### On three-color Ramsey number of paths

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#### Abstract

Let  $G_1, G_2, \ldots, G_t$  be graphs. The multicolor Ramsey number  $R(G_1, G_2, \ldots, G_t)$  is the smallest positive integer n such that if the edges of complete graph  $K_n$  are partitioned into t disjoint color classes giving t graphs  $H_1, H_2, \ldots, H_t$ , then at least one  $H_i$  has a subgraph isomorphic to  $G_i$ . In this paper, we prove that if  $(n, m) \neq (3, 3), (3, 4)$  and  $m \geq n$ , then  $R(P_3, P_n, P_m) = R(P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ . Consequently  $R(P_3, mK_2, nK_2) = 2m + n - 1$  for  $m \geq n \geq 3$ .

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## 1 Introduction

In this paper, we only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a graph G, the vertex set, edge set, maximum degree and minimum degree of G are denoted by V(G), E(G),  $\Delta(G)$  and  $\delta(G)$  (or simply V, E,  $\Delta, \delta$ ), respectively. As usual, the complete graph of order p is denoted by  $K_p$  and a complete bipartite graph with partite set (X, Y) such that |X| = m and |Y| = n is denoted by  $K_{m,n}$ . For two disjoint subsets X and Y of the vertices of a graph G, we use E(X, Y) to denote the set of all edges with one end point in X and the other in Y. For a vertex v and an induced subgraph Hof G the set of all neighbors of v in H are denoted by  $N_H(v)$ . Throughout this paper, we denote a cycle and a path on m vertices by  $C_m$  and  $P_m$ , respectively. Also for a 3-edge coloring (say green, red and blue) of a graph G, we denote by  $G^g$  (resp.  $G^r$  and  $G^b$ ) the induced subgraph by the edges of color green (resp. red and blue).

For given graphs  $G_1, G_2, \ldots, G_t$  the multicolor Ramsey number  $R(G_1, G_2, \ldots, G_t)$ , is the smallest positive integer n such that if the edges of complete graph  $K_n$  are partitioned into

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t disjoint color classes giving t graphs  $H_1, H_2, \ldots, H_t$ , then at least one  $H_i$  has a subgraph isomorphic to  $G_i$ . The existence of such a positive integer is guaranteed by Ramsey's classical result [12]. Since 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. For  $t \ge 3$ , there is a few results about multicolor Ramsey number  $R(G_1, G_2, \ldots, G_t)$ . A survey including some results on Ramsey number of graphs, can be found in [11].

The multicolor Ramsey number  $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$  is not known for  $t \ge 3$ . In the case t = 2, a well-known theorem of Gerencsér and Gyárfás [7] states that  $R(P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ , where  $m \ge n \ge 2$ . Faudree and Schelp in [5] determined  $R(P_{n_1}, P_{2n_2+\delta}, \ldots, P_{2n_t})$  where  $\delta \in \{0, 1\}$ and  $n_1$  is sufficiently large. As an improvement of this result in [10] the authors determined  $R(C_{n_1}, P_{2n_2+\delta}, \ldots, P_{2n_t})$  where  $\delta \in \{0, 1\}$  and  $n_1$  is sufficiently large. In addition, in [5] the authors determined  $R(P_{n_1}, P_{n_2}, P_{n_3})$  for the case  $n_1 \ge 6(n_2 + n_3)^2$  and they conjectured that

$$R(P_n, P_n, P_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd,} \\ \\ 2n-2 & \text{if } n \text{ is even.} \end{cases}$$

This conjecture was established by Gyárfás et al. [8] for sufficiently large n. In asymptotic form, this was proved by Figaj and Luczak in [6] as a corollary of more general results about the asymptotic results on the Ramsey number for three long even cycles.

It is a natural question to ask whether similar conclusion is true if  $K_{R(P_m,P_n)}$  is replaced by some weaker structures. One such result was obtained in [9] where it was proved that in every 2-coloring of the edges of the complete 3-partite graph  $K_{n,n,n}$  there is a monochromatic  $P_{(1-o(1))2n}$ . The following conjecture involving the minimum degree, was formulated by Schelp [13].

**Conjecture 1** Suppose that n is large enough and G is a graph on  $R(P_n, P_n)$  vertices with minimum degree larger than  $\frac{3}{4}|V(G)|$ . Then in any 2-coloring of the edges of G there is a monochromatic  $P_n$ .

Schelp also noticed that the condition on the minimum degree is sharp. Indeed, suppose that 3n - 1 = 4m and consider a graph whose vertex set is partitioned into four parts  $A_1, A_2, A_3, A_4$  with  $|A_i| = m$ . There are no edges from  $A_1$  to  $A_2$  and from  $A_3$  to  $A_4$ . Edges between  $A_1, A_3$  and  $A_2, A_4$  are red, edges between  $A_1, A_4$  and  $A_2, A_3$  are blue and for i = 1, 2, 3, 4 the edges of with two end points in  $A_i$  are colored arbitrary. In this coloring the longest monochromatic path has 2m vertices, much smaller then 2n, while the minimum degree is  $\frac{3}{4}|V(G)| - 1$ . Thus, this makes the conjecture surprising, even a minuscule increase in the minimum degree results in a dramatic increase in the length of the longest monochromatic path. Schelp [14] proved that there exists a c < 1 for which Conjecture 1 holds if the minimum degree is raised to c|V(G)|. The main result of this paper is the following.

