UNIFORM ESTIMATES OF NONLINEAR SPECTRAL GAPS

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ABSTRACT. By generalizing the path method, we show that nonlinear spectral gaps of a finite connected graph are uniformly bounded from below by a positive constant which is independent of the target metric space. We apply our result to an *r*-ball $T_{d,r}$ in the *d*-regular tree, and observe that the asymptotic behavior of nonlinear spectral gaps of $T_{d,r}$ as $r \to \infty$ does not depend on the target metric space, which is in contrast to the case of a sequence of expanders. We also apply our result to the *n*-dimensional Hamming cube H_n and obtain an estimate of its nonlinear spectral gap with respect to an arbitrary metric space, which is asymptotically sharp as $n \to \infty$.

1. INTRODUCTION

Let G = (V, E) be a graph, whose set of vertices and unoriented edges are denoted by V and E, respectively. In this paper, graphs are always assumed to be simple, connected and finite. We denote by \vec{E} the set $\{(x, y) \in V \times V \mid \{x, y\} \in E\}$. A weight function on G is a symmetric function $m : V \times V \to [0, \infty)$ whose support equals \vec{E} . The pair (G, m) is called a weighted graph. A weight function m induces a weight m(x) of each vertex $x \in V$ by $m(x) = \sum_{y \in V} m(x, y)$. We use the convention that $m(\emptyset) = \sum_{x \in V} m(x)$. We call the following special weight function m the uniform weight function on G:

$$m(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \vec{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Throughout this paper, graphs and metric spaces are always assumed to contain at least two distinct points.

Definition 1.1. Let (G, m) be a weighted graph, and (X, d_X) be a metric space. We define the *nonlinear spectral gap* $\lambda(G, X)$ of G with respect to X to be the reciprocal of the smallest constant C > 0 such that the following Poincaré inequality holds for any $f: V \to X$:

$$\frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)d_X(f(x), f(y))^2 \le C \sum_{x,y \in V} m(x,y)d_X(f(x), f(y))^2.$$

Although the notation $\lambda_1^{\text{Gro}}(G, X)$ was used for $\lambda(G, X)$ in the first author's previous paper [12], we use this notation to avoid the notational complexity. We will see later that the constant C > 0 in the definition always exists. Hence nonlinear

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spectral gaps are always positive real numbers. If $X = \mathbb{R}$, $\lambda(G, \mathbb{R})$ equals the first positive eigenvalue $\mu_1(G)$ of the combinatorial Laplacian Δ on G, which acts on a real-valued function f on V as

$$\Delta f(x) = f(x) - \sum_{y \in V} \frac{m(x, y)}{m(x)} f(y), \quad x \in V.$$

Nonlinear spectral gaps play important roles both in geometric group theory and metric geometry. For example, they are related to the fixed point property of discrete groups ([5], [8], [9], [10], [16], [22]), measure concentration of metric measure spaces ([6]) and coarse embeddability of a metric space into another metric space ([5], [14], [15]).

Let us see the difference between $\lambda(G, X)$ and $\mu_1(G) = \lambda(G, \mathbb{R})$. For a Hilbert space \mathcal{H} , by summing over the coordinates with respect to some orthonormal basis, it is straightforward to see that $\lambda(G, \mathcal{H}) = \lambda(G, \mathbb{R}) = \mu_1(G)$ for any graph G. On the other hand, as we see in Proposition 1.2, the asymptotic behavior of nonlinear spectral gaps of a sequence of expanders changes drastically if the target metric space changes. A sequence of (uniformly weighted) finite connected graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ is called a *sequence of expanders* if it satisfies the following properties:

- (1) $\lim_{n\to\infty} |V_n| = \infty$.
- (2) There exists d such that $\deg(v) \leq d$ for all $v \in V_n$ and all n.
- (3) There exists $\lambda > 0$ such that $\mu_1(G_n) \ge \lambda$ for all n.

