The Dilworth Number of Auto-Chordal-Bipartite Graphs

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Abstract

The mirror (or bipartite complement) mir(B) of a bipartite graph B = (X, Y, E) has the same color classes X and Y as B, and two vertices $x \in X$ and $y \in Y$ are adjacent in mir(B) if and only if $xy \notin E$. A bipartite graph is chordal bipartite if none of its induced subgraphs is a chordless cycle with at least six vertices. In this paper, we deal with chordal bipartite graphs whose mirror is chordal bipartite as well; we call these graphs auto-chordal bipartite graphs (ACB graphs for short). We describe the relationship to some known graph classes such as interval and strongly chordal graphs and we present several characterizations of ACB graphs. We show that ACB graphs have unbounded Dilworth number, and we characterize ACB graphs with Dilworth number k.

1 Introduction

Given a finite relation \mathcal{R} between two sets X and Y, a corresponding graph can classically be defined in several ways; X and Y are often considered either as stable sets or as cliques of a graph, and \mathcal{R} describes the edges between X and Y. First, when both X and Y are stable sets, \mathcal{R} defines a bipartite graph $bip(\mathcal{R})$ with edges between X and Y. The maximal bicliques of this bipartite graph can be organized by inclusion into a lattice, called a concept lattice $\mathcal{L}(\mathcal{R})$ [11] (or Galois lattice [5]). Second, when both X and Y are cliques, the corresponding graph is co-bipartite. Third, when without loss of generality, X is a clique and Y is a stable set, the corresponding graph is a split graph. Finally, \mathcal{R} defines a hypergraph where, without loss of generality, X is the vertex set and Y describes the hyperedges.

Naturally, there are strong relationships between different realizations of \mathcal{R} . One example of this correspondence, which is central to this paper, is the

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one between chordal bipartite graphs and strongly chordal graphs (see [7] and Lemma 2 of Section 2): A bipartite graph B = (X, Y, E) is chordal bipartite if and only if the graph obtained from B by completing X to a clique is strongly chordal.

In this paper, we will also use the complement relation $\overline{\mathcal{R}}$, which we called the mirror relation [2]. The mirror (or bipartite complement) mir(B) of a bipartite graph B = (X, Y, E) has the same color classes X and Y as B, and two vertices $x \in X$ and $y \in Y$ are adjacent in mir(B) if and only if $xy \notin E$. Several papers use this mirror notion, with various names and notations. Most of them in fact investigate 'auto-mirror' relations (i.e., both the relation and its mirror relation are in the same class). Such relations were used e.g. by [12] to describe bipartite graphs whose vertex set can be partitioned into a stable set and a maximal biclique; by [13] to decompose a bipartite graph in a manner similar to modular decomposition; by [17] to investigate the chain dimension of a bipartite graph, remarking the (obvious) fact that a bipartite graph is a chain graph (i.e., is $2K_2$ -free) if and only if its mirror is also a chain graph; by [1] and [10] to characterize split graphs of Dilworth number 2 (the Dilworth number of a graph is the maximum number of its vertices such that the neighborhood of one vertex is not contained in the closed neighborhood of another [20]; see Section 3); by [18] to characterize split graphs of Dilworth number 3; by [2] to characterize lattices with an articulation point.

Recently, [3] characterized concept lattices which are planar and whose mirror lattice is also planar: this is the case if and only if the corresponding bipartite graph as well as its mirror is chordal bipartite. We call these graphs auto-chordal-bipartite graphs (ACB graphs for short); these are the main topic of this paper. Though chordal bipartite graphs have given rise to a wealth of publications, to the best of our knowledge ACB graphs have not been studied.

By Lemma 2, ACB graphs correspond to split graphs which are strongly chordal and whose mirror is strongly chordal as well (*auto-strongly-chordal* graphs). One special class of auto-strongly-chordal graphs which is well-known is that of interval graphs whose complement is an interval graph as well (*auto-interval graphs*); this special class of split graphs was characterized by [1] using results from [10] as those having Dilworth number at most 2.

This paper is organized as follows: In Sections 2 and 3, we give some necessary notations, definitions and previous results. In Section 4, we show that the Dilworth number of ACB graphs is unbounded. We address the question of determining both the Dilworth number with respect to X and to Y, and show that both numbers can be arbitrarily large and that the gap between the two numbers can also be arbitrarily large. In Section 5, the main result of this paper is a characterization of ACB graphs with Dilworth number at most k in terms of forbidden induced subgraphs. Finally, in Section 6, we discuss some algorithmic aspects of ACB graphs.

2 Notions and Preliminary Results

2.1 Some Basic Graph and Hypergraph Notions

Throughout this paper, all graphs are finite, simple (i.e., without loops and multiple edges) and undirected. For a graph G = (V, E), let $\overline{G} = (V, \overline{E})$ denote the complement graph with $\overline{E} = \{xy : x \neq y, xy \notin E\}$. Isomorphism of graphs G_1, G_2 will be denoted by $G_1 \sim G_2$. As usual, $N(x) = \{y \in V : xy \in E\}$ is the open neighborhood of x, and $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of x

For $X \subseteq V$, G[X] denotes the subgraph induced by X. For a set \mathcal{F} of graphs, G is \mathcal{F} -free if none of the induced subgraphs of G is in \mathcal{F} . A clique is a set of pairwise adjacent vertices. A stable set or independent set is a set of pairwise non-adjacent vertices.

For two vertex-disjoint graphs G_1 and G_2 , G_1+G_2 denotes the disjoint union of them; iK_2 denotes the disjoint union of i edges, $i \geq 2$.

 C_k , $k \geq 4$, denotes the chordless cycle on k vertices. A graph is *chordal* if it is C_k -free for every $k \geq 4$. P_k , $k \geq 4$, denotes the chordless path on k vertices. For $k \geq 3$, a (complete) k-sun, denoted S_k , consists of a clique with k vertices, say q_0, \ldots, q_{k-1} , and another k vertices, say s_0, \ldots, s_{k-1} , such that s_0, \ldots, s_{k-1} form a stable set and every s_i is adjacent to exactly q_i and q_{i+1} (index arithmetic modulo k). Later on, S_3 , S_4 and $\overline{S_3}$ (also called net) play a special role. A chordal graph is strongly chordal [8] if it is S_k -free for every k > 3.

