Graphical representations of graphic frame matroids

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Abstract

A frame matroid M is *graphic* if there is a graph G with cycle matroid isomorphic to M. In general, if there is one such graph, there will be many. Zaslavsky has shown that frame matroids are precisely those having a representation as a biased graph; this class includes graphic matroids, bicircular matroids, and Dowling geometries. Whitney characterized which graphs have isomorphic cycle matroids, and Matthews characterized which graphs have isomorphic graphic bicircular matroids. In this paper, we give a characterization of which biased graphs give rise to isomorphic graphic frame matroids.

1 Introduction

A *biased graph* Ω consists of a pair $(G, \mathcal{B})$, where G is a graph and $\mathcal{B}$ is a collection of cycles of G, called *balanced*, obeying the *theta property*. A *theta* graph consists of a pair of distinct vertices and three internally disjoint paths between them; the *theta property* is the property that no theta subgraph contains exactly two balanced cycles. Cycles not in $\mathcal{B}$ are called *unbalanced*. We write $\Omega = (G, \mathcal{B})$ and say G is the *underlying graph* of Ω. Throughout graphs are finite, and may have loops and parallel edges.

Biased graphs were introduced by Zaslavsky in [12], and in [13] Zaslavsky defined a natural matroid with ground set the edges of a biased graph $(G, \mathcal{B})$, which we may describe in terms of its circuits as follows. A set $C \subseteq E(G)$ is a circuit in this matroid if in $(G, \mathcal{B})$, C induces one of: a balanced cycle, two edge-disjoint unbalanced cycles intersecting in only one vertex, two vertex-disjoint unbalanced cycles along with a path connecting them, or a theta subgraph with all cycles unbalanced.

A matroid is *frame* if it may be extended such that it possesses a basis B_0 (a frame) such that every element is spanned by at most two elements of B_0. Zaslavsky [14] has shown

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that the class of frame matroids is precisely that of matroids arising from biased graphs as described above (whence these have also been called *bias* matroids). Given a biased graph $\Omega = (G, \mathcal{B})$ we denote by $F(\Omega)$ or $F(G, \mathcal{B})$ the frame matroid arising from Ω. Observe that given a graph G, if $\mathcal{B}$ contains all cycles in G, then $F(G, \mathcal{B})$ is the cycle matroid $M(G)$ of G, and that $F(G, \emptyset)$ is the bicircular matroid of G. Frame matroids also include Dowling geometries [2] (see also, for example, [3] and [12]).

Whitney [9] characterised when two graphs give rise to the same graphic matroid, and Matthews [4] characterized which graphs have isomorphic graphic bicircular matroids. To state Whitney's result we first need some definitions. Given a graph H and a set of edges Y, we let $H|Y$ denote the subgraph of H with edge set Y and no isolated vertices. Let H be a graph, and let (X_1, X_2) be a partition of $E(H)$ such that $V(H|X_1) \cap V(H|X_2) = \{u_1, u_2\}$. We say that H' is obtained by a *Whitney flip* of H on $\{u_1, u_2\}$ if H' is a graph obtained by identifying vertices u_1, u_2 of $H|X_1$ with vertices u_2, u_1 of $H|X_2$, respectively. A graph H' is *2-isomorphic to* H if H' is obtained from H by a sequence of the operations: Whitney flips, identifying two vertices from distinct components of a graph, or partitioning a graph into components each of which is a block of the original graph.

Theorem 1.1 (Whitney's 2-Isomorphism Theorem, [9])**.** *Let G and H be graphs without isolated vertices. Then $M(G) \cong M(H)$ if and only if G and H are 2-isomorphic.*

Six families of biased graphs whose frame matroids are graphic are defined and exhibited in Section 2. A biased graph in any of these families is obtained from a graph G by a simple operation, and it is easily checked that the frame matroid arising from the resulting biased graph is isomorphic to the cycle matroid of G. For ease of reference, we name them: (1) balanced, (2) fat thetas, (3) curlings, (4) pinches, (5) 4-twistings, and (6) consecutive odd-twistings. We call the corresponding operation in each case by the same name. Our main result says that every graphic frame matroid comes from a biased graph in one of these families.

Theorem 1.2. *Let G be a 2-connected graph and Ω a biased graph with $F(\Omega) = M(G)$. Then there is a graph H 2-isomorphic to G such that either Ω is balanced with underlying graph H, or Ω is obtained from H as a fat theta, a curling, a pinch, a 4-twisting, or a consecutive odd-twisting.*

Increasing the connectivity of the graph G in Theorem 1.2 reduces the possible biased graph representations of $F(\Omega)$. Asking that G be 3-connected removes one family from the list of possibilities, and simplifies the curling operation. The following is an immediate consequence of Theorem 1.2.