Proposition 1.2. Suppose that a sequence of (uniformly weighted) graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ satisfies the above properties (1) and (2). Then there exists a metric space X such that

$$\lambda(G_n, X) \lesssim_d \frac{1}{(\log |V_n|)^2}.$$

Here, $A \leq B$ or $B \geq A$ means that there exists a universal constant C > 0 such that $A \leq CB$. We write $A \simeq B$ if both $A \leq B$ and $B \leq A$ hold. If we have $A \leq C_p B$ for a constant $C_p > 0$ which depends only on some parameter p, we write $A \leq_p B$ or $B \geq_p A$. We write $A \simeq_p B$ if both $A \leq_p B$ and $B \leq_p A$ hold.

In particular, Proposition 1.2 shows that for any sequence of expanders $\{G_n\}$, for some metric space X, we have

(1.1)
$$\frac{\lambda(G_n, X)}{\mu_1(G_n)} \lesssim_d \frac{1}{(\log |V_n|)^2}.$$

In fact, this estimate of the ratio between linear and nonlinear spectral gaps turns out to be a sharp order of magnitude by a simple application of Bourgain's embedding theorem:

(1.2)
$$\frac{\lambda(G,X)}{\mu_1(G)} \gtrsim \frac{1}{(\log|V|)^2}$$

(see Proposition 2.4).

A purpose of this paper is to establish a more accurate lower estimate of the nonlinear spectral gap of a given graph, which is independent of target metric spaces, by generalizing the path-method developed by Jerrum and Sinclair [11], Diaconis and Stroock [3], and Quastel [18], and Diaconis and Saloff-Coste [4] (see also Saloff-Coste [19]). Although the original path method is well-known in the context of random walks as one giving only rough estimates of the linear spectral gap, we will see that our generalization gives asymptotically sharp estimates of the nonlinear spectral gaps in some examples.

A path on a graph G = (V, E) is a finite sequence

$$(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n), (x_n, x_{n+1})$$

in \vec{E} such that all the vertices x_1, \ldots, x_{n+1} are distinct. For two distinct vertices $x, y \in V$, we denote by $\Gamma(x, y)$ the set of all paths from x to y. For a weight function $w: V \times V \to [0, \infty)$ and a path $\gamma: (x_1, x_2), (x_2, x_3), \ldots, (x_n, x_{n+1})$ on G, we define

$$|\gamma|_w = \sum_{i=1}^n \frac{1}{w(x_i, x_{i+1})}.$$

Theorem 1.3. Let (G,m) be a weighted graph. Let w be another weight function on G. We assign one path $\gamma(x,y) \in \Gamma(x,y)$ to each ordered pair $(x,y) \in V \times V$ of distinct vertices. We define

$$A(w) = \max_{e \in \vec{E}} \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(e)} w(e) \sum_{(x,y) \text{ s.t. } \gamma(x,y) \ni e} |\gamma(x,y)|_w m(x) m(y) \right\}.$$

Then we have

$$\lambda(G, X) \ge \frac{1}{A(w)}$$

for any metric space X.

For the case $X = \mathbb{R}$, Theorem 1.3 is the usual path method.

In Section 3, we apply Theorem 1.3 to the *n*-dimensional Hamming cube H_n and an *r*-balls $T_{d,r}$ in the *d*-regular tree both equipped with uniform weights.

For Hamming cubes, as a corollary of Theorem 1.3, we obtain

(1.3)
$$\lambda(H_n, X) \gtrsim \frac{1}{n^2}$$

for any metric space X. Since it is known that

$$\mu_1(H_n) = \frac{2}{n},$$

this estimate is not asymptotically sharp for $X = \mathbb{R}$. However, as we see in Proposition 3.1 this is asymptotically sharp as an estimate for arbitrary metric spaces. Note that the estimate (1.2) only yields

$$\frac{\lambda(H_n, X)}{\mu_1(H_n)} \gtrsim \frac{1}{n^2}$$

As an another application of Theorem 1.3, we show that

(1.4)
$$\lambda(T_{d,r}, X) \asymp_d \frac{1}{(d-1)^r}$$

for any metric space X (Corollary 3.4). Hence, the asymptotic behavior of nonlinear spectral gaps of $T_{d,r}$ as $r \to \infty$ does not depend upon the target metric space in contrast to the case of a sequence of expanders (1.1). This implies that

$$\frac{\lambda(T_{d,r}, X)}{\mu_1(T_{d,r})} \asymp_d 1,$$

which is much better than the estimate (1.2).