A bipartite graph B is a graph whose vertex set can be partitioned into two stable sets X and Y, which we refer to as its color classes. We use the notation B = (X, Y, E). A biclique in B = (X, Y, E) is a subgraph induced by sets $X' \subseteq X$ and $Y' \subseteq Y$ having all possible edges between elements of X' and Y'. In a bipartite graph B = (X, Y, E), vertices of a path P_k alternate between X and Y; a P_7 with its end-vertices in X (with its end-vertices in Y, respectively) is called an $X - P_7$ (a $Y - P_7$, respectively).

A bipartite graph B is a chordal bipartite graph if B is C_{2k} -free for all $k \geq 3$ [14]. A chain graph is a $2K_2$ -free bipartite graph; obviously, every chain graph is chordal bipartite. A co-bipartite graph is the complement of a bipartite graph, i.e., a graph whose vertex set can be partitioned into two cliques X and Y. A graph G is a split graph if its vertex set can be partitioned into a clique K and a stable set I, also denoted as G = (K, I, E). The following is well-known:

Lemma 1 ([9]). The following are equivalent:

- (i) G is a split graph.
- (ii) G and \overline{G} are chordal.
- (iii) G is $(2K_2, C_4, C_5)$ -free.

For a given bipartite graph B=(X,Y,E), let $split_X(B)$ ($split_Y(B)$, respectively) denote the split graph resulting from B by completing X (Y, respectively) to a clique. For example, if $B=C_{2k}$ then $split_X(B)=S_k$ and if $B=3K_2$ then $split_X(B)=\overline{S_3}$, $k\geq 3$.

The following fact is well-known ([7], see also [4]):

Lemma 2 ([7]). A bipartite graph B = (X, Y, E) is chordal bipartite if and only if $split_X(B)$ ($split_Y(B)$, respectively) is strongly chordal.

For a split graph G = (K, I, E) with split partition into a clique K and a stable set I, let bip(G) = (K, I, E') denote the corresponding bipartite graph with edge set $E' = \{xy : x \in K, y \in I, xy \in E\}.$

The reader is referred to [4] and [20] for further graph notions.

2.2Mirror of Relations, Hypergraphs, Bipartite Graphs and Split Graphs

The notion of mirror of relations, hypergraphs, bipartite graphs and split graphs is closely related to the complement and is defined as follows:

- (1) Let $\mathcal{R} \subseteq X \times Y$ be a relation between sets X and Y. The mirror relation of \mathcal{R} , denoted $mir(\mathcal{R}) = \overline{\mathcal{R}}$, is the complement relation $\overline{\mathcal{R}} \subseteq X \times Y$ such that $(x,y) \in \overline{\mathcal{R}}$ if and only if $(x,y) \notin \mathcal{R}$.
- (2) Let $H = (V, \mathcal{E})$ be a hypergraph. The mirror of H is the complement hypergraph $mir(H) = \overline{H} = (V, {\overline{e} : e \in \mathcal{E}}).$
- (3) Let B = (X, Y, E) be a bipartite graph. The mirror (or bipartite complement) of B is the bipartite graph mir(B) = (X, Y, E') such that for all $x \in X$, $y \in Y$, $xy \in E'$ if and only if $xy \notin E$. Thus, for example, $mir(C_6) = 3K_2$, $mir(C_8) = C_8$, and $mir(P_7) = P_7$.
- (4) Let G = (K, I, E) be a split graph with split partition into a clique K and stable set I. The mirror of G is the split graph mir(G) = (K, I, E')where for all $x \in K$ and for all $y \in I$, $xy \in E'$ if and only if $xy \notin E$.

Figure 1 illustrates the mirror of the bipartite graphs $C_6, C_8, 2K_2, 3K_2$ and

their split graphs $S_3, S_4, P_4, \overline{S_3}$. Note that $mir(S_3) = \overline{S_3}$ and $mir(S_4) = S_4$. Moreover, the following equalities obviously hold:

Proposition 1.

- (1) For any bipartite graph B = (X, Y, E), $\overline{split_X(B)} = mir(split_Y(B))$.
- (2) For any split graph G = (K, I, E), $mir(bip(G)) = bip(\overline{G})$ as well as $\overline{G} =$ $split_I(mir(bip(G))).$

Recall that a bipartite graph is auto-chordal bipartite (ACB for short) if B and mir(B) are chordal bipartite. Since for every $k \geq 5$, C_{2k} contains an induced subgraph $3K_2$, it follows:

Proposition 2. A bipartite graph B is an ACB graph if and only if B is $(3K_2, C_6, C_8)$ -free.

bipartite B	mir(B)	$Split_X(B)$	$mir(Split_X(B))$
C_6	$\int \int $	S_3	$\overline{S_3}$
C_8	C_8	S_4	S_4
\bigcup_{2K_2}	\bigcup_{2K_2}	P_4	P_4
$3K_2$	C_6	$\overline{S_3}$	S_3

Figure 1: Correspondences between bipartite graphs, split graphs, and their mirrors.

Strongly chordal graphs whose complement graph is strongly chordal as well are called *auto-strongly-chordal* in this paper. The next proposition follows from Lemma 1 and the following three facts: Auto-strongly-chordal graphs are exactly the S_k -free and $\overline{S_k}$ -free split graphs; for $k \geq 5$, sun S_k contains net $\overline{S_3}$ as induced subgraph; $S_4 \sim \overline{S_4}$.

Proposition 3. The following are equivalent:

- (i) G is auto-strongly-chordal.
- (ii) G is a $(S_3, \overline{S_3}, S_4)$ -free split graph.
- (iii) G is $(2K_2, C_4, C_5, S_3, \overline{S_3}, S_4)$ -free.

Together with Lemma 2 this gives:

Corollary 1. A bipartite graph B = (X, Y, E) is an ACB graph if and only if $split_X(B)$ ($split_Y(B)$, respectively) is auto-strongly-chordal.

