GENERALIZED WREATH PRODUCTS OF GRAPHS AND GROUPS

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ABSTRACT. Inspired by the definition of generalized wreath product of permutation groups, we define the generalized wreath product of graphs, containing the classical Cartesian and wreath product of graphs as particular cases. We prove that the generalized wreath product of Cayley graphs of finite groups is the Cayley graph of the generalized wreath product of the corresponding groups.

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1. Introduction

The idea of constructing new graphs starting from smaller component graphs is very natural. Products of graphs were widely studied in the literature for their theoretical interest in Combinatorics, Probability, Harmonic Analysis, but also for their practical applications. Standard products include the Cartesian product, direct product, strong product, lexicographic product [12, 16, 17] (see also the beautiful handbook [10]). In [15], the zig-zag product was introduced in order to produce constant-degree expanders of arbitrary size (see the surveys [11, 14] for definition, properties and further references on expander graphs). The zig-zag product and the simpler replacement product play also an important role in Geometric Group Theory, since it turns out that, when applied to Cayley graphs of two finite groups, they provide the Cayley graph of the semidirect product of these groups [1, 8, 11, 13], with a suitable choice of the corresponding generating sets. An analogous result holds for the classical wreath product of graphs (Theorem 2.6, Section 2).

Inspired by the paper [2], where the definition of generalized wreath product of permutation groups is given as a generalization of the classical direct and wreath product of permutation groups, we define the *generalized wreath product of graphs* (note that in [9] a different notion of generalized wreath product of graphs is presented). It is remarkable that, with a particular choice of the generating sets, our construction of the generalized wreath product applied to Cayley graphs of finite groups gives the Cayley graph of the generalized wreath product of the groups (Theorem 3.14, Section 3), providing a strong generalization of Theorem 2.6.

Key words and phrases. Wreath and generalized wreath product, Cayley graph, Poset block structure, Ancestral set.

2. Preliminaries

Let us start by recalling the definition of Cayley graph of a finitely generated group with respect to some symmetric generating set. We denote by 1_G the identity element of a group G.

Definition 2.1. Let G be a group generated by a finite set S, and suppose that S is symmetric, i.e., if $s \in S$, then also $s^{-1} \in S$, and that $1_G \notin S$. The Cayley graph Cay(G,S) of G with respect to S is the graph whose vertex set is G, and where two vertices g and g' are adjacent (we will use the notation $g \sim g'$) if there exists a generator $s \in S$ such that gs = g'. The graph Cay(G,S) is clearly a connected regular graph of degree |S|.

Note that we assume $1_G \notin S$ in order to avoid loops in the graph Cay(G, S).

Let us recall the definition of Cartesian product (see, for instance, [17], or [10, 12], where a more general construction containing it as a particular case is introduced) and wreath product of graphs [9].

Definition 2.2. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two finite graphs. The Cartesian product $\mathcal{G}_1 \square \mathcal{G}_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (v_1, v_2) and (w_1, w_2) are adjacent if:

- (1) either $v_1 = w_1$ and $v_2 \sim w_2$ in \mathcal{G}_2 ;
- (2) or $v_2 = w_2$ and $v_1 \sim w_1$ in \mathcal{G}_1 .

It follows from the definition that if \mathcal{G}_1 is a d_1 -regular graph on n_1 vertices and \mathcal{G}_2 is a d_2 -regular graph on n_2 vertices, then the graph $\mathcal{G}_1 \square \mathcal{G}_2$ is a $(d_1 + d_2)$ -regular graph on $n_1 n_2$ vertices. Notice also that the graphs $\mathcal{G}_1 \square \mathcal{G}_2$ and $\mathcal{G}_2 \square \mathcal{G}_1$ are isomorphic.

Definition 2.3. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two finite graphs. The wreath product $\mathcal{G}_1 \wr \mathcal{G}_2$ is the graph with vertex set $V_2^{V_1} \times V_1 = \{(f, v) | f : V_1 \to V_2, v \in V_1\}$, where two vertices (f, v) and (f', v') are connected by an edge if:

- (1) (edges of the first type) either $v = v' =: \overline{v}$ and f(w) = f'(w) for every $w \neq \overline{v}$, and $f(\overline{v}) \sim f'(\overline{v})$ in \mathcal{G}_2 ;
- (2) (edges of the second type) or f(w) = f'(w), for every $w \in V_1$, and $v \sim v'$ in \mathcal{G}_1 .