Finally, we briefly review some results concerning the estimates of nonlinear spectral gaps with respect to the class of metric spaces called CAT(0) spaces. In order to estimate $\lambda(G, X)$ from below, Izeki and Nayatani [10] introduced an invariant $0 \leq \delta(X) \leq 1$ for a complete CAT(0) space X, and showed that

$$\frac{1}{2}(1-\delta(X))\mu_1(G) \le \lambda(G,X) \le \mu_1(G).$$

Hence, if $\delta(X) < 1$ we have

$$\mu_1(G_n) \asymp \lambda(G_n, X)$$

for any sequence of graphs $\{G_n\}_{n=1}^{\infty}$. This implies in particular that a sequence of expanders in a usual sense is also a sequence of expanders with respect to X.

Many estimates of δ have been done up to now (see [10], [9], [7], [20], and [21]). According to these estimates, the class of CAT(0) spaces with $\delta < 1$ includes Hilbert spaces, Hadamard manifolds, trees, complete CAT(0) cube complexes, and geodesically complete CAT(0) spaces which admit proper cocompact isometric group actions such as Bruhat-Tits buildings associated to semi-simple algebraic groups. On the other hand, the first author [12] proved the existence of a complete CAT(0) space X and a sequence $\{G_n\}_{n=1}^{\infty}$ of expanders such that

(1.5)
$$\lim_{n \to \infty} \lambda(G_n, X) = 0.$$

In particular, such a CAT(0) space X satisfies $\delta(X) = 1$. This result means that a drastic change in the nonlinear spectral gap may happen even within the class of CAT(0) spaces.

Related estimates of the nonlinear spectral gap in comparison with the linear spectral gap are also found in Naor-Silberman ([16]). They showed that if a metric space X has a finite Nagata dimension and G is any graph,

$$\frac{\lambda(G,X)}{\mu_1(G)^2} \gtrsim_X 1$$

holds.

The paper is organized as follows. In Section 2, we develop techniques to estimate nonlinear spectral gaps uniformly from below. First, we present a simple method to estimate nonlinear spectral gaps by using estimates of Euclidean distortions. Then we prove Theorem 1.3. In Section 3, we apply Theorem 1.3 to Hamming cubes and trees and obtain (1.3) and (1.4).

In this section, we prove Theorem 1.3. Before proving it, we present one simple argument to obtain a uniform lower bound of non-linear spectral gaps by using embeddings into a Hilbert space. This type of estimation is also found in Naor and Silberman [16].

Definition 2.1. Let (X, d_X) and (Y, d_Y) be metric spaces. For an injective mapping $f: X \to Y$, the *distortion* of f is defined to be the product

$$\sup_{x,y \in X, \ x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \cdot \sup_{x,y \in X, \ x \neq y} \frac{d_X(x, y)}{d_Y(f(x), f(y))}.$$

If a mapping $f: X \to Y$ is not injective, the distortion of f is defined to be ∞ . The *distortion* $c_Y(X)$ is the infimum of the distortions of all mappings from X to Y. We denote by $c_p(X)$ the distortion of X into L_p .

Proposition 2.2. Let (G, m) be a finite connected weighted graph with $n \ge 2$ vertices, and let (X, d_X) be a metric space. We define

$$c_2(X,n) = \max\{c_2(X') \mid X' \subset X, |X'| \le n\}.$$

Then we have

$$\lambda(G, X) \ge \frac{1}{c_2(X, n)^2} \mu_1(G).$$

Proof. Let $f: V \to X$ be an arbitrary mapping. Then, by the definition of $c_2(X, n)$, for any $\varepsilon > 0$, there exists a mapping $\varphi : f(V) \to \ell_2$ such that

$$d_X(f(x), f(y)) \le \|\varphi \circ f(x) - \varphi \circ f(y)\| \le (c_2(X, n) + \varepsilon)d_X(f(x), f(y))$$

holds for any $x, y \in V$. Hence, we have

$$\frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)d_X(f(x), f(y))^2$$

$$\leq \frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y) \|\varphi \circ f(x) - \varphi \circ f(y)\|^2$$

$$\leq \frac{1}{\mu_1(G)} \sum_{x,y \in V} m(x,y) \|\varphi \circ f(x) - \varphi \circ f(y)\|^2$$

$$\leq \frac{(c_2(X, n) + \varepsilon)^2}{\mu_1(G)} \sum_{x,y \in V} m(x,y)d_X(f(x), f(y))^2.$$

Since $\varepsilon > 0$ is arbitrary, this proves the proposition.