Summarizing, we obtain:

Corollary 2. For a bipartite graph B = (X, Y, E), the following are equivalent:

- (i) B is an ACB graph.
- (ii) B is $(3K_2, C_6, C_8)$ -free.
- (iii) B is $3K_2$ -free chordal bipartite.
- (iv) $split_X(B)$ and $\overline{split_X(B)}$ are strongly chordal.

3 Dilworth Number of Hypergraphs, Graphs, Bipartite Graphs and Split Graphs

3.1 Dilworth Number and Poset Width

The width of a poset is the maximum number of its pairwise incomparable elements. In [20], Chapter 8.5 deals with the Dilworth number of graphs and the width of posets. We need the following notions of the Dilworth number:

3.1.1 Dilworth Number of Hypergraphs

Let $H = (V, \mathcal{E})$ be a hypergraph. The *Dilworth number dilw*(H) of H is the maximum number of pairwise incomparable hyperedges in \mathcal{E} with respect to set inclusion. Obviously:

$$dilw(H) = dilw(mir(H)).$$

Note that hypergraphs with pairwise incomparable hyperedges are also called Sperner hypergraphs. A Sperner hypergraph $H = (V, \mathcal{E})$ is k-critical if H has k hyperedges and if for all $v \in V$, deleting v in H leads to a non-Sperner hypergraph.

3.1.2 Dilworth Number of Graphs

For graphs, the *vicinal preorder* \leq on the vertex set of a graph G is defined as

$$x \leq y \text{ if } N(x) \subseteq N[y].$$

The Dilworth number dilw(G) of G is the maximum integer k such that there are k pairwise incomparable nodes with respect to \leq in G. Obviously, $dilw(G) = dilw(\overline{G})$ holds [10]. In [10], for a subset $S \subseteq V$, the Dilworth number $\nabla_G(S)$ with respect to S is defined as the maximum number of elements of S that are pairwise incomparable in the vicinal preorder of S.

3.1.3 Dilworth Number of Bipartite Graphs

In a bipartite graph B=(X,Y,E), for every $x\in X$ and $y\in Y$, the neighborhoods N(x) and N(y) are incomparable. Thus, it only makes sense to compare neighborhoods of vertices from X (from Y, respectively). Moreover, for $x,x'\in X$, $N(x)\subseteq N[x']$ if and only if $N(x)\subseteq N(x')$. Thus, the vicinal preorder for bipartite graphs can be defined as follows:

- (i) for $x, x' \in X$, $x \leq x'$ if $N(x) \subseteq N(x')$ and analogously,
- (ii) for $y, y' \in Y$, $y \prec y'$ if $N(y) \subseteq N(y')$.

Let $\nabla_B(X)$ ($\nabla_B(Y)$, respectively) be the Dilworth number of the corresponding neighborhood hypergraph $\mathcal{N}_X = \{N(x) : x \in X\}$ ($\mathcal{N}_Y = \{N(y) : y \in Y\}$, respectively), which we will refer to as the X-Dilworth number (Y-Dilworth number, respectively). We define the bipartite Dilworth number of B as

$$bip - dilw(B) := \max(\nabla_B(X), \nabla_B(Y)).$$

Obviously, for a bipartite graph B = (X, Y, E), $bip - dilw(B) \le \nabla_B(X) + \nabla_B(Y)$ holds. The Dilworth number dilw(B) of a bipartite graph B can be as large as the sum of the X- and Y-Dilworth numbers of B, as is the case for the C_{2k} with $k \ge 3$.

3.1.4 Dilworth Number of Split Graphs

For a split graph G = (K, I, E), the Dilworth number can be defined in a very similar way. Since K is a clique, I is independent, and for every $x \in K$ and $y \in I$, $N(y) \subseteq N[x]$ holds, it is natural to define $\nabla_G(K)$ ($\nabla_G(I)$, respectively) as the maximum number of pairwise incomparable neighborhoods of vertices in K (of vertices in I, respectively). Then, similarly as for bipartite graphs, let

$$dilw(G) := \max(\nabla_G(K), \nabla_G(I)).$$

Thus, for a split graph G and its bipartite version bip(G), dilw(G) = bip - dilw(bip(G)) holds and is the same as the maximum of the Dilworth numbers of the corresponding neighborhood hypergraphs.

3.2 Related Work on Small Dilworth Numbers

3.2.1 Dilworth Number 1

Recall that a bipartite graph is called a chain graph if it is $2K_2$ -free. Obviously, the following holds:

Proposition 4. A bipartite graph B = (X, Y, E) is a chain graph if and only if $\nabla_B(X) = \nabla_B(Y) = 1$.

The corresponding class for split graphs is the class of threshold graphs; in [6], it is shown:

Proposition 5. Let G be a split graph. The following are equivalent:

- (i) G is a threshold graph.
- (ii) dilw(G) = 1.
- (iii) G is $(2K_2, C_4, P_4)$ -free.

Note that $split_X(2K_2) = P_4$.

3.2.2 Dilworth Number 2 for Split Graphs

Interval graphs form a famous graph class; it is well-known that these graphs are strongly chordal, and a graph is an interval graph if and only if it is chordal and its complement graph is a comparability graph (see e.g. [4]). Since a graph G is a permutation graph if and only if G and \overline{G} are comparability graphs, it follows that G and \overline{G} are interval graphs if and only if G is a split graph and a permutation graph. Various papers deal with such graphs [1, 15].

Let $rising\ sun\ denote$ the graph resulting from S_4 by deleting one of its simplicial vertices, and let co- $rising\ sun\ denote$ its complement graph (see Figure 2). Földes and Hammer [10] proved the following property:

Proposition 6 ([10]). A split graph G is an interval graph if and only if G is $(S_3, \overline{S_3}, rising sun)$ -free.