Corollary 1.3. *Let G be a 3-connected graph with at least four vertices and Ω a biased graph with $F(\Omega) = M(G)$. Then either Ω is balanced with underlying graph G, or obtained from G as a simple curling, a pinch, a 4-twisting, or a consecutive odd-twisting.*

If we further demand that G be 4-connected, we are still left with two possible families of biased graph representations. The following is an immediate consequence of Corollary 1.3.

Corollary 1.4. *Let G be a 4-connected graph with at least five vertices and Ω a biased graph with $F(\Omega) = M(G)$. Then either Ω is balanced with underlying graph G, or obtained from G as a simple curling or a pinch.*

Finally, we note that for any $k \geq 4$, if G is a k-connected graph on n vertices, then in general there may be up to $\binom{n}{2} + n$ non-isomorphic biased graphs Ω with $F(\Omega)$ isomorphic to $M(G)$ (obtained as pinches and simple curlings of G). Corollary 1.4 says, however, that these will be all the biased graph representations of $M(G)$.

The remainder of this paper is organized as follows. First we exhibit the six families of biased graphs whose frame matroids are graphic (appearing in Theorem 1.2). We then show that the frame matroids arising from these biased graphs are indeed graphic, and that every graphic frame matroid arises from a biased graph in one of these families.

2 Six families of biased graphs with graphic frame matroids

We now describe six families of biased graphs whose frame matroids are graphic. For any positive integer n, set $[n] = \{1, 2, \cdots, n\}$.

1. Balanced biased graphs. Let $\Omega = (H, \mathcal{B})$ be a biased graph. If every cycle of H is in $\mathcal{B}$, then Ω is *balanced*. Clearly, $F(\Omega) = M(H)$, so $F(\Omega)$ is graphic.

2. Fat thetas. To describe our second family, let H_1, H_2, H_3 be non-empty graphs with distinct vertices $x_i, y_i \in V(H_i)$. Let H be obtained from H_1, H_2, H_3 by identifying y_i and x_{i+1} to a vertex w_i for every $i \in [3]$ (where the indices are modulo 3; see the left of Figure 1). Let $\Omega = (\Gamma, \mathcal{B})$ be a biased graph, where Γ is obtained from H_1, H_2, H_3 by

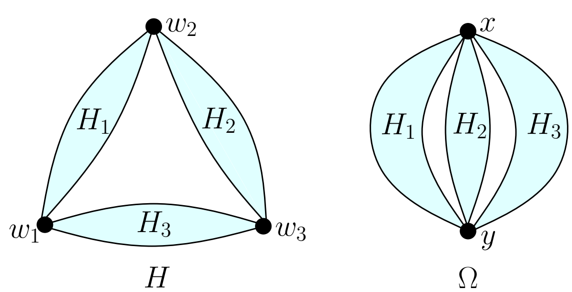

Figure 1: A fat theta.

identifying x_1, x_2, x_3 to a vertex x and identifying y_1, y_2, y_3 to a vertex y. A cycle of Γ is in $\mathcal{B}$ if and only if $E(C)$ is completely contained in one of H_1, H_2 or H_3 (see the right of Figure 1). Then we say that Ω is a *fat theta* obtained from H.

A biased graph Ω is a *signed graph* if its edges can be labelled by 1 or -1 such that a cycle C is balanced in Ω if and only if $E(C)$ contains an even number of edges labelled -1. In all figures of signed graphs we adopt the following convention. A shaded area around a vertex denotes that all the edges in that area incident with that vertex are labelled with -1. Bold edges are also labelled -1. All unmarked edges are labelled 1.

3. Curlings. Let H be a 2-connected graph, $v \in V(H)$, and suppose that there are distinct vertices $v_1, \dots, v_k$ and connected subgraphs $H_1, \dots, H_k$ of H such that $V(H_i) \cap V(H|(E(H) - E(H_i))) = \{v, v_i\}$. Suppose moreover that every edge incident with v is contained in some H_i. Let Ω be the signed graph obtained from H by first labeling all edges incident with v by -1 and then changing any such edge $e = uv$ to uv_i when $u \in H_i$ (if $u = v_i$ this produces a loop at v_i), and keeping all other edges not incident with v unchanged and labelled by 1 (see Figure 2). Then we say that Ω is a *curling* of H. If, for every i, every edge in H_i is between v and v_i then we call Ω a *simple curling.*