Combining this proposition with the following well-known Bourgain's embedding theorem, we obtain a uniform estimate on nonlinear spectral gaps.

Theorem 2.3 (Bourgain [1]). For every n-point metric space (X, d_X) , we have

$$c_2(X) \lesssim \log n$$

Proposition 2.4. Let (G, m) be a connected weighted graph with n vertices, and let (X, d_X) be a metric space. Then we have

$$\frac{\lambda(G, X)}{\mu_1(G)} \gtrsim \frac{1}{(\log n)^2}.$$

Though we do not see the graph structure in the proof of Proposition 2.4, this is asymptotically sharp as is shown by a sequence $\{G_n\}_{n=1}^{\infty}$ of expanders, for which we have $\mu_1(G_n) \simeq 1$ and $\lambda(G_n, X) \leq_d (\log |G_n|)^{-2}$ for some metric space X by Proposition 1.2. Here we give the proof of Proposition 1.2 in Introduction.

Proof of Proposition 1.2. We assume that $d \geq 3$ since when d = 2, a graph G_n is a path graph, and in this case the proposition follows from Remark 3.5. Let (X, d_X) be a metric space containing all G_n isometrically, and let $f_n : V_n \to X$ be an isometric embedding for each n. Since any r-ball in G_n contains at most $\sum_{i=0}^{i=r} (d-1)^i \asymp_d (d-1)^r$ vertices, at least $|V_n|/2$ vertices $y \in V_n$ satisfy $d_X(x, y) \gtrsim_d \log |V_n|$ for any $x \in V_n$. Thus, at least half of the pairs $(x, y) \in V_n \times V_n$ satisfy

$$d_X(f_n(x), f_n(y)) \gtrsim_d \log |V_n|$$

Hence, we have

$$\frac{\sum_{x,y\in V_n} m(x,y) d_X(f_n(x), f_n(y))^2}{\frac{1}{m(\emptyset)} \sum_{x,y\in V_n} m(x)m(y) d_X(f_n(x), f_n(y))^2} \le \frac{d^2 |V_n|^2}{\sum_{x,y\in V_n} d_X(f_n(x), f_n(y))^2}$$

which proves the proposition.

However, as we will see later, for a specific sequence of graphs, we can obtain a more accurate estimate by generalizing the path method.

Proof of Theorem 1.3. Let $f: V \to X$ be a map. Then the triangle inequality and the Cauchy-Schwarz inequality yield the following.

$$d_X (f(x), f(y))^2 \le \left\{ \sum_{(u,v)\in\gamma(x,y)} d_X(f(u), f(v)) \right\}^2$$

$$\le \left(\sum_{(u,v)\in\gamma(x,y)} w(u,v)^{-1} \right) \left(\sum_{(u,v)\in\gamma(x,y)} d_X(f(u), f(v))^2 w(u,v) \right)$$

$$= |\gamma(x,y)|_w \sum_{(u,v)\in\gamma(x,y)} d_X(f(u), f(v))^2 w(u,v).$$

Multiplying $\frac{m(x)m(y)}{m(\emptyset)}$ both sides of the above inequality and summing over all $(x, y) \in V \times V$, we obtain the following.

$$\begin{split} \frac{1}{m(\emptyset)} & \sum_{(x,y)\in V\times V} m(x)m(y)d(f(x), f(y))^2 \\ & \leq \sum_{(x,y)\in V\times V} \sum_{(u,v)\in\gamma(x,y)} \frac{1}{m(\emptyset)} |\gamma(x,y)|_w m(x)m(y)w(u,v)d(f(u), f(v))^2 \\ & = \sum_{(u,v)\in \vec{E}} \sum_{(x,y)\text{ s.t. } \gamma(x,y)\ni(u,v)} \frac{1}{m(\emptyset)} |\gamma(x,y)|_w m(x)m(y)w(u,v)d(f(u), f(v))^2 \\ & = \sum_{(u,v)\in \vec{E}} \left[m(u,v)d(f(u), f(v))^2 \right. \\ & \times \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(u,v)} w(u,v) \sum_{(x,y)\text{ s.t. } \gamma(x,y)\ni(u,v)} |\gamma(x,y)|_w m(x)m(y) \right\} \right] \end{split}$$