Corollary 3 ([1]). If G is a split graph then G and \overline{G} are both interval graphs if and only if G is $(S_3, \overline{S_3}, rising sun, co-rising sun)-free.$

As already mentioned, for split graphs G = (K, I, E), Földes and Hammer in [10] defined the *I*-Dilworth number $\nabla_G(I)$ for the independent set *I*. They showed:

Proposition 7 ([10]). Let G = (K, I, E) be a split graph. Then G is an interval graph if and only if $\nabla_G(I) \leq 2$.

Corollary 4. A graph G and its complement are interval graphs if and only if the Dilworth number of G is at most 2.

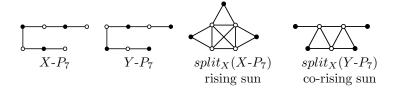


Figure 2: Correspondences between the P_7 s and their split graphs.

Thus, S_3 , $\overline{S_3}$, rising sun, and co-rising sun are the minimal split graphs of Dilworth number 3.

Remark. Surprisingly, both papers [10] and [1] have nearly the same title, namely "Split graphs of Dilworth number 2", and in Theorem 5 of [1], the result of [10] is cited in a wrong way, namely as "G is an interval and split graph if and only if its Dilworth number is at most 2" (but the results in [1] are correct).

3.2.3 Dilworth Number 2 for Bipartite Graphs

Recall that $split_X(C_6) = S_3$ and $split_X(3K_2) = \overline{S_3}$. Obviously, for an X- P_7 P, $split_X(P)$ is the rising sun, and for a Y- P_7 P, $split_X(P)$ is the co-rising sun, as illustrated by Figure 2.

Split graphs which are interval graphs are characterized in Proposition 6 as being $(S_3, \overline{S_3}, rising sun)$ -free and in Proposition 7 as having *I*-Dilworth number at most 2. Let us translate Proposition 6 into terms of ACB graphs.

Proposition 8. For a bipartite graph B = (X, Y, E), $split_X(B)$ is an interval graph if and only if B is $(3K_2, C_6, X-P_7)$ -free.

Since C_8 contains P_7 , it follows:

Corollary 5. For an ACB graph B = (X, Y, E), $split_X(B)$ is an interval graph if and only if B is X-P₇-free.

Corollary 3 corresponds to a more restricted class of ACB graphs:

Proposition 9. Let B = (X, Y, E) be a bipartite graph. The following are equivalent:

- (i) B is $(3K_2, C_6, P_7)$ -free.
- (ii) B is a P₇-free ACB graph.
- (iii) $split_X(B)$ and $split_Y(B)$ are both interval graphs.

Recall that for a split graph G=(K,I,E) with clique K and stable set I, bip(G) denotes the bipartite graph resulting from G by turning K into a stable set

Proposition 10. Let G be a split graph. The following are equivalent:

- (i) G is $(S_3,\overline{S_3},rising\ sun,co-rising\ sun)$ -free.
- (ii) bip(G) is $(3K_2, C_6, P_7)$ -free.
- (iii) bip(G) is a P_7 -free ACB graph.

In [1], a linear-time recognition algorithm for split graphs of Dilworth number greater than 2 is given. This approach can easily be adapted to decide whether a bipartite graph has X-Dilworth number (Y-Dilworth number, respectively) more than 2.

3.2.4 Dilworth Numbers 3 and 4 for Split Graphs

In [18], Nara characterized split graphs of Dilworth number at most 3 by a list of forbidden induced subgraphs G_1, \ldots, G_{16} which, together with their complement graphs, represent all 4-critical split graphs; for $k \geq 2$, a split graph G = (K, I, E) is called k-critical (with respect to I) in [18] if $\nabla_G(I) = k$ and for all $v \in I$, $\nabla_{G-v}(I-v) \leq k-1$.

Theorem 1 ([18]). The Dilworth number of a split graph G is at most 3 if and only if G and \overline{G} are (G_1, \ldots, G_{16}) -free.

It should be noted that the graph G_{11} of Nara's list in [18] contains the 4-sun S_4 as an induced subgraph (which is graph G_{12} of his list) and thus is not 4-critical. But we verified Theorem 1 by determining all 4-critical (and 5-critical) Sperner hypergraphs using a well adapted backtracking. It turned out that there are 15 4-critical and 178 5-critical Sperner hypergraphs.

Up to G_{11} all other 15 graphs of Nara's list are determined by the 4-critical Sperner hypergraphs and hence are 4-critical. The 4-critical split graphs $H_1 - H_{10}$ and their mirrors in Figure 3 represent the 15 4-critical Sperner hypergraphs.

Theorem 1 can also be formulated in the following way:

Theorem 2. The Dilworth number of a split graph G is at most 3 if and only if G and \overline{G} are (H_1, \ldots, H_{10}) -free and $(mir(H_1), \ldots, mir(H_{10}))$ -free.

4 Arbitrarily large X- and Y-Dilworth Numbers of ACB Graphs

In the following, we show that for ACB graphs G, both $\nabla_G(X)$ and $\nabla_G(Y)$ can be arbitrarily large. For $k \geq 2$, let $D_k = (X, Y, E)$ be the bipartite graph with vertex sets $X = X_{3k-4} = \{x_1, \dots, x_{3k-4}\}$ and $Y = Y_{3k-4} = \{y_1, \dots, y_{3k-4}\}$ where $x_i y_j$ is an edge if and only if

$$|i-j| < k-2 \text{ and } \{i,j\} \cap \{k-1,\ldots,2k-2\} \neq \emptyset.$$

Figure 4 shows graph D_5 . In order to show the subsequent Theorem 3, we color the vertices x_i and y_j green if $i, j \in \{k-1, \ldots, 2k-2\}$ and red otherwise.

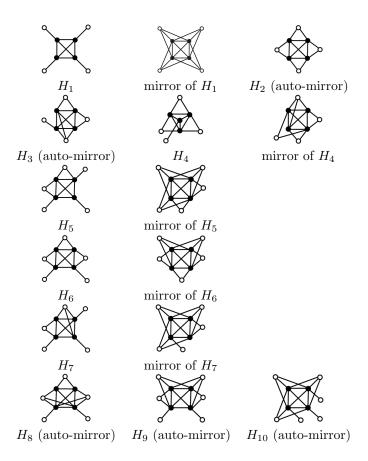


Figure 3: 4-critical split graphs $H_1 - H_{10}$ and their mirrors.