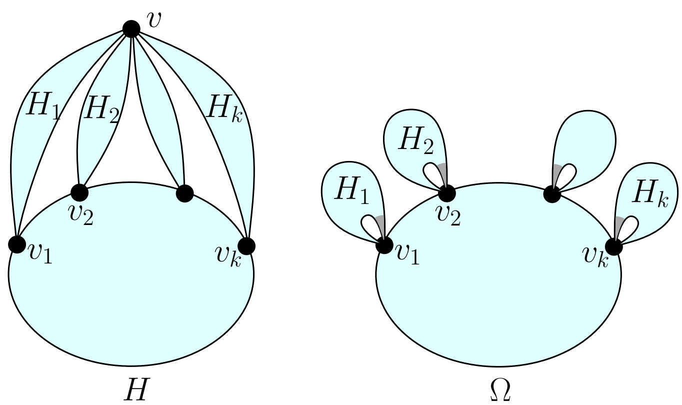

Figure 2: A curling.

4. Pinches. If Ω is obtained from H by identifying two vertices v_1 and v_2 to a new vertex v and labeling all edges incident with v_1 by -1 and all other edges by 1, then we say Ω is a *pinch*. An edge with endpoints v_1, v_2 becomes an unbalanced loop incident to v (Figure 3).

5. 4-twistings. Let H_1, H_2, H_3, H_4 be graphs (not necessarily all non-empty) with distinct vertices $x_i, y_i, z_i \in V(H_i)$. Let H be obtained from H_1, H_2, H_3, H_4 by identifying x_i, y_{3-i}, z_{i+2} to a vertex w_i for every $i \in [4]$ (where the indices are modulo 4). Let Ω be a signed graph obtained from H_1, H_2, H_3, H_4 by identifying x_1, x_2, x_3, x_4 to a vertex x, identifying y_1, y_2, y_3, y_4 to a vertex y and identifying z_1, z_2, z_3, z_4 to a vertex z, and with all edges originally incident with x_1, y_2 or z_3 labelled by -1 and all other edges labelled by 1 (see Figure 4). Then we say that Ω is a *4-twisting* of H.

6. Consecutive odd-twistings. Let $H_1, \dots, H_k$ (for $k \geq 3$), be graphs with distinct vertices $x_i, y_i, z_i \in V(H_i)$ for $i \in [k]$. Let H be a graph obtained from $H_1, \dots, H_k$ by

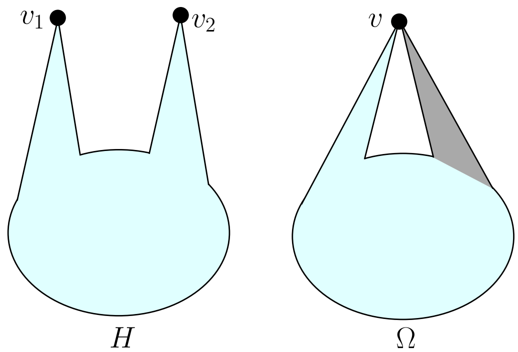

Figure 3: A pinch.

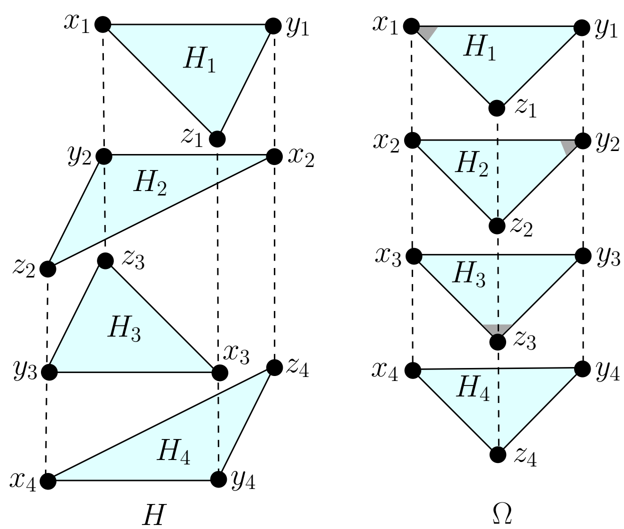

Figure 4: A 4-twisting. Vertices on a same dashed line are identified.