It follows from the definition that, if \mathcal{G}_1 is a regular graph on n_1 vertices with degree d_1 and \mathcal{G}_2 is regular graph on n_2 vertices with degree d_2 , then the graph $\mathcal{G}_1 \wr \mathcal{G}_2$ is a $(d_1 + d_2)$ -regular graph on $n_1 \cdot n_2^{n_1}$ vertices.

The wreath product of graphs represents a graph-analogue of the classical wreath product of groups (Theorem 2.6). To show that, we need to recall the basic definition of semidirect product of groups. Let A and B be two finite groups, and suppose that an action by automorphisms of B on A is defined, i.e., there exists a group homomorphism $\phi: B \to Aut(A)$. For every $a \in A$ and $b \in B$, we denote by a^b the image of a under the action of $\phi(b)$ and, similarly, we denote by $a^B = \{a^b \mid b \in B\}$ the orbit of a under the action of the group B.

Definition 2.4. The semidirect product $A \times B$ is the group whose underlying set is $A \times B = \{(a,b) \mid a \in A, b \in B\}$, and whose group operation is defined by

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2^{b_1}, b_1 b_2),$$
 for all $a_1, a_2 \in A, b_1, b_2 \in B$.

It is easy to check that the identity of $A \times B$ is given by $(1_A, 1_B)$, where 1_A and 1_B are the identity in A and B, respectively, and that $(a, b)^{-1} = ((a^{-1})^{b^{-1}}, b^{-1})$, for all $a \in A, b \in B$. Note that the subgroup $A \times \{1_B\}$ of $A \times B$ is isomorphic to A, it is normal in $A \times B$ and the action of B on A by conjugation coincides with the original action of B on A. In formulas, we have $(1_A, b)(a, 1_B)(1_A, b)^{-1} = (a^b, 1_B)$, for all $a \in A, b \in B$.

Definition 2.5. Let A and B be two finite groups. The set $B^A = \{f : A \to B\}$ can be endowed with a group structure with respect to the pointwise multiplication: $(f_1f_2)(a) = f_1(a)f_2(a)$. The wreath product $A \wr B$ is the semidirect product $B^A \rtimes A$, where A acts on B^A by shifts, i.e., if $f \in B^A$, one has

$$f^{a}(x) = f(a^{-1}x), \text{ for all } a, x \in A.$$

We introduce some notation. If $f \in B^A$ and $A = \{a_1, a_2, \dots, a_{n_A}\}$, then we write $f = (f_1, f_2, \dots, f_{n_A})$, where we denote by f_i the element $f(a_i) \in B$, for each $i = 1, \dots, n_A$. In particular, an element of $B^A \times A$ will be written as $((f_1, \dots, f_{n_A}), a)$.

Theorem 2.6. Let A and B be two finite groups and let S_A and S_B be symmetric generating sets for A and B, respectively. Then

$$Cay(A, S_A) \wr Cay(B, S_B) = Cay(A \wr B, S),$$

where S is the generating set of $A \wr B$ given by

$$S = \{((s_b, 1_B, \dots, 1_B), 1_A), ((1_B, \dots, 1_B), s_a) \mid s_a \in S_A, s_b \in S_B\}.$$

Proof. It is easy to check that S is a symmetric generating set of $A \wr B = B^A \rtimes A$. More precisely, any element $(f, a) = ((f_1, \ldots, f_{n_A}), a) \in B^A \rtimes A$, with $f_i \in B$ and $a \in A$, can be decomposed as

$$((f_1, 1_B, \dots, 1_B), 1_A)((1_B, f_2, 1_B, \dots, 1_B), 1_A) \cdots ((1_B, \dots, 1_B, f_{n_A}), 1_A)((1_B, \dots, 1_B), a).$$

Now if $a = \prod_{k=1}^r s_k^{m_k}$, with $s_k \in S_A$ and $m_k \in \mathbb{N}$, then one has

$$((1_B,\ldots,1_B),a)=\prod_{k=1}^r((1_B,\ldots,1_B),s_k)^{m_k}.$$

Similarly, if $f_1 = \prod_{h=1}^l s_h^{m_h}$, with $s_h \in S_B$ and $m_h \in \mathbb{N}$, it holds:

$$((f_1, 1_B, \dots, 1_B), 1_A) = \prod_{h=1}^l ((s_h, 1_B, \dots, 1_B), 1_A)^{m_h}.$$

Finally, we have

$$((1_B, \dots, 1_B, f_i, 1_B, \dots, 1_B), 1_A) = ((1_B, \dots, 1_B), a_i a_1^{-1})((f_i, 1_B, \dots, 1_B), a_1 a_i^{-1})$$

and so we conclude that S generates $A \wr B$.