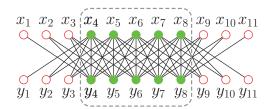


Figure 4: Graph D_5 having X- and Y-Dilworth number 5: $\nabla_B(X) = \nabla_B(Y) = 5$.

Note that in D_k , X consists of a green central interval of k vertices surrounded by two red intervals, each of k-2 vertices, and the same holds for Y. Note also that each edge of D_k has at least one green vertex, and subgraphs induced by one of the red intervals and its green neighbors form chain graphs, i.e., are $2K_2$ -free.

Theorem 3. For every $k \geq 2$, D_k is an ACB graph with Dilworth numbers $\nabla_{D_k}(X) = \nabla_{D_k}(Y) = k$.

Proof. Obviously, $|N(x_i)| = 2(k-2)+1$ if $i \in \{k-1,\ldots,2k-2\}$. Moreover, $N(x_1) \subseteq \cdots \subseteq N(x_{k-1})$ and $N(x_{2k-2}) \supseteq \cdots \supseteq N(x_{3k-4})$. Thus $\nabla_{D_k}(X) = k$ and by symmetry also $\nabla_{D_k}(Y) = k$.

In order to show that D_k is an ACB graph, we need some lemmas:

Lemma 3. Let $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}$ be edges but $x_{i_1}y_{j_2}$ and $x_{i_2}y_{j_1}$ not be edges in D_k . Then $i_1 < i_2$ implies $j_1 < j_2$.

Proof. Assume to the contrary that

$$i_1 < i_2 \text{ and } j_1 > j_2.$$
 (1)

By symmetry, we may assume that x_{i_1} is green. Since $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}$ are edges

$$i_1 - (k-2) \le j_1 \le i_1 + (k-2),$$

 $i_2 - (k-2) \le j_2 \le i_2 + (k-2)$

and consequently in view of (1)

$$i_1 - (k-2) < i_2 - (k-2) \le j_2 < j_1 \le i_1 + (k-2),$$

i.e., $x_{i_1}y_{j_2}$ is an edge, a contradiction.

Lemma 4. Let $x_i y_{j_1}, x_i y_{j_3}$ be edges in D_k . Then $j_1 < j_2 < j_3$ implies that $x_i y_{j_2}$ is an edge in D_k .

Proof. Let $j_1 < j_2 < j_3$.

Case 1. x_i is green. By supposition, the interval $\{x : |i-x| \le k-2\}$ contains j_1, j_3 and hence it contains also j_2 . But this implies that $x_i y_{j_2}$ is an edge.

Case 2. x_i is red. Then y_{j_1} and y_{j_3} are green and hence also y_{j_2} is green. By symmetry, we may assume that i < k - 1. By supposition, the interval $\{x : k - 1 \le x \le i + (k - 2)\}$ contains j_1, j_3 and hence it contains also j_2 . But this implies that $x_i y_{j_2}$ is an edge.

Lemma 5. The graph D_k is $3K_2$ -free.

Proof. Assume the contrary and let $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}, x_{i_3}y_{j_3}$ be the edges of an induced $3K_2$. We may assume that $i_1 < i_2 < i_3$. Then, by Lemma 3, $j_1 < j_2 < j_3$. First we show that x_{i_2} is green. Indeed, assume that it is red. By symmetry we may assume that $i_2 < k-1$. But then also $i_1 < k-1$ which means that x_{i_1} is red. Then y_{j_1} and y_{j_2} have to be green. Since $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}$ are edges,

$$k-1 \le j_1 \le i_1 + (k-2),$$

 $k-1 \le j_2 \le i_2 + (k-2).$

With $i_1 < i_2$ it follows that

$$k-1 \le j_1 \le i_1 + (k-2) < i_2 + (k-2),$$

i.e., $x_{i_2}y_{j_1}$ is an edge, a contradiction.

Thus x_{i_2} and analogously y_{j_2} are green. By symmetry we may assume that x_{i_1} is green. Then $k-1 \le i_1 < i_2 \le 2k-2$ and $k-1 \le j_2 \le 2k-2$.

Case 1. $j_2 < 2k - 2$. Then

$$-(k-2) = (k-1) - (2k-3) < i_2 - i_1 < (2k-3) - (k-1) = k-2,$$

i.e., $x_{i_1}y_{j_2}$ is an edge, a contradiction.

Case 2. $j_2 = 2k - 2$. Then y_{j_3} is red because of $j_2 = 2k - 2 < j_3$. It follows that x_{i_3} is green, i.e., $k - 1 \le i_1 < i_2 < i_3 \le 2k - 2$, and further

$$-(k-2) \le (2k-2) - (2k-2) \le j_2 - i_3 \le (2k-2) - (k+1) < k-2$$

i.e., $x_{i_3}y_{j_2}$ is an edge, a contradiction.

Now the proof of Theorem 3 is completed by the following:

Lemma 6. The graph D_k is C_6 -free and C_8 -free.

Proof. Assume that D_k contains an induced $C_6 = x_{i_1}y_{j_1}x_{i_2}y_{j_2}x_{i_3}y_{j_3}x_{i_1}$. Let $\{l_1, l_2, l_3\} = \{j_1, j_2, j_3\}$ where $l_1 < l_2 < l_3$, i.e., the "l-set" is the "ordered j-set". Since the degrees of the vertices $x_{i_1}, x_{i_2}, x_{i_3}$ in the cycle C_6 are exactly 2, by Lemma 4 the neighborhoods of $x_{i_1}, x_{i_2}, x_{i_3}$ have to be either $\{y_{l_1}, y_{l_2}\}$ or $\{y_{l_2}, y_{l_3}\}$, hence two of these vertices have the same neighborhood in the cycle, a contradiction. In the same way one can show that D_k does not contain a C_8 .

Theorem 3 states that ACB graphs and hence also auto-strongly-chordal graphs have unbounded Dilworth number.