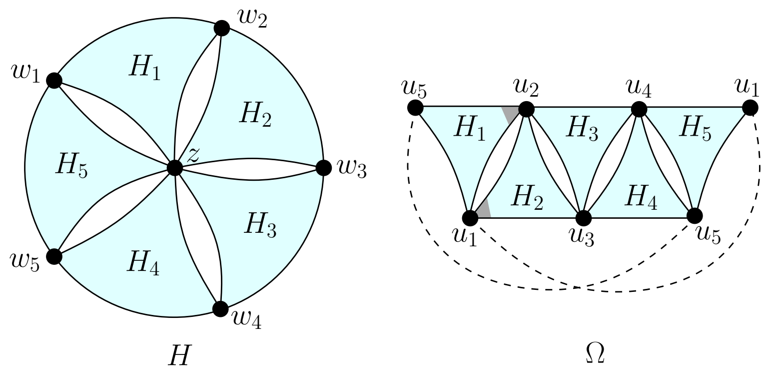

Figure 5: A consecutive odd-twisting. Vertices on a same dashed line are identified.

identifying $z_1, z_2, \dots, z_k$ to a vertex z and for each $i \in [k]$ identifying y_{i-1} and x_i to a vertex w_i (where the indices are modulo k). Let Ω be the signed graph obtained from $H_1, \dots, H_k$ by identifying y_{i-1}, z_i, x_{i+1} to a vertex u_i for every $i \in [k]$ (where the indices are modulo k), and with all edges originally incident with y_1 or x_2 labelled by -1 and all other edges labelled by 1 (see Figure 5). Then we say that Ω is a *consecutive twisting* of H. If k is odd then Ω is a *consecutive odd-twisting* of H.

3 All graphic frame matroids arise from these six families

In preparation for the proof of our main result, we now introduce some notation. Let H be a graph and $X \subseteq V(H)$. We say X is a *vertex-cut* of H if $H \backslash X$ has at least one more component than H. When $|X| = 1$, we also say X is a *cut-vertex* of H. A *block* of H is a maximal connected subgraph which has no cut-vertex. An *end-block* is a block containing at most one cut-vertex.

In the rest of the paper, let G be a 2-connected graph and let Ω be a biased graph with $F(\Omega) = M(G)$. We let Γ denote the underlying graph of Ω and $E = E(\Gamma)$.

A *handcuff* consists of a pair of cycles C_1 and C_2, and a path P connecting C_1 and C_2 such that P meets C_i at u_i and nowhere else and C_1 meets C_2 only at $\{u_1\} \cap \{u_2\}$. If $u_1 \neq u_2$ then the handcuff is *loose*; otherwise it is *tight*. A subgraph or edge set of Ω is *balanced* if each cycle in it is balanced; otherwise it is *unbalanced*. Moreover, it is *contra-balanced* if it has no balanced cycles. A vertex v of a biased graph Ω is a *blocking vertex* if $\Omega \backslash v$ is balanced.

Zaslavsky has characterized those biased graphs Ω for which $F(\Omega)$ is binary.

Theorem 3.1 (Zaslavsky [11]). *Let Ω be a biased graph. Then $F(\Omega)$ is binary if and only if each connected component of Ω has one of the following forms.*

(1) *It is balanced.*

(2) *It is a fat theta.*

(3) *It is a signed graph with more than one unbalanced block, and each unbalanced block B_i has a vertex v_i such that $B_i \backslash v_i$ is balanced and v_i is a cut-vertex separating B_i from all other unbalanced blocks.*

(4) *It is a signed graph with just one unbalanced block, and has no two vertex-disjoint unbalanced cycles.*

Therefore, any biased graph Ω with graphic frame matroid has one of the forms (1)-(4) of Theorem 3.1. Evidently, when Ω is balanced, by Whitney's 2-Isomorphism Theorem Γ is 2-isomorphic to G. That is, when Ω has the form in Theorem 3.1(1), Ω is balanced with underlying graph Γ 2-isomorphic to G. Next we consider a biased graph Ω that has one of forms (2)-(4) of Theorem 3.1.

First we consider an Ω that has form Theorem 3.1(2). Assume that Ω is a fat theta obtained from balanced graphs $\Omega_1, \Omega_2, \Omega_3$ by identifying u_1, u_2, u_3 to a vertex u and v_1, v_2, v_3 to a vertex v, where $u_i, v_i \in V(\Gamma_i)$ (where Γ_i is the underlying graph of Ω_i). Let H be the graph obtained from $\Gamma_1, \Gamma_2, \Gamma_3$ by identifying u_i with v_{i+1} for any $i \in [3]$, where the subscripts are modulo 3. Evidently, $M(H) = F(\Omega)$; and consequently, by Whitney's 2-Isomorphism Theorem H is 2-isomorphic to G as $F(\Omega) = M(G)$ implying $M(H) = M(G)$. So we only need to consider Ω with forms (3) and (4) of Theorem 3.1. These cases will be discussed in Sections 3.1 and 3.2 respectively. We end this section with two results that will be used without reference sometimes. The first one appears in [10].