What we have to prove now is that an edge in the graph product $Cay(A, S_A) \wr Cay(B, S_B)$

corresponds exactly to the multiplication by an element of S in $A \wr B$.

Consider an edge of the first type in $Cay(A, S_A) \wr Cay(B, S_B)$: such an edge connects the vertices $((f_1, \ldots, f_k, \ldots, f_{n_A}), a_k)$ and $((g_1, \ldots, g_k, \ldots, g_{n_A}), a_k)$. By definition, it must be $f_i = g_i$ for every $i \neq k$, whereas f_k and g_k are vertices adjacent in $Cay(B, S_B)$. It follows that there exists $s_k \in S_B$ such that $f_k s_k = g_k$. Then one gets $((g_1, \ldots, g_k, \ldots, g_{n_A}), a_k)$ from $((f_1, \ldots, f_k, \ldots, f_{n_A}), a_k)$ by multiplying by $((s_k, 1_B, \ldots, 1_B), 1_A)$. In fact:

$$((f_1, \dots, f_k, \dots, f_{n_A}), a_k) \cdot ((s_k, 1_B, \dots, 1_B), 1_A) = ((f_1, \dots, f_k, \dots, f_{n_A}) \cdot (s_k, 1_B, \dots, 1_B)^{a_k}, a_k \cdot 1_A)$$

$$= ((f_1, \dots, f_k, \dots, f_{n_A}) \cdot (1_B, \dots, 1_B), \underbrace{s_k}_{k-\text{th place}}, 1_B, \dots, 1_B), a_k) = ((g_1, g_2, \dots, g_k, \dots, g_{n_A}), a_k).$$

This implies that edges of the first type correspond to multiplication by elements of the form $((s_b, 1_B, \ldots, 1_B), 1_A)$, with $s_b \in S_B$.

Consider now an edge of the second type in $Cay(A, S_A) \wr Cay(B, S_B)$: such an edge connects the vertices $((f_1, \ldots, f_{n_A}), a_k)$ and $((g_1, \ldots, g_{n_A}), a_h)$. By definition, it must be $f_i = g_i$ for every $i = 1, \ldots, n_A$, whereas a_k and a_h are vertices adjacent in $Cay(A, S_A)$. It follows that there exists $s_a \in S_A$ such that $a_k s_a = a_h$. Then one gets $((g_1, \ldots, g_{n_A}), a_h)$ from $((f_1, \ldots, f_{n_A}), a_k)$ by multiplying by $((1_B, \ldots, 1_B), s_a)$. In fact:

$$((f_1, \ldots, f_{n_A}), a_k) \cdot ((1_B, \ldots, 1_B), s_a) = ((f_1, \ldots, f_{n_A}) \cdot (1_B, \ldots, 1_B)^{a_k}, a_k \cdot s_a) = ((f_1, \ldots, f_{n_A}) \cdot (1_B, \ldots, 1_B), a_h) = ((g_1, \ldots, g_{n_A}), a_h).$$

This ensures that edges of the second type correspond to multiplication by elements of the form $((1_B, \ldots, 1_B), s_a)$, with $s_a \in S_A$.

Example 2.7. Consider the graphs \mathcal{G}_1 and \mathcal{G}_2 in Fig. 1.

$$\underbrace{\begin{array}{ccc} u_1 & v_1 \\ \mathcal{G}_1 \end{array}} \underbrace{\begin{array}{ccc} u_2 & v_2 \\ \mathcal{G}_2 \end{array}}$$

Figure 1.

Then the wreath product $\mathcal{G}_1 \wr \mathcal{G}_2$ is the octagonal graph in Fig. 2.