Corollary 6. For all $k, l \geq 2$, there is an ACB graph G = (X, Y, E) with Dilworth numbers $\nabla_G(X) = k$ and $\nabla_G(Y) = l$.

Proof. In the case k=l, the graph D_k fulfills Corollary 6 by Theorem 3. Thus we may assume that $k>l\geq 2$. Let $D_{k,l}$ be the graph resulting from D_k by omitting the k-l green vertices y_k,\ldots,y_{2k-l-1} (thus, keeping only l green vertices in Y). As an induced subgraph of an ACB graph it is an ACB graph as well. As for D_k , we have also in this graph $N(x_1)\subseteq\cdots\subseteq N(x_{k-1})$, $N(x_{2k-2})\supseteq\cdots\supseteq N(x_{3k-4})$, $N(y_1)\subseteq\cdots\subseteq N(y_{k-1})$, $N(y_{2k-2})\supseteq\cdots\supseteq N(y_{3k-4})$. Moreover, the neighborhoods of x_{k-1},\ldots,x_{2k-2} are distinct and of the same size. Hence $\nabla_{D_{k,l}}(X)=k$. Similarly, the neighborhoods of the l vertices $y_{k-1},y_{2k-l},y_{2k-l+1},\ldots,y_{2k-2}$ are distinct and of the same size. Hence $\nabla_{D_{k,l}}(Y)=l$.

In the next section, the following graphs play an essential role: Let B_k denote the bipartite graph with vertex sets $X_k = \{x_1, \ldots, x_k\}$ and $Y_k = \{y_1, \ldots, y_{2k-2}\}$ such that for every $i \in \{1, \ldots, k\}$, $N(x_i) = \{y_i, \ldots, y_{i+k-2}\}$. Note that B_2 is the $2K_2$, B_3 is the P_7 , which is the bipartite counterpart of the rising sun $(X-P_7)$ and of the co-rising sun $(Y-P_7)$, and B_4 is the bipartite counterpart of graph G_5 from [18]. Figure 5 illustrates the bipartite graphs B_3 and B_4 .

Corollary 7. For every $k \geq 2$, B_k is an ACB graph with $\nabla_{B_k}(X) = k$ and $\nabla_{B_k}(Y) = 2$.

Proof. Obviously, for all $k \geq 2$, B_k is an induced subgraph of D_k , namely induced by the vertices $x_{k-1}, \ldots, x_{2k-2}$ and $y_1, \ldots, y_{k-1}, y_{2k-2}, \ldots, y_{3k-4}$. By Theorem 3, this implies that B_k is an ACB graph. Moreover, clearly, $\nabla_{B_k}(X_k) \leq \nabla_{D_k}(X_k) = k$. In B_k , for every $i \neq j$, the neighborhoods of $N(x_i)$ and $N(x_j)$ are incomparable because they are distinct and of the same size. Thus $\nabla_{B_k}(X) = k$, and obviously $\nabla_{B_k}(Y) = 2$.

5 Characterization of ACB Graphs with bounded Dilworth Number

Let $k \geq 2$. An ACB graph G = (X,Y,E) is called k-critical (with respect to Dilworth number) if bip-dilw(G)=k and for all $v \in X \cup Y$, $bip-dilw(G-v) \leq k-1$ holds. The main result of this section is Theorem 4 which shows that the k-critical ACB graphs are the graphs B_k , introduced in the preceding section.

The graphs B_k have the following nice property:

Proposition 11. For every $k \geq 2$, B_{k-1} is an induced subgraph of B_k .

Proof. For all $i \in \{1, ..., k-1\}$, $N(x_i)$ contains y_{k-1} . Thus, if one deletes y_{k-1}, y_{2k-2} and x_k in B_k , the resulting subgraph is isomorphic to B_{k-1} . \square Now we are ready to state our main theorem:

Theorem 4. The bipartite graph G is a k-critical ACB graph if and only if G is isomorphic to B_k .

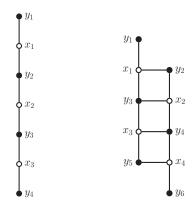


Figure 5: Critical bipartite graphs B_3 and B_4 with $\nabla_{B_3}(X) = 3$, $\nabla_{B_3}(Y) = 2$, $\nabla_{B_4}(X) = 4$, and $\nabla_{B_4}(X) = 2$.

Proof. By Corollary 7, B_k is an ACB graph. It is easy to check that B_k is k-critical. This proves the if-part. In order to prove the only-if-part let G = (X, Y, E) be a k-critical ACB graph. Let $\mathcal{F} = \{N(x) : x \in X\}$ and let $\mathcal{F}^* = \{N(y) : y \in Y\}$. Note that $H^* = (X, \mathcal{F}^*)$ is the dual hypergraph of the hypergraph $H = (Y, \mathcal{F})$.

Let without loss of generality, $dilw(Y, \mathcal{F}) = k$. Since G is k-critical, |X| = k. Moreover, \mathcal{F} as well as \mathcal{F}^* do not contain multiple members, G does not contain isolated vertices and the members of \mathcal{F} are pairwise incomparable with respect to inclusion.

Lemma 7. We have $dilw(X, \mathcal{F}^*) = 2$.

Proof. First assume that $dilw(X, \mathcal{F}^*) = 1$. Let $\mathcal{F}^* = \{X_1, \dots, X_l\}$ with $X_i = N(y_i)$, $i = 1, \dots, l$, and $X_1 \subset \dots \subset X_l$. Note that $X_l = X$. For $x \in X$, let $\iota(x) = \min\{i : x \in X_i\}$. Let $a, b \in X$ and let, without loss of generality, $\iota(a) \leq \iota(b)$. Then $\{y_{\iota(b)}, y_{\iota(b)+1}, \dots, y_l\} = N(b) \subseteq N(a) = \{y_{\iota(a)}, y_{\iota(a)+1}, \dots, y_l\}$, a contradiction.