Lemma 3.2 ([10], Theorem 6)**.** *A biased graph is a signed graph if and only if it has no contra-balanced theta subgraphs.*

The last result of this section is an immediate consequence of Theorem 3.1 and Lemma 3.2.

Corollary 3.3. *Let Ω be an unbalanced signed graph such that $F(\Omega)$ is a connected binary matroid. Then Ω has no balanced loops and at least one of the following holds.*

(1) *Ω consists of one unbalanced block.*

(2) *Ω has more than one unbalanced blocks and a block is unbalanced if and only if it is an end-block. Moreover, when $F(\Omega)$ is 3-connected, each unbalanced block is an unbalanced loop.*

3.1 Ω with form Theorem 3.1(3)

In this section, we mainly characterize those signed graphs Ω representing the 2-connected graph G with form Theorem 3.1(3), that is, Ω is a signed graph with more than one unbalanced block, and each unbalanced block B_i has a vertex v_i such that $B_i \backslash v_i$ is balanced and v_i is a cut-vertex separating B_i from all other unbalanced blocks. It follows from Corollary 3.3 that a block is unbalanced if and only if it is an end-block.

First we show that when Ω is a curling, $F(\Omega)$ is graphic.

Lemma 3.4. *Let Ω be a curling of H defined as Section 2. Then $M(H) = F(\Omega)$.*

Proof. Let C be an arbitrary cycle of H. When $v \notin C$, the set C is also a balanced cycle of H. So we may assume $v \in C$ and $e_1 = vu_1, e_2 = vu_2 \in C$. When u_1, u_2 are in the same H_i, C is also a balanced cycle of Ω; otherwise, C is a contra-balanced handcuff of Ω. Therefore, every circuit of $M(H)$ is a circuit of $F(\Omega)$.

On the other hand, let C be an arbitrary circuit of $F(\Omega)$. Evidently, C is a balanced cycle or a contra-balanced handcuff of Ω as Ω is a signed graph with no contra-balanced theta subgraph. In either case, by the definition of Ω, it is easy to verify that C is a cycle of H. Hence, every circuit of $F(\Omega)$ is a circuit of $M(H)$. □

Secondly, we show that when Ω is a biased graph representing $M(G)$ with more than one unbalanced block, there is a graph H 2-isomorphic to G such that Ω is obtained as a curling of H. To prove this we need some definitions and results first.

Assume that Ω is a signed graph, and (V_1, V_2) is an arbitrary partition of V. Let $\delta = (V_1, 1; V_2, -1)$ be a labeling of V such that any vertex in V_1 is labelled by 1 and any vertex in V_2 labelled by -1. Then $\delta(\Omega)$ is a *switching* of Ω with any edge relabelled by the product of its end-vertices' labeling and its original labeling in Ω. Evidently, $F(\Omega) = F(\delta(\Omega))$.

Lemma 3.5. *Let Ω be a balanced signed graph. Then by switching all edges of Ω can be labelled by* 1*.*

Proof. It suffices to show that the result holds when Γ is connected. Let T be a spanning tree of Γ. Then for some switching $\delta(\Omega)$, every edge of T is labelled by 1. For every edge e not in T, the unique cycle in $T \cup \{e\}$ is balanced, thus e is also labelled with 1 in $\delta(\Omega)$. It follows that all edges in $\delta(\Omega)$ are labelled by 1. □

Lemma 3.6. *Let G be a* 2*-connected graph and Ω be an unbalanced signed graph with a blocking vertex v and satisfying $F(\Omega) = M(G)$. Then there is a graph H 2-isomorphic to G such that Ω is obtained from H by a pinch.*

Proof. By Lemma 3.5, it is easy to see that by some switching we can assume that all edges of Ω labelled by -1 are incident with v. Moreover, since Ω is unbalanced, some edges incident with v are labelled by 1 and some edges incident with v are labelled by -1. Let H be the graph obtained from Ω by splitting v into v_1 and v_2 such that any edge $e = vu$ labelled by -1 is changed to $e = v_1u$ and any edge $e = vu$ labelled by 1 is changed to $e = v_2u$ and with all other edges not incident with v unchanged. Every unbalanced loop at v becomes a v_1v_2 edge. Evidently, $F(\Omega) = M(H)$; and hence, H 2-isomorphic to G as $F(\Omega) = M(G)$. □

A graph H is a *path graph* if H is connected and its blocks-cut-vertices graph is a path. The proof of Lemma 3.7 is similar to the proof of Lemma 3.6.