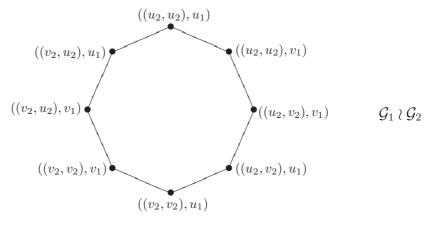
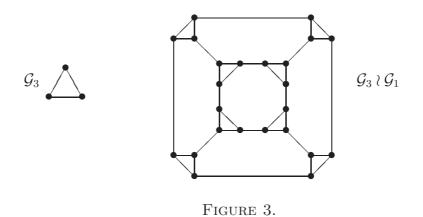


FIGURE 2.

If we regard the graphs \mathcal{G}_1 and \mathcal{G}_2 as the Cayley graphs of two cyclic groups of two elements, then the generators given by Theorem 2.6 have order 2, but they do not commute: the wreath product $\mathcal{G}_1 \wr \mathcal{G}_2$ is the Cayley graph of the wreath product of these groups, which is isomorphic to the dihedral group of 8 elements.

Now let \mathcal{G}_3 be a triangular graph. In Fig. 3 the wreath product $\mathcal{G}_3 \wr \mathcal{G}_1$ is represented.



Note that the graph $\mathcal{G}_3 \wr \mathcal{G}_1$ is obtained in [8] as the replacement product of the 3-dimensional Hamming cube with a triangular graph.

Remark 2.8. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two finite graphs. The *lexicographic* product $\mathcal{G}_1 \circ \mathcal{G}_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (v_1, v_2) and (w_1, w_2) are adjacent if:

- (1) either $v_1 \sim w_1$ in \mathcal{G}_1 ;
- (2) or $v_1 = w_1$ and $v_2 \sim w_2$ in \mathcal{G}_2 .

It follows from the definition that if \mathcal{G}_1 is a d_1 -regular graph on n_1 vertices and \mathcal{G}_2 is a d_2 -regular graph on n_2 vertices, then the graph $\mathcal{G}_1 \circ \mathcal{G}_2$ is a $(d_1n_2 + d_2)$ -regular graph on n_1n_2 vertices.

Sometimes the lexicographic product of graphs, whose automorphism group contains the wreath product of the automorphism groups of the factors, is called wreath product. It has nothing to do with the wreath product of Definition 2.3.

3. Generalized wreath product of graphs

Before introducing the notion of generalized wreath product of graphs (Definition 3.9), we recall the definition of poset block structure and generalized wreath product of permutation groups introduced in [2]. We will follow the same notation for the action to the right presented there. See also [3, 5, 7], where the Gelfand pairs associated with the action of a generalized wreath product of groups on a poset block structure are studied, in connection with Markov chain Theory.

Let (I, \preceq) be a finite poset, with |I| = n. For every $i \in I$, the following subsets of I can be defined:

- $A(i) = \{j \in I : j \succ i\} \text{ and } A[i] = A(i) \sqcup \{i\};$
- $H(i) = \{j \in I : j \prec i\} \text{ and } H[i] = H(i) \sqcup \{i\}.$

A subset $J \subseteq I$ is said ancestral if, whenever $i \succ j$ and $j \in J$, then $i \in J$. Note that by definition A(i) and A[i] are ancestral, for each $i \in I$. The set A(i) is called the ancestral set of i, whereas the set H(i) is called the hereditary set of i.

For each $i \in I$, let X_i be a finite set, with $|X_i| \ge 2$. For $J \subseteq I$, put $X_J = \prod_{i \in J} X_i$. In particular, we put $X = X_I$. If $K \subseteq J \subseteq I$, let π_K^J denote the natural projection from X_J onto X_K . In particular, we set $\pi_J = \pi_J^I$ and $x_J = x\pi_J$, for every $x \in X$. Moreover, we will use X^i for $X_{A(i)} = \prod_{j \in A(i)} X_j$ and π^i for $\pi_{A(i)}$.