Thus,

$$\frac{1}{m(\emptyset)} \sum_{(x,y)\in V\times V} m(x)m(y)d(f(x), f(y))^2$$

$$\leq \max_{(u,v)\in\vec{E}} \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(u,v)} w(u,v) \sum_{(x,y)\text{s.t.}\gamma(x,y)\ni(u,v)} |\gamma(x,y)|_w m(x)m(y) \right\}$$

$$\times \sum_{(u,v)\in\vec{E}} m(u,v)d(f(u), f(v))^2$$

$$= A(w) \sum_{(u,v)\in\vec{E}} m(u,v)d(f(u), f(v))^2,$$

which proves the theorem.

3. Nonlinear spectral gaps of Hamming cubes and trees

In this section, we applied Theorem 1.3 to Hamming cubes and trees. Let H_n be the *n*-dimensional Hamming cube equipped with the uniform weight. In [19], it was shown that

$$\mu_1(H_n) \gtrsim \frac{1}{n^2}$$

by using the path method. However, it is not asymptotically sharp since it is known that

$$\mu_1(H_n) = \frac{2}{n}.$$

On the other hand, Theorem 1.3 guarantees that the estimation in [19] also works for nonlinear spectral gaps with respect to any target metric spaces. Thus we actually have

(3.1)
$$\lambda(H_n, X) \gtrsim \frac{1}{n^2}$$

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for an arbitrary metric space X. This is the right order of magnitude for general metric spaces because if we take the identity mappings $\iota_n : H_n \to H_n$, we see that

(3.2)
$$\lambda(H_n, H_n) \le \frac{\sum_{x,y \in V} m(x,y) d(\iota_n(x), \iota_n(y))^2}{\frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x) m(y) d(\iota_n(x), \iota_n(y))^2} = \frac{4}{n(n+1)}.$$

Proposition 3.1. Let H_n be the n-dimensional Hamming cube equipped with the uniform weight. Let X be an arbitrary metric space. Then, we have

$$\lambda(H_n, X) \gtrsim \frac{1}{n^2},$$

and this estimate is asymptotically sharp.

In fact, we will show in a forthcoming paper [13] that the right-hand side of (3.2) gives the optimal value of the infimum of the nonlinear spectral gaps:

$$\inf\{\lambda(H_n, X) : X \text{ is a metric space}\} = \lambda(H_n, H_n) = \frac{4}{n(n+1)}$$

Now, we proceed to another application of Theorem 1.3. For an integer $d \ge 2$, let T_d be the *d*-regular tree. For an integer $r \ge 0$, let $T_{d,r}$ be an *r*-ball in T_d . We consider the uniform weight function m on $T_{d,r}$. Combination of Bourgain's estimate ([2]) of the Euclidean distortion

$$c_2(T) \asymp \sqrt{\log \log n}$$

for an n-point tree T with Proposition 2.2 only yields that

$$\frac{\lambda(T,X)}{\mu_1(T)} \gtrsim \frac{1}{\log\log n}$$

for an arbitrary metric space X, which implies that for $T_{d,r}$, we have

(3.3)
$$\frac{\lambda(T_{d,r}, X)}{\mu_1(T_{d,r})} \gtrsim_d \frac{1}{\log r}.$$

Unfortunately, the right hand side of (3.3) tends to 0 as r goes to infinity. The following proposition gives a more accurate estimate.

Proposition 3.2. Let (X, d) be any metric space. Then we have

$$\lambda(T_{d,r}, X) \ge \frac{d-2}{d^2(d-1)} \frac{\{(d-1)^r - 1\}}{(d-1)^r} \times (d-1)^{-r}$$

for any $d \geq 3$ and $r \geq 1$.

