2d+5} := (1) \left[\prod_{i=2}^{d-1} (d+1+i)\right] (d+2),$$

$$B_{2d+2}^{2d+5} := (2d+1) \prod_{i=2}^{d-1} (d+1+i),$$

$$B_{2d+3}^{2d+5} := \left[\prod_{i=1}^{d-3} (d+2+i)\right] (d+1, d+2, 1),$$

$$B_{2d+4}^{2d+5} := \left[\prod_{i=1}^{d-1} (1+i)\right] (2d),$$

$$B_{2d+5}^{2d+5} := (2d+1) \left[\prod_{i=2}^{d-2} (d+1+i)\right] (d+1, d+2)$$

to form a d-appropriate set $\mathcal{B}_d^{2d+5}$: we have $B_{2d-1}^{2d+5} \xrightarrow{1} B_{2d+1}^{2d+5} \xrightarrow{-1} B_{2d+3}^{2d+5} \xrightarrow{1} B_{2d+5}^{2d+5}$,

$$P(B_{2d+5}^{2d+5}, B_{2}^{2d+5}) = (2d+1, d+1) \left[\prod_{i=3}^{d} (2d+2-i)\right] (2d)$$

and $B_{2d}^{2d+5} \xrightarrow{1} B_{2d+2}^{2d+5} \xrightarrow{-1} B_{2d+4}^{2d+5} \xrightarrow{1} B_{1}^{2d+5}$.

Finally, suppose that $k \geq 2d+7$ and for every odd $q \in [2d+1, k-2]$ there is a d-appropriate sequence $\mathcal{S}_d^q$ of length q with $\mathcal{T}^{2d} \leq \mathcal{S}_d^q$. By Lemma 19 then $\mathcal{S}_d^{k-6}\mathcal{S}_d^6$ is a d-appropriate sequence of length k satisfying $\mathcal{T}^{2d} \leq \mathcal{S}_d^k$.

Theorem 21. *If $\Delta \in [3, \infty)$, $k \in [6, \infty)$, and either k is even or $k \geq 2\Delta + 1$, then $\chi'_a(G \times C_k) \leq 2\Delta + 1$.*

Proof. The statement follows immediately from Lemma 17 and Theorem 20.

Theorem 22. *If $\Delta \in [3, \infty)$ and $k \in [4, \infty)$, then $\chi'_a(G \times P_k) \leq 2\Delta + 1$.*

Proof. Let $\beta : E(G \times C_{2k}) \to [1, 2\Delta + 1]$ be a proper avd coloring constructed using a Δ-appropriate sequence of length $2k$ (see Theorem 21) and let $\gamma : E(G \times P_k) \to [1, 2\Delta+1]$ be the restriction of β. Since $S_\gamma(u, 1) \neq S_\gamma(v, 2)$ and $S_\gamma(u, k-1) \neq S_\gamma(v, k)$ for arbitrary $u, v \in V(G)$, we can proceed similarly as in the proof of Theorem 12.

Theorem 21 does not cover the case $\Delta = 2$. However, if G is a connected graph of maximum degree 2, then G is either a cycle or a path. In the rest of this section we deal with the direct product of two cycles or of two paths. (The direct product of a path and a cycle was analyzed in [5].)

Let $d \in [2, \infty)$, $c \in [2d+1, \infty)$ and $k \in [3, \infty)$. A sequence $\prod_{i=1}^{k}(A_i)$ of d-subsets of the set $[1, c]$ is a *cyclic avd* (d, c)*-sequence* provided that

$$A_i \cap A_{(i+1)_k} = \emptyset,\ A_i \neq A_{(i+2)_k},\ i \in [1, k].$$

Note that a cyclic avd $(d, 2d+1)$-sequence is just a cyclic avd d-sequence in the terminology of [5]. In that paper it is proved:

Proposition 23. *If $l \in \{5, 6\} \cup [8, \infty)$, there exists a cyclic avd $(2, 5)$-sequence of length l.*

Lemma 24. *If G is a $[1, \Delta - 1]$-neighbor irregular graph, $c \in [2\Delta + 1, \infty)$, $k \in [3, \infty)$ and there is a cyclic avd (Δ, c)-sequence of length k, then $\chi'_a(G \times C_k) \leq c$.*