Let \mathcal{A} be the set of ancestral subsets of I. If $J \in \mathcal{A}$, then the equivalence relation \sim_J on X is defined as

$$x \sim_J y \iff x_J = y_J, \quad \text{for } x, y \in X.$$

Definition 3.1. A poset block structure is a pair $(X, \sim_{\mathcal{A}})$, where

- (1) $X = \prod_{(I, \prec)} X_i$, with (I, \preceq) a finite poset and $|X_i| \geq 2$, for each $i \in I$;
- (2) $\sim_{\mathcal{A}}$ denotes the set of equivalence relations on X defined by all the ancestral subsets of I.

For each $i \in I$, let G_i be a permutation group on X_i and let F_i be the set of all functions from X^i into G_i . For $J \subseteq I$, we put $F_J = \prod_{i \in J} F_i$ and set $F = F_I$. An element of F will be denoted $f = (f_i)_{i \in I}$, with $f_i \in F_i$.

Definition 3.2. For each $f \in F$, the action of f on X is defined as follows: if $x = (x_i)_{i \in I} \in X$, then

$$xf = y$$
, where $y = (y_i)_{i \in I} \in X$ and $y_i = x_i(x\pi^i f_i)$, for each $i \in I$.

It is easy to verify that this is a faithful action of F on X, i.e, if xf = xg for every $x \in X$, then f = g. Therefore (F, X) is a permutation group, called the *generalized wreath* product of the permutation groups $(G_i, X_i)_{i \in I}$ and denoted $\prod_{(I, \preceq)} (G_i, X_i)$.

Definition 3.3. An automorphism of a poset block structure $(X, \sim_{\mathcal{A}})$ is a permutation σ of X such that, for every equivalence relation $\sim_{\mathcal{J}}$ in $\sim_{\mathcal{A}}$,

$$x \sim_J y \iff (x\sigma) \sim_J (y\sigma), \quad \text{for all } x, y \in X.$$

The following fundamental results are proven in [2]. We denote by $Sym(X_i)$ the symmetric group acting on X_i .

Theorem 3.4. The generalized wreath product of the permutation groups $(G_i, X_i)_{i \in I}$ is transitive on X if and only if (G_i, X_i) is transitive for each $i \in I$.

Theorem 3.5. Let (X, \sim_A) be the poset block structure associated with the poset (I, \preceq) . Let F be the generalized wreath product $\prod_{(I,\preceq)} Sym(X_i)$. Then F is the automorphism group of (X, \sim_A) . Given $f \in F$ and $J \subset I$ ancestral, a map $f_J : X_J \to X_J$ is defined such that $f\pi_J = \pi_J f_J$. The following lemma holds.

Lemma 3.6 ([2]). Let $f, h \in F$. Then fh = t, with

$$t_i = f_i \cdot f_{A(i)} h_i, \quad \text{for every } i \in I,$$

where the product of f_i and $f_{A(i)}h_i$ is pointwise.

Proof. Let $x \in X$. We have:

$$(xfh)_i = (xf)_i (xf\pi^i h_i)$$

$$= x_i (x\pi^i f_i) (x\pi^i f_{A(i)} h_i)$$

$$= x_i (x\pi^i (f_i \cdot f_{A(i)} h_i))$$

$$= x_i (x\pi^i t_i).$$

Remark 3.7. If (I, \leq) is a finite poset, with \leq the identity relation (Fig. 4), then the generalized wreath product is the permutation direct product. In this case, we have $A(i) = \emptyset$, for each $i \in I$, so that an element f of F is given by $f = (f_i)_{i \in I}$, where the function f_i is identified with an element of G_i , so that its action on x_i does not depend on any other coordinate of x.

FIGURE 4.

Remark 3.8. If (I, \preceq) is a finite chain (Fig. 5), then the generalized wreath product is the classical permutation wreath product $(G_1, X_1) \wr (G_2, X_2) \wr \cdots \wr (G_n, X_n)$. In this case, we have $A(i) = \{1, 2, \dots, i-1\}$, for each $i \in I$, so that an element $f \in F$ is given by $f = (f_i)_{i \in I}$, with

$$f_i: X_1 \times \cdots \times X_{i-1} \longrightarrow G_i$$

In other words, the action of f on x_i depends on its "ancestral" coordinates x_1, \ldots, x_{i-1} .

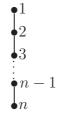


Figure 5.

Inspired by the definition of generalized wreath product of permutation groups, we introduce here the notion of generalized wreath product of graphs.