Proof. Since $T_{d,r}$ is a tree, each $\Gamma(x, y)$ contains only one path $\gamma(x, y)$. We define a weight function $w: V \times V \to (0, \infty)$ on $T_{d,r}$ by setting

$$w(x,y) = (d-1)^{i+1}, \quad (x,y) \in \vec{E},$$

where *i* is the graph distance from the center vertex *o* to $\{x, y\}$.

Let $e = (u, v) \in \vec{E}$ be an arbitrary ordered edge. Then e separates V into two connected components, U containing u and W containing v. According to Theorem 1.3, we need to estimate

$$A(w,e) = \frac{1}{m(\emptyset)} \frac{1}{m(e)} w(e) \sum_{(x,y) \text{ s.t. } \gamma(x,y) \ni e} |\gamma(x,y)|_w m(x) m(y)$$

from above. We can assume that $o \in U$. Let k be the graph distance between o and v. Then we have

$$\begin{split} |W| &= \sum_{l=0}^{r-k} (d-1)^l = \frac{(d-1)^{r-k+1} - 1}{d-2} \le (d-1)^{r-k+1}, \\ |U| &\le |V| = 1 + \sum_{l=1}^r d(d-1)^{l-1} \le \frac{d}{d-2} (d-1)^r, \\ m(\emptyset) &= |\vec{E}| = 2 \sum_{l=1}^r d(d-1)^{l-1} = \frac{2d}{d-2} \left\{ (d-1)^r - 1 \right\}, \\ m(v) &\le d, \\ |\gamma(x,y)|_w \le 2 \sum_{i=1}^r \frac{1}{(d-1)^i} \le \frac{2}{d-2} \end{split}$$

for every $v, x, y \in V$. Thus,

$$\begin{split} A(w,e) &\leq \frac{d-2}{2d\left\{(d-1)^r - 1\right\}} \cdot \frac{1}{1} \cdot (d-1)^k \cdot \sum_{(x,y) \text{ s.t. } \gamma(x,y) \ni e} \frac{2}{d-2} d^2 \\ &\leq \frac{d-2}{d\left\{(d-1)^r - 1\right\}} \cdot (d-1)^k \cdot (d-1)^{r-k+1} \frac{d}{d-2} (d-1)^r \frac{1}{d-2} d^2 \\ &= \frac{d^2(d-1)}{d-2} \frac{(d-1)^r}{\{(d-1)^r - 1\}} \cdot (d-1)^r, \end{split}$$

which proves the proposition.

Proposition 3.3. Let (X, d) be any metric space. Then we have

$$\begin{split} \lambda(T_{d,r},X) &\leq 2 \bigg(\frac{d \left\{ (d-1)^{r-1} - 1 \right\}}{d-2} + (d-1)^{r-1} \\ &+ \frac{\left\{ (d-1)^{r-1} - 1 \right\} (d-1)^r}{(d-1)^r - 1} + \frac{(d-2) \left\{ (d-1)^{2r-1} \right\}}{d \left\{ (d-1)^r - 1 \right\}} \bigg)^{-1} \end{split}$$

for any $d \geq 3$ and $r \geq 1$.

Proof. We take an edge $e = \{o, v\} \in E$ containing the center. The edge e divides V into two components U containing o and $W = U^c$. Let $f : V \to X$ be a mapping sending U to p and W to q, where $p, q \in X$ are distinct points.

The weight of each vertex is either d or 1, and we have

$$\begin{split} |\{u \in U \mid m(u) = d\}| &= \sum_{i=0}^{r-1} (d-1)^i = \frac{(d-1)^r - 1}{d-2}, \\ |\{u \in U \mid m(u) = 1\}| &= (d-1)^r, \\ |\{w \in W \mid m(w) = d\}| &= \sum_{i=0}^{r-2} (d-1)^i = \frac{(d-1)^{r-1} - 1}{d-2} \frac{(d-1)^r - 1}{d-2}, \\ |\{w \in W \mid m(u) = 1\}| &= (d-1)^{r-1}. \end{split}$$