Lemma 3.7. *Let Ω be an unbalanced signed graph with a blocking vertex v. Then there is a graph H with $F(\Omega) = M(H)$ and such that Γ is obtained by identifying two vertices v_1 and v_2 of H to v. Moreover, if Γ is 2-connected and H is a path graph that is not 2-connected, then each end-block contains exactly one of v_1 and v_2.*

For a path graph H, arbitrarily choose two vertices v_1, v_2 from its end-blocks such that when H is not 2-connected neither v_1 nor v_2 is a cut-vertex of H and they are not in the same end-block. Add an edge e connecting v_1 and v_2 to obtain a new graph H_1. Let H_1' be a graph 2-isomorphic to H_1 and v_1', v_2' be the end-vertices of e in H_1'. Evidently, graph $H' = H_1' - e$ is 2-isomorphic to H and any v_1v_2-path in H is changed to a $v_1'v_2'$-path in H' although the order of edges may be different. In this case we say that H' is a path graph *2-isomorphic to H with v_1v_2-paths changed to $v_1'v_2'$-paths.*

Lemma 3.8. *Let Ω be a signed graph with Γ connected and such that a block is unbalanced if and only if it is an end-block. Assume each unbalanced block B_i has a vertex v_i such that $B_i \backslash v_i$ is balanced and v_i is a cut-vertex separating B_i from all other unbalanced blocks. Let $B_1, \cdots, B_k$ be all end-blocks of Ω and for each $i \in [k]$ set $E_i = E(B_i), \Gamma_i = \Gamma|E_i$. Then the following hold.*

(1) *For some switch $\delta(\Omega)$, every edge labelled by -1 is in some B_i and incident with v_i for some $i \in [k]$.*

(2) *For each $i \in [k]$, there is a path graph H_i' such that B_i is obtained from H_i' by identifying v_{i1}' and v_{i2}' to v_i with all edges originally incident with v_{i1}' labelled by -1 and all other edges not incident with v_{i1}' labelled by 1 and satisfying $F(B_i) = M(H_i')$.*

(3) *For each $i \in [k]$, let H_i be a path graph 2-isomorphic to H_i' with $v_{i1}'v_{i2}'$-paths changed to $v_{i1}v_{i2}$-paths. First add a new isolated vertex v to the graph $\Gamma' = \Gamma \backslash (E_1 \cup \cdots \cup E_k)$, and then add $H_1, \cdots, H_k$ to Γ' by identifying v_{i2} with v_i and $v_{11}, \cdots, v_{k1}$ with v. Let H denote the new graph. Then $F(\Omega) = M(H)$.*

Proof. Evidently, (1) is an immediate consequence of Lemma 3.5. Moreover, since each B_i is a signed graph with a blocking vertex v_i, (2) follows immediately from Lemma 3.7. To show (3), let C be an arbitrary cycle of H. When $C \cap (E_1 \cup \cdots \cup E_k) = \emptyset$, the set C is also a balanced cycle of Ω. When $C \subseteq E_i$ for some $i \in [k]$, since H_i is 2-isomorphic to H_i' and $F(B_i) = M(H_i')$, the set C is a circuit of $F(B_i)$. So we may assume that $C \cap E_i, C \cap (E \backslash E_i) \neq \emptyset$ for some $i \in [k]$. Evidently, there is only one integer $i \neq j \in [k]$ such that $C \cap E_j \neq \emptyset$, and for any $s \in \{i, j\}$, the set $C \cap E_s$ is a $v_{s1}v_{s2}$-path of H_s; and consequently, $C \cap E_s$ is a $v_{s1}'v_{s2}'$-path of H_s' as H_s is a path graph 2-isomorphic to H_s' with $v_{s1}'v_{s2}'$-paths changed to $v_{s1}v_{s2}$-paths. Thus, by (1) and (2) C is a contra-balanced handcuff of Ω. So every circuit of $M(H)$ is a circuit of $F(\Omega)$.