Let \mathcal{S}, \mathcal{T} be two sets. Given two functions $f, g: \mathcal{S} \longrightarrow \mathcal{T}$, we will use the notation $f \equiv g$ to say that f(x) = g(x) for every $x \in \mathcal{S}$. Similarly, we will write $f \equiv g$ in $A \subset \mathcal{S}$ to say that f(x) = g(x) for every $x \in A$.

Definition 3.9. Let (I, \preceq) be a finite poset, with |I| = n, and let $\mathcal{G}_i = (V_i, E_i)$ be a finite graph, for every $i \in I$. The generalized wreath product of the graphs $\{\mathcal{G}_i\}_{i \in I}$ is the graph \mathcal{G} with vertex set

$$V_{\mathcal{G}} = \{ (f_1, f_2, \dots, f_n) \mid f_i : \prod_{j \in A(i)} V_j \to V_i, \text{ for each } i \in I \}$$

and where two vertices $f = (f_1, f_2, \dots, f_n)$ and $h = (h_1, h_2, \dots, h_n)$ are adjacent if there exists $i \in I$, with $A(i) = \{i_1, \dots, i_p\}$, such that:

(1)
$$f_j \equiv h_j$$
, for every $j \neq i$;
(2) $f_i \equiv h_i$ in $\prod_{j \in A(i)} V_j \setminus \{(f_{i_1}, \dots, f_{i_p})\}$, and $f_i(f_{i_1}, \dots, f_{i_p}) \sim h_i(f_{i_1}, \dots, f_{i_p})$ in V_i .

The elements f_{i_l} , for l = 1, ..., p, are defined recursively, starting from indices whose ancestral set in (I, \preceq) is empty; more precisely, they represent vertices of V_{i_l} obtained by evaluating the functions f_{i_l} on $(f_{j_1}, \ldots, f_{j_m})$, where $A(i_l) = \{j_1, \ldots, j_m\}$, and so on.

In other words, $f = (f_1, \ldots, f_n)$ and $h = (h_1, \ldots, h_n)$ are adjacent if there exists $i \in I$ such that $f_j \equiv h_j$ for each $j \neq i$ and f_i coincides with h_i , except when evaluated on the p-tuple $(f_{i_1}, \ldots, f_{i_p})$, where $A(i) = \{i_1, \ldots, i_p\}$. Note also that, if $A(i) = \emptyset$, the condition (2) means that it must be $f_i \sim h_i$ in \mathcal{G}_i .

We have $|V_{\mathcal{G}}| = \prod_{i=1}^n |V_i|^{\prod_{j \in A(i)} |V_j|}$, where we put $\prod_{j \in A(i)} |V_j| = 1$ if $A(i) = \emptyset$. Moreover, if \mathcal{G}_i is a d_i -regular graph for every $i \in I$, then \mathcal{G} is a regular graph of degree $\sum_{i \in I} d_i$.

Example 3.10. Consider the case where |I|=4, with the poset (I, \preceq) and the graphs \mathcal{G}_i represented in Fig. 6.

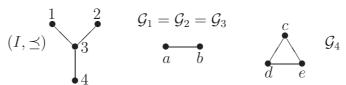


FIGURE 6.

In this case, $A(1) = A(2) = \emptyset$, $A(3) = \{1, 2\}$ and $A(4) = \{1, 2, 3\}$, so that

$$V_{\mathcal{G}} = \{ (f_1, f_2, f_3, f_4) \mid f_1 \in V_1, f_2 \in V_2, f_3 : \{a, b\}^2 \to \{a, b\}, f_4 : \{a, b\}^3 \to \{c, d, e\} \}.$$

The function f_3 will be represented as a 4-tuple of elements in $\{a, b\}$, whereas f_4 will be an 8-tuple of elements in $\{c, d, e\}$ (coordinates are ordered lexicographically). We have $|V_{\mathcal{G}}| =$ $2 \cdot 2 \cdot 2^4 \cdot 3^8$. Consider, for instance, the vertex $f = (a, b, (b, b, a, a), (c, e, d, c, e, d, e, e)) \in V_{\mathcal{G}}$. Its 5 neighbors in \mathcal{G} are the vertices:

(1)
$$(b, b, (b, b, a, a), (c, e, d, c, e, d, e, e))$$
, since $f_1 = a \sim b$ in \mathcal{G}_1 ;