Thus, we have

$$\begin{split} \frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)d(f(x), f(y))^2 \\ &= \frac{d-2}{2d\left\{(d-1)^r - 1\right\}} \times 2d(p,q)^2 \bigg\{ d^2 \frac{\left\{(d-1)^{r-1} - 1\right\} \left\{(d-1)^r - 1\right\}}{(d-2)^2} \\ &+ d \frac{(d-1)^{r-1} \left\{(d-1)^r - 1\right\}}{d-2} + d \frac{\left\{(d-1)^{r-1} - 1\right\} (d-1)^r}{d-2} + (d-1)^{2r-1} \bigg\} \\ &= d(p,q)^2 \bigg\{ \frac{d\left\{(d-1)^{r-1} - 1\right\}}{d-2} + (d-1)^{r-1} \\ &+ \frac{\left\{(d-1)^{r-1} - 1\right\} (d-1)^r}{(d-1)^r - 1} + \frac{(d-2)\left\{(d-1)^{2r-1}\right\}}{d\left\{(d-1)^r - 1\right\}} \bigg\}. \end{split}$$

On the other hand,

$$\sum_{x,y \in V} m(x,y) d(f(x), f(y))^2 = 2d(p,q)^2.$$

Hence,

$$\begin{split} \frac{\sum_{x,y\in V} m(x,y) d(f(x),f(y))^2}{\frac{1}{m(\emptyset)} \sum_{x,y\in V} m(x)m(y) d(f(x),f(y))^2} \\ &= 2 \bigg\{ \frac{d\left\{ (d-1)^{r-1} - 1 \right\}}{d-2} + (d-1)^{r-1} + \frac{\left\{ (d-2)\left\{ (d-1)^{2r-1} \right\}}{d-2} \right\}^{-1}, \\ &\frac{\left\{ (d-1)^{r-1} - 1 \right\} (d-1)^r}{(d-1)^r - 1} + \frac{\left(d-2 \right)\left\{ (d-1)^{2r-1} \right\}}{d\left\{ (d-1)^r - 1 \right\}} \bigg\}^{-1}, \end{split}$$

which proves the proposition.

The following corollary is straightforward from Proposition 3.2 and 3.3. Corollary 3.4. Let X be any metric space, and let $d \ge 3$. Then, we have

$$\lambda(T_{d,r}, X) \asymp_d \frac{1}{(d-1)^r}.$$

Remark 3.5. When d = 2, $T_{2,r}$ is just a path graph with 2r+1 vertices. In this case nonlinear spectral gaps are never less than the linear spectral gap as we see below. Let $P_n = (V, E)$ be a path graph with n + 1 vertices. More precisely, suppose that $V = \{v_0, \ldots, v_n\}$ and $(v_i, v_j) \in \vec{E}$ if and only if |i - j| = 1. Let m be an arbitrary weight function on P_n , and let (X, d) be a metric space. For any $f : V \to X$, we define a map $\varphi_f : V \to \mathbb{R}$ by setting

$$\varphi_f(v_i) = \begin{cases} 0, & \text{if } i = 0, \\ \sum_{1 \le l \le i} d(f(v_{l-1}), f(v_l)), & \text{if } 2 \le i \le n \end{cases}$$

Then by the triangle inequality we have

$$\frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)d(f(x), f(y))^2$$

$$\leq \frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)|\varphi_f(x) - \varphi_f(y)|^2$$

$$\leq \frac{1}{\mu_1(P_n)} \sum_{x,y \in V} m(x,y)|\varphi_f(x) - \varphi_f(y)|^2$$

$$= \frac{1}{\mu_1(P_n)} \sum_{x,y \in V} m(x,y)d(f(x), f(y))^2,$$

which implies

$$\lambda(P_n, X) \ge \mu_1(P_n).$$

We remark that if the weight function m is uniform, it is known that

$$\mu_1(P_n) = 1 - \cos\frac{\pi}{n}.$$

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Added in proof. After we have completed this work, we found a uniform lower estimate of the nonlinear spectral gap for a regular graph has been obtained by Mendel and Naor [15, Lemma 2.1]. According to their estimate we have

$$\lambda(H_n, X) \ge \frac{1}{n4^n}.$$

However, this is not sharp by Proposition 3.1. Their estimate cannot be applied for $T_{d,r}$ since this is not a regular graph.

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