Proof. Let $\prod_{i=1}^{k}(A_i)$ be a cyclic avd (Δ, c)-sequence and for $i \in [1, k]$ let F_i be the subgraph of $G \times C_k$ induced by the set $V(G) \times \{i, (i+1)_k\}$. Since F_i is isomorphic to $G \times K_2$, by Theorem 6 there is a (symmetric) proper edge coloring $\alpha_i : E(F_i) \to A_i$, $i \in [1, k]$. Then, clearly, the common extension $\alpha : E(G \times C_k) \to [1, c]$ of the colorings α_i, $i \in [1, k]$, is proper. Suppose that $uv \in E(G)$ and $d_G(u) = d = d_G(v)$. Then $d = \Delta$ and

$$S_\alpha(u, i) = A_{(i-1)_k} \cup A_i \neq A_i \cup A_{(i+1)_k} = S_\alpha(v, (i+1)_k),\ i \in [1, k].$$

Thus, α is an avd coloring and $\chi'_a(G \times C_k) \leq c$.

Note that $\chi'_a(C_m \times C_n)$ is known in the following cases treated in [5]:

- at least one of m, n is even and greater than 4,
- both m, n are odd and greater than 7,
- $m = n \in [3, 4]$.

Theorem 25. *If $(m, n) \in [3, \infty) \times [3, \infty)$ and $(\{m\} \cup \{n\}) \cap ([3, \infty) \setminus \{3, 4, 7\}) \neq \emptyset$, then $\chi'_a(C_m \times C_n) = 5$.*

Proof. Since $C_m \times C_n \cong C_n \times C_m$, without loss of generality we may suppose that $n \in ([3, \infty) \setminus \{3, 4, 7\})$. Then, by Proposition 23, there is a cyclic avd $(2, 5)$-sequence of length n. Moreover, the graph C_m is $[1, 1]$-neighbor irregular, and so, by Lemma 24 with $c = 5$, $\chi'_a(C_m \times C_n) \leq 5$. Thus, we are done using Proposition 3.

Theorem 26. *If $(m, n) \in [3, \infty) \times [3, \infty)$, then $5 \leq \chi'_a(C_m \times C_n) \leq 6 = \Delta(C_m \times C_n) + 2$.*

Proof. If $l \in \{3, 4, 7\}$, there is a cyclic avd $(2, 6)$-sequence $C(2, 6, l)$ of length l, for example

$$C(2,6,3) := (\{1,2\},\{3,4\},\{5,6\}),$$
$$C(2,6,4) := (\{1,2\},\{3,4\},\{1,5\},\{3,6\}),$$
$$C(2,6,7) := (\{1,2\},\{3,4\},\{2,5\},\{1,3\},\{2,4\},\{3,5\},\{4,6\}).$$

So, having in mind Theorem 25, the statement follows from Proposition 3 and Lemma 24 with $c = 6$.

There are pairs (m, n), for which the upper bound in Theorem 26 applies. Namely, according to [5], $\chi'_a(C_3 \times C_3) = 6 = \chi'_a(C_4 \times C_4)$.

Finally, we turn to the direct product of two paths. From [5] it is known that $\chi'_a(P_m \times P_n) = 2$ if $(m, n) \in \{(2, 3), (3, 2)\}$ and $\chi'_a(P_m \times P_n) = 3$ if $\min(m, n) = 2$ and $\max(m, n) \geq 4$. By Theorem 9 we have $\chi'_a(P_m \times P_n) = 4$ provided that $\min(m, n) = 3$.

Theorem 27. *If $(m, n) \in [4, \infty) \times [4, \infty)$, then $\chi'_a(P_m \times P_n) = 5$.*

Proof. In [5] it is shown that there is a proper avd coloring $\beta : E(P_m \times C_{n+2}) \to [1, 5]$ satisfying $S_\beta(u, i+) \cap S_\beta(v, (i+1)_{n+2}+) = \emptyset$ for any $u, v \in V(P_m)$ and any $i \in [1, n+2]$. Therefore, similarly as in the proof of Theorem 12, the restriction $\gamma : V(P_m \times P_n) \to [1, 5]$ of the coloring β is a proper avd coloring. Thus, Proposition 3 yields $\chi'_a(P_m \times P_n) = 5$.

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