On the other hand, assume that C is an arbitrary circuit of $F(\Omega)$. Then C is a balanced cycle or a contra-balanced handcuff of Ω as Ω is a signed-graph. When C is a balanced cycle, no matter whether $C \subseteq E_i$ or $C \cap (E_1 \cup \cdots \cup E_k) = \emptyset$, by the definition of H it is easy to see that the set C is also a cycle of H as H_i is 2-isomorphic to H_i'. So we may assume that C is a contra-balanced handcuff of Ω. Without loss of generality we may assume $C \cap E_i, C \cap E_j \neq \emptyset$. Then for any $s \in \{i, j\}$, the set $C \cap E_s$ is a $v_{s1}v_{s2}$-path of H_s as H_s is a path graph 2-isomorphic to H_s' with $v_{s1}'v_{s2}'$-paths changed to $v_{s1}v_{s2}$-paths. Therefore, C is also a cycle of H. So every circuit of $F(\Omega)$ is a circuit of $M(H)$. □

Therefore, when Ω has form Theorem 3.1(3), it follows from Lemma 3.8 that Ω is obtained as a curling. Moreover, when G is 3-connected, by Corollary 3.3 each unbalanced block of Ω is a loop. Thus, by Lemma 3.8 we have the following result.

Corollary 3.9. *Let G be a 3-connected graph and let Ω be an unbalanced signed graph with $F(\Omega) = M(G)$. Assume that Ω has more than one unbalanced block. Then G is obtained from Γ by adding a new isolated vertex v to Γ and changing all loops to links connecting v and their original end-vertices.*

3.2 Ω with form Theorem 3.1(4)

In this section, we mainly characterize the signed graphs Ω representing the 2-connected graph G with form Theorem 3.1(4). These have just one unbalanced block and no two vertex-disjoint unbalanced cycles.

While Slilaty [7] characterized those signed graphs having no blocking vertex and no two vertex disjoint unbalanced cycles having graphic frame matroid in terms of projective-planar signed graphs and 1, 2, and 3-sums of balanced signed graphs, an application of a theorem on lift matroids gives us a different structural characterization. The *lift matroid* $L(\Omega)$ of a signed graph Ω was defined by Zaslavsky in [13]. Its circuits are the sets of edges of one of the following two types: balanced cycles and the union of two unbalanced cycles meeting in at most one vertex. In his Ph.D. thesis Shih proved the following characterisation of graphic lift matroids (see also [6], Theorem 4.1).

Theorem 3.10 (Theorem 1, Chapter 2 in [8])**.** *Let G be a graph and let Ω be a signed graph such that $M(G) = L(\Omega)$. Then there exists a graph H 2-isomorphic to G such that one of the following holds.*

(1) *Ω is obtained from H by a pinch.*

(2) *Ω is obtained from H by a 4-twisting.*

(3) *Ω is obtained from H by a consecutive twisting.*

Since $L(\Omega) = F(\Omega)$ when Ω has no vertex-disjoint unbalanced cycles, the signed graph we want to find consisting of one unbalanced block without vertex-disjoint unbalanced cycles has the form of one of Theorem 3.10(1)-(3). However, the signed graph Ω in Theorem 3.10(3) may have vertex-disjoint unbalanced cycles, so we only need to find all signed graphs having no vertex-disjoint unbalanced cycles. Evidently, when Ω is obtained through 3.10(1), that is, obtained as a pinch, Ω has no two vertex-disjoint unbalanced cycles. On the other hand, note that Ω has no vertex-disjoint unbalanced cycles if and only if each cycle of G is connected in Γ. Thus, we only need to determine under which conditions a cycle of H is connected in Γ, for the graph H (2-isomorphic to G) given in Theorem 3.10.

Lemma 3.11. *Suppose that Ω is obtained from H by a 4-twisting as in Theorem 3.10(2). Then every cycle of H is connected in Γ.*

Proof. Let C be an arbitrary cycle of H. Assume to the contrary that C is not connected in Γ. Then C is a union of two vertex-disjoint cycles C_1 and C_2 of Γ as $M(G) = L(\Omega)$. Moreover, by the definition of Γ, either $|C_1 \cap \{x, y, z\}| = 1$ or $|C_2 \cap \{x, y, z\}| = 1$. By symmetry we may assume that the former holds. Then C_1 is a cycle of H, a contradiction to the fact that C_1 is a proper subset of the cycle C of H. □

Lemma 3.12. *Suppose that Ω is obtained from H by a consecutive twisting as in Theorem 3.10(3). If G is 2-connected, then every cycle in H is connected in Γ if and only if for some $i \in [k]$ no path connects x_i and y_i in $H_i \backslash z_i$ when k is even.*