- (2) (a, a, (b, b, a, a), (c, e, d, c, e, d, e, e)), since $f_2 = b \sim a$ in \mathcal{G}_2 ;
- (3) (a, b, (b, a, a, a), (c, e, d, c, e, d, e, e)), since $f_3(f_1, f_2) = f_3(a, b) = b \sim a$ in \mathcal{G}_3 ;
- (4) (a, b, (b, b, a, a), (c, e, d, d, e, d, e, e)) and (a, b, (b, b, a, a), (c, e, d, e, e, d, e, e)), since $f_4(f_1, f_2, f_3(f_1, f_2)) = f_4(a, b, b) = c \sim d, e \text{ in } \mathcal{G}_4.$

Remark 3.11. If (I, \preceq) is the poset (I, \preceq_1) (resp. (I, \preceq_2)) in Fig. 7, one obtains the classical Cartesian product of Definition 2.2 (resp. the classical wreath product of Definition 2.3).

$$(I, \preceq_1) \quad \bullet \quad \bullet \quad 1 \quad 2 \quad \qquad \downarrow_2 \quad (I, \preceq_2)$$

Figure 7.

Remark 3.12. In [6], finite posets are used to define products of finite Markov chains, called generalized crested products, and to develop their spectral analysis. Notice that this construction generalizes the crested products of Markov chains introduced in [4].

We are going to prove that the generalized wreath product of Cayley graphs of finite groups is the Cayley graph of the generalized wreath product of the groups.

Let (I, \preceq) be a finite poset, with |I| = n, and let G_i be a finite group, for each $i \in I$. Let S_i be a symmetric generating set for G_i and consider the Cayley graph $\mathcal{G}_i = Cay(G_i, S_i)$. In order to see the correspondence, we regard the group G_i as a permutation group on itself, acting on its elements by right multiplication (according with the notation of [2]). Definition 3.9 can be reformulated as follows.

Definition 3.13. Let (I, \preceq) be a finite poset, with |I| = n, and let $\mathcal{G}_i = Cay(G_i, S_i)$, where G_i is a finite group and S_i is a symmetric generating set of G_i , for all $i \in I$. We construct the graph \mathcal{G} with vertex set

$$V_{\mathcal{G}} = \{ (f_1, f_2, \dots, f_n) \mid f_i : \prod_{j \in A(i)} G_j \to G_i, \text{ for each } i \in I \}$$

and where the vertices (f_1, f_2, \ldots, f_n) and (h_1, h_2, \ldots, h_n) are adjacent if there exists $i \in I$, with $A(i) = \{i_1, \ldots, i_p\}$, such that:

- (1) $f_j \equiv h_j$, for every $j \neq i$; (2) $f_i \equiv h_i$ in $\prod_{j \in A(i)} G_j \setminus \{(f_{i_1}^{-1}, \dots, f_{i_p}^{-1})\}$, and the vertices $f_i(f_{i_1}^{-1}, \dots, f_{i_p}^{-1})$ and $h_i(f_{i_1}^{-1}, \dots, f_{i_p}^{-1})$ are adjacent in \mathcal{G}_i .

Also in this case, the elements f_{i_l} , for $l=1,\ldots,p$, are defined recursively, so that they represent elements of the group G_{i_l} obtained by evaluating the functions f_{i_l} on $(f_{j_1}^{-1}, \ldots, f_{j_m}^{-1})$, where $A(i_l) = \{j_1, \dots, j_m\}.$

The following theorem is a strong generalization of Theorem 2.6.

Theorem 3.14. The generalized wreath product \mathcal{G} of the graphs $\{\mathcal{G}_i = Cay(G_i, S_i)\}_{i \in I}$ is the Cayley graph of the generalized wreath product G of the groups $\{G_i\}_{i \in I}$, with respect to the generating set

$$S = \{ \overline{f_i} = (\mathbf{1}_1, \dots, \mathbf{1}_{i-1}, \overline{s_i}, \mathbf{1}_{i+1}, \dots, \mathbf{1}_n), i \in I \},$$

where $\overline{s_i}$ is a function taking the value $s_i \in S_i$ on $(1_{G_{1_1}}, \ldots, 1_{G_{i_p}})$, with $A(i) = \{i_1, \ldots, i_p\}$, and the value 1_{G_i} elsewhere, whereas $\mathbf{1}_q$ is the constant function taking the value 1_{G_q} on $\prod_{u \in A(q)} G_u$, for each $q \neq i$.