Now assume that $dilw(X, \mathcal{F}^*) \geq 3$. Then there are $\alpha, \beta, \gamma \in Y$ such that $A = N(\alpha)$, $B = N(\beta)$ and $C = N(\gamma)$ are pairwise incomparable. We study several cases which are defined depending on the following conditions:

$$A \subseteq B \cup C, \tag{2}$$

$$B \subseteq A \cup C, \tag{3}$$

$$C \subseteq A \cup B. \tag{4}$$

Case 1. At least two of the relations (2)-(4) are satisfied. Let, without loss of generality, (2) and (3) be true. Then $A \setminus B \subseteq C$ and $B \setminus A \subseteq C$. There is some $d \in A \cap B \setminus C$ because otherwise $A \subseteq C$ (and $B \subseteq C$). Since A and B are incomparable there exist elements $a \in A \setminus B$ and $b \in B \setminus A$. Then $a, \alpha, d, \beta, b, \gamma$ form a C_6 , a contradiction.

Case 2. Exactly one of the relations (2) – (4) is satisfied. Let, without loss of generality, (2) be true. Then $A \setminus B \subseteq C$ and there are elements $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. Let $a \in A \setminus B$. Since N(a) and N(c) are incomparable there is some $\delta \in N(c) \setminus N(a)$. We have $\delta \neq \alpha$ since $\alpha \in N(a)$ and $\delta \neq \beta$ since $\beta \notin N(c)$. Moreover, $\delta \neq \gamma$ since $\gamma \in N(a)$ in view of $\alpha \in A \setminus B \subseteq C$.

Case 2.1. $\delta \notin N(b)$. Then $a\alpha$, $b\beta$, $c\delta$ form a $3K_2$, a contradiction.

Case 2.2. $\delta \in N(b)$. There is some $d \in A \cap B \setminus C$ because otherwise $A \subseteq C$.

Case 2.2.1 $d \notin N(\delta)$. Then $b, \beta, d, \alpha, a, \gamma, c, \delta$ form a C_8 , a contradiction.

Case 2.2.2 $d \in N(\delta)$. Then $a, \gamma, c, \delta, d, \alpha$ form a C_6 , a contradiction.

Case 3. None of the relations (2) - (4) is satisfied. Then there are elements $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. But $a\alpha$, $b\beta$, $c\gamma$ form a $3K_2$, a contradiction which finally shows Lemma 7.

Lemma 8. The family \mathcal{F}^* does not have any single maximal element.

Proof. Assume that there is some $\alpha \in Y$ such that $N(y) \subseteq N(\alpha)$ for all $y \in Y$. Then $\nabla_{G-\alpha}(X) = k$, a contradiction.

From Lemmas 7 and 8 it follows that \mathcal{F}^* has exactly two maximal elements, say $A = N(\alpha)$ and $B = N(\beta)$.

Lemma 9. There is no $\gamma \in Y$ such that $N(\gamma) \subseteq A \cap B$.

Proof. Assume that there is some $\gamma \in Y$ such that $C = N(\gamma) \subseteq A \cap B$. Let $a \in A \setminus B$, $b \in B \setminus A$ and $c \in C$. Note that $c \neq a$ and $c \neq b$. Let $\delta \in N(a) \setminus N(c)$ and $\varepsilon \in N(b) \setminus N(c)$.

Case 1. $\varepsilon \notin N(a)$ and $\delta \notin N(b)$. Then $a\delta$, $c\gamma$, $b\varepsilon$ form a $3K_2$, a contradiction. **Case 2.** $\varepsilon \in N(a)$ or $\delta \in N(b)$. Without loss of generality, we may assume that $\varepsilon \in N(a)$. Then $a, \alpha, c, \beta, b, \varepsilon$ form a C_6 , a contradiction.

Let $\mathcal{F}_A^* = \{C \in \mathcal{F}^* : C \subseteq A\}$ and $\mathcal{F}_B^* = \{C \in \mathcal{F}^* : C \subseteq B\}$. Since A and B are the maximal elements of \mathcal{F}^* and in view of Lemma 9 the families \mathcal{F}_A^* and \mathcal{F}_B^* form a partition of \mathcal{F}^* .

Lemma 10. The families \mathcal{F}_A^* and \mathcal{F}_B^* are chains with respect to inclusion.

Proof. Assume e.g. that \mathcal{F}_A^* contains two incomparable elements C and D. Since $C \nsubseteq B$ and $D \nsubseteq B$ the members C, D and B are three pairwise incomparable elements of \mathcal{F}^* , a contradiction to Lemma 7.

Let

$$\mathcal{F}_A^* = \{A_1, \dots, A_r\}$$
 where $A_1 \subset \dots \subset A_r = A$, $\mathcal{F}_B^* = \{B_1, \dots, B_s\}$ where $B_1 \subset \dots \subset B_s = B$

and let α_i, β_j be those elements of Y for which $N(\alpha_i) = A_i$ and $N(\beta_j) = B_j$, $i = 1, \ldots, r, j = 1, \ldots, s$. Note that $Y = \{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$. For an element $x \in X$, let

$$\iota_A(x) = \begin{cases} \min\{i : x \in A_i\} & \text{if } x \in A, \\ \infty & \text{otherwise} \end{cases}$$

and

$$\iota_B(x) = \begin{cases} \min\{j : x \in B_j\} & \text{if } x \in B, \\ \infty & \text{otherwise.} \end{cases}$$

For brevity, we set $A_0 = B_0 = \emptyset$ and $A_{r+1} = B_{s+1} = X$.

Lemma 11. We have $|A_i \setminus A_{i-1}| = |B_j \setminus B_{j-1}| = 1, i = 1, ..., r+1, j = 1, ..., s+1.$

Proof. Assume e.g. that $|A_i \setminus A_{i-1}| \ge 2$ and let a, a' be two different elements of $|A_i \setminus A_{i-1}|$. Let, without loss of generality, $\iota_B(a) \le \iota_B(a')$.

Case 1. $\iota_B(a) = \iota_B(a') = \infty$, i.e., $a, a' \notin B$. Let, without loss of generality, $\iota_A(a) \leq \iota_A(a')$. Obviously, $N(a) = \{\alpha_{\iota_A(a)}, \alpha_{\iota_A(a)+1}, \ldots, \alpha_r\} \supseteq \{\alpha_{\iota_A(a')}, \alpha_{\iota_A(a')+1}, \ldots, \alpha_r\} = N(a)$, a contradiction to the incomparability of N(a) and N(a').