Proof. First we prove the "only if" part. Assume to the contrary that $k = 2n$ and for any $i \in [k]$ there is a path connecting x_i and y_i in $H_i \backslash z_i$. Then G' has a cycle $C = P_{x_1,y_1}P_{x_2,y_2} \cdots P_{x_k,y_k}$, where P_{x_i,y_i} is a path of H_i connecting x_i and y_i with $z_i \notin P_{x_i,y_i}$. However, $C_1 = P_{x_1,y_1}P_{x_3,y_3} \cdots P_{x_{2n-1},y_{2n-1}}$ and $C_2 = P_{x_2,y_2}P_{x_4,y_4} \cdots P_{x_{2n},y_{2n}}$ are vertex-disjoint cycles of Γ such that $(E(C_1), E(C_2))$ is a partition of $E(C)$, a contradiction.

Secondly, we prove the "if" part. Let C be an arbitrary cycle of H. Evidently, when C is completely contained in some H_i, the set C is also a cycle of Γ. So we may assume that C intersects at least two H_i's. Assume that at most one $H_i|C$ uses z_i. Under this case C must have the structure $P_{x_1,y_1}P_{x_2,y_2} \cdots P_{x_k,y_k}$, where P_{x_i,y_i} is a path of H_i connecting x_i and y_i; and hence, either $k = 2n+1$ or $k = 2n$ and there is exactly one integer $i \in [k]$ with $z_i \in P_{x_i,y_i}$, say $i = 1$. When $k = 2n+1$, we have

$$C' = P_{x_1,y_1}P_{x_3,y_3} \cdots P_{x_{2n+1},y_{2n+1}}P_{x_2,y_2}P_{x_4,y_4} \cdots P_{x_{2n},y_{2n}}$$

is a cycle of Γ with $E(C) = E(C')$ if $z_1 \notin P_{x_1,y_1}$ and C' is the union of two edge-disjoint cycles of Γ sharing vertex z_1 otherwise. When $k = 2n$ (so $z_1 \in P_{x_1,y_1}$), we have $C_1 = P_{x_1,y_1}P_{x_3,y_3} \cdots P_{x_{2n-1},y_{2n-1}}$ and $C_2 = P_{x_2,y_2}P_{x_4,y_4} \cdots P_{x_{2n},y_{2n}}$ are two edge-disjoint cycles of Γ with a unique common vertex u_1. Hence, we can assume that there are exactly two $H_i|C$ using z_i. Without loss of generality we may assume that $C = P_{z_1,y_1}P_{x_2,y_2} \cdots P_{x_m,z_m}$, where $P_{z_1,y_1}, P_{x_i,y_i}(2 \leq i \leq m-1), P_{x_m,z_m}$ are paths of H_1, H_i, H_m, respectively. When $m = 2s$, the set

$$C' = P_{z_1,y_1}P_{x_3,y_3} \cdots P_{x_{2s-1},y_{2s-1}}P_{z_{2s},x_{2s}}P_{y_{2s-2},x_{2s-2}} \cdots P_{y_2,x_2}$$

is a cycle of Γ with $E(C) = E(C')$; and when $m = 2s+1$, the set

$$C' = P_{z_1,y_1} P_{x_3,y_3} \cdots P_{x_{2s-1},y_{2s-1}} P_{x_{2s+1},z_{2s+1}} P_{y_{2s},x_{2s}} P_{y_{2s-2},x_{2s-2}} \cdots P_{y_2,x_2}$$

is a cycle of Γ with $E(C) = E(C')$. □

Lemma 3.13. *Suppose that Ω is obtained from H by a consecutive twisting, where k is even. If G is 2-connected and every cycle of H is connected in Γ, then Ω is a pinch.*

Proof. By Lemma 3.12, for some $i \in [k]$ no path connects x_i and y_i in $H_i \backslash z_i$. Then every unbalanced cycle in Ω uses z_i, i.e. z_i is a blocking vertex of Ω. Lemma 3.6 implies that there is a graph H' 2-isomorphic to G such that Ω is obtained from H' by a pinch. □

Therefore, by Lemmas 3.11, 3.12 and 3.13, and the analysis in the paragraph following Theorem 3.10, those signed graphs Ω having $F(\Omega)$ isomorphic to $M(G)$ with form Theorem 3.1(4) (that is, having just one unbalanced block and without two vertex-disjoint unbalanced cycles) are obtained by a pinch, a 4-twisting, or a consecutive odd-twisting.

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