Proof. Generalizing the argument developed in the proof of Theorem 2.6, one can check that S is a generating set of the group G. Hence, we have to show that an adjacency in the generalized wreath product of the Cayley graphs can be obtained by multiplication by an element of S.

Suppose that the vertices $f = (f_1, f_2, ..., f_n)$ and $h = (h_1, h_2, ..., h_n)$ are adjacent, i.e., there exists $i \in I$ satisfying the conditions of Definition 3.13. Without loss of generality, we can suppose that

$$A(i) = \{l_1 < \dots < l_r < k_1 < \dots < k_s\}, \quad \text{with } r + s = p,$$

with $A(l_i) = \emptyset$, $A(k_j) = \{m_{j,1}, \ldots, m_{j,t_j}\}$ and $A(k_j) \subset A(k_{j+1})$. In particular, observe that it must be $t_j \leq p$ and $A(k_j) \subset A(i)$. By definition, it must be $f_j \equiv h_j$, for $j \neq i$; moreover, we have $f_i \equiv h_i$ in $\prod_{j \in A(i)} G_j \setminus \{(f_{i_1}^{-1}, \ldots, f_{i_p}^{-1})\}$, and there exists $s_i \in S_i$ such that $f_i(f_{i_1}^{-1}, \ldots, f_{i_p}^{-1}) = g_i$ and $h_i(f_{i_1}^{-1}, \ldots, f_{i_p}^{-1}) = g_i s_i$. We have to show the following identity:

(1)
$$(f_1, f_2, \dots, f_i, \dots, f_n)\overline{f_i} = (h_1, h_2, \dots, h_i, \dots, h_n).$$

The identity (1) is clearly true for every coordinate $j \neq i$. We use Lemma 3.6 in order to verify it for the coordinate i. Let $x = (x_1, \ldots, x_n) \in \prod_{i \in I} G_i$. Observe that, for every $c = 1, \ldots, r$, we can put $f_{l_c} = g_{l_c}$, for some $g_{l_c} \in G_{l_c}$, since $A(l_c) = \emptyset$.

The action of $(f_1, f_2, \dots, f_i, \dots, f_n)\overline{f_i}$ on x_i is given by

$$x_i \cdot (f_i(x_{l_1},\ldots,x_{l_r},x_{k_1},\ldots,x_{k_s}))$$

$$\cdot ((x_{l_1}g_{l_1},\ldots,x_{l_r}g_{l_r},x_{k_1}(f_{k_1}(x_{m_{1,1}},\ldots,x_{m_{1,t_1}})),\ldots,x_{k_s}(f_{k_s}(x_{m_{s,1}},\ldots,x_{m_{s,t_s}})))\overline{s_i}).$$

Now if

$$(x_{l_1},\ldots,x_{l_r},x_{k_1},\ldots,x_{k_s})=(g_{l_1}^{-1},\ldots,g_{l_r}^{-1},(f_{k_1}(f_{m_1,1}^{-1},\ldots,f_{m_1,t_1}^{-1}))^{-1},\ldots,(f_{k_s}(f_{m_s,1}^{-1},\ldots,f_{m_s,t_s}^{-1}))^{-1}),$$

then the argument of $\overline{s_i}$ is $(1_{G_{l_1}}, \ldots, 1_{G_{l_r}}, 1_{G_{k_1}}, \ldots, 1_{G_{k_s}})$, so that one gets

$$x_i(f_1, f_2, \dots, f_i, \dots, f_n)\overline{f_i} = x_ig_is_i$$

and so $(f\overline{f_i})_i = h_i$.

Otherwise, the argument of $\overline{s_i}$ is different from $(1_{G_{l_1}}, \ldots, 1_{G_{l_r}}, 1_{G_{k_1}}, \ldots, 1_{G_{k_s}})$, so that one gets $x_i(f_1, f_2, \ldots, f_i, \ldots, f_n)\overline{f_i} = x_ig_i1_{G_i} = x_ig_i$ and so $(f\overline{f_i})_i = f_i = h_i$.

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