Case 2. $\iota_B(a) < \infty$, i.e., $a \in B$. Then $N(a) = \{\alpha_i, \dots, \alpha_r, \beta_{\iota_B(a)}, \dots, \beta_s\} \supseteq N(a')$, a contradiction.

From Lemma 11 it follows that r = s and that the elements of X can be numbered in such a way that $A_i = \{x_1, \ldots, x_i\}, i = 1, \ldots, r+1$.

Lemma 12. We have
$$B_i = \{x_{r+1}, x_r, \dots, x_{r-i+2}\}, i = 1, \dots, r+1.$$

Proof. Assume the contrary and choose i maximal such that $B_i \neq \{x_{r+1}, x_r, \ldots, x_{r-i+2}\}$. Then $i \leq r$ since $B_{r+1} = A_{r+1} = X$. By the maximality of i we have $B_{i+1} = \{x_{r+1}, x_r, \ldots, x_{r-i+1}\}$ and by Lemma 11, $B_i = \{x_{r+1}, x_r, \ldots, x_{r-i+1}\} \setminus \{x_j\}$ for some $j \in \{r-i+2, \ldots, r+1\}$. We have $j \neq r+1$ since otherwise $B_i \subseteq A_r$. Since $x_j \notin B_i$, the element x_j is not contained in B_1, \ldots, B_i . Thus $N(x_j) = \{\alpha_j, \ldots, \alpha_r, \beta_r, \ldots, \beta_{i+1}\} \subseteq \{\alpha_{r-i+1}, \ldots, \alpha_r, \beta_r, \ldots, \beta_{i+1}, \beta_i\} \subseteq N(x_{r-i+1})$, a contradiction.

Now the equalities $A_i=\{x_1,\ldots,x_i\}$ and $B_i=\{x_{r+1},x_r,\ldots,x_{r-i+2}\},\ i=1,\ldots,r,$ imply

$$N(x_1) = \{\alpha_1, \dots, \alpha_r\},\$$

$$N(x_i) = \{\alpha_i, \dots, \alpha_r, \beta_r, \dots, \beta_{r-i+2}\}, i = 2, \dots, r,\$$

$$N(x_{r+1}) = \{\beta_r, \dots, \beta_1\}.$$

Thus $G \sim B_r$. But since G is k-critical, $G \sim B_k$ follows which finally shows Theorem 4.

Corollary 8. The Dilworth number of an ACB graph B = (X, Y, E) is at most k if and only if B is B_{k+1} -free.

Thus, Theorem 4 yields a characterization by forbidden induced subgraphs both of the family of ACB graphs with bipartite Dilworth number at most k, and of the split graph counterpart with Dilworth number at most k. Since by Proposition 11, B_i is an induced subgraph of B_k for each $i \in \{2, ..., k\}$, this defines a hierarchy of properly included ACB graphs, with B_k as separating example between the class of Dilworth number k and the class of Dilworth number k-1. The corresponding hierarchy of split graphs is easily derived.

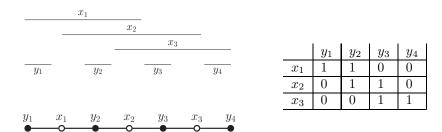


Figure 6: Bipartite graph $B = Y-P_7$, an interval representation of $split_X(B)$, and its corresponding Γ -free matrix.

6 Conclusion

In this paper, we have studied ACB graphs with respect to their Dilworth number. One of our open questions is the recognition complexity of ACB graphs. Chordal bipartite graphs can be recognized in $O(\min(n \log n, n^2))$ [16, 19, 21], by computing a doubly lexical ordering of the bipartite adjacency matrix of the (bipartite) graph G, which is Γ -free if and only if G is chordal bipartite (a Γ is a submatrix with three 1's, and one 0 in the bottom-right corner). See [20] for a detailed discussion of this aspect. A linear-time recognition for chordal bipartite graphs remains a long-standing open question.

ACB graphs can be recognized in $O(n^2)$ time checking whether both the graph and its mirror are chordal bipartite. In Subsection 3.2.3, we discussed how to recognize whether a bipartite graph has X-Dilworth number (Y-Dilworth number, respectively) more than 2 in linear time. An alternative linear-time approach for this would be to check whether $split_X(B)$ is an interval graph.

Let us remark that, given this interval realization, we can construct in linear time a Γ -free matrix of B as follows: order the vertices of Y as they appear from left to right in the interval representation of $split_X(B)$; order the vertices of X using the left endpoints of their intervals, and order these from left to right. Figure 6 gives an example for the P_7 , which has X-Dilworth number 3 and Y-Dilworth number 2.

This result is noteworthy, as to the best of our knowledge the chordal bipartite graphs for which there is a linear-time algorithm to produce a Γ -free matrix are the (trivial) chain graphs.

Corollary 9. It can be recognized in linear time if a given bipartite graph is $(3K_2, C_6, P_7)$ -free, or if it is $(3K_2, C_6, X - P_7)$ -free

Another open question is the complexity of computing the Dilworth number of an ACB graph: Given a Γ -free matrix for a (chordal) bipartite graph, the neighborhood order can be computed in linear time [20]. This order in turn yields the Dilworth number. Thus as discussed above, if the X- or Y- Dilworth number is 2, the Dilworth number can be determined in linear time; if both the X- and the Y- Dilworth numbers are > 2, $O(n^2)$ time is required to compute the Dilworth number.

We conclude this paper with the following open questions:

- 1. Can one recognize ACB graphs in linear time?
- 2. Determine all (k, l)-critical ACB graphs G = (X, Y, E), i.e., all ACB graphs for which $\nabla_G(X) = k$, $\nabla_G(Y) = l$ and $\nabla_{G-v}(X) + \nabla_{G-v}(Y) < k+l$ for all $v \in X \cup Y$.

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