**Theorem 1.1** If  $m \ge n$  and  $(n, m) \ne (3, 3), (3, 4)$ , then  $R(P_3, P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ . Moreover,  $R(P_3, P_3, P_3) = R(P_3, P_3, P_4) = 5$ .

In other words,  $R(P_3, P_n, P_m) = R(P_n, P_m)$  for  $m \ge n$  and  $(n, m) \ne (3, 3), (3, 4)$ . Clearly  $R(P_n, P_m)$  is a lower bound for  $R(P_3, P_n, P_m)$  and so we shall always prove just the claimed upper bound for the Ramsey number.

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$$R(P_3, P_n, P_m)$$
 for  $m \ge n$  and  $n \le 7$ 

In this section, we provide the exact values of  $R(P_3, P_n, P_m)$  when  $3 \le n \le 7$  and  $m \ge n$ . First, we recall a result of Faudree and Schelp.

**Theorem 2.1** ([5]) If G is a graph with |V(G)| = nt + r where  $0 \le r < n$  and G contains no path on n + 1 vertices, then  $|E(G)| \le t\binom{n}{2} + \binom{r}{2}$  with equality if and only if either  $G \cong tK_n \cup K_r$  or if n is odd, t > 0 and  $r = (n \pm 1)/2$ 

$$G \cong lK_n \cup \left( K_{(n-1)/2} + \overline{K}_{((n+1)/2 + (t-l-1)n+r)} \right),$$

for some  $0 \leq l < t$ .

By Theorem 2.1, it is easy to obtain the following corollary.

**Corollary 2.2** For all integer  $n \geq 3$ ,

$$ex(n, P_4) = \begin{cases} n & \text{if } n = 0 \pmod{3}, \\ n-1 & \text{if } n = 1, 2 \pmod{3}. \end{cases}$$

$$ex(n, P_5) = \begin{cases} 3n/2 & \text{if } n = 0 \pmod{4}, \\ 3n/2 - 2 & \text{if } n = 2 \pmod{4}, \\ (3n-3)/2 & \text{if } n = 1, 3 \mod{4}, \\ (3n-3)/2 & \text{if } n = 1, 3 \mod{4}, \end{cases}$$

$$ex(n, P_6) = \begin{cases} 2n & \text{if } n = 0 \pmod{5}, \\ 2n-2 & \text{if } n = 1, 4 \pmod{5}, \\ 2n-3 & \text{if } n = 2, 3 \mod{5}. \end{cases}$$

**Theorem 2.3** ([3, 4])  $R(P_3, P_4, P_m) = m + 1$  for  $m \ge 6$  and  $R(P_3, P_5, P_m) = m + 1$  for  $m \ge 8$ .

**Theorem 2.4** (i)  $R(P_3, P_3, P_m) = m$  for  $m \ge 5$ . (ii)  $R(P_3, P_4, P_m) = m + 1$  for  $4 \le m \le 5$ . (iii)  $R(P_3, P_5, P_m) = m + 1$  for  $5 \le m \le 7$ . **Proof.** (i) Let  $G = K_m$  be 3-edge colored green, red and blue such that G does not contain green or red  $P_3$ . It is clear to see that  $G^b$  is connected and  $\delta(G^b) \ge m - 3$ . Thus  $G^b$  has a Hamiltonian path(see [1]) and so a  $P_m$ .

(ii) Let  $G = K_{m+1}$  be 3-edge colored green, red and blue such that  $P_3 \notin G^g$  and  $P_4 \notin G^r$ . First let m = 4. Using corollary 2.2 we may assume that  $|E(G^g)| \leq 2$  and  $|E(G^r)| \leq 4$ . If  $|E(G^r)| = 4$ , then by Theorem 2.1  $G^r \cong K_3 \cup K_2$  or  $G^r \cong K_{1,4}$  which clearly the complement of  $G^r$  with respect to G is colored green and blue and so it contains a blue copy of  $P_4$ . Thus we may assume that  $|E(G^r)| \leq 3$  and so  $|E(G^b)| \geq 5$ . Using corollary 2.2  $G^b$  contains  $P_4$ . By a similar argument one can show that  $R(P_3, P_4, P_5) = 6$ .

(iii) Let  $G = K_{m+1}$  be 3-edge colored green, red and blue such that  $P_3 \notin G^g$  and  $P_5 \notin G^r$ . First let  $m \neq 5$ . By a result in [11],  $R(P_3, C_4, P_m) = m + 1$  for  $m \in \{6, 7\}$  and so we may assume that G contains a red  $C_4$ . Set  $A = V(C_4)$  and  $B = V(G) \setminus A$ . Since  $P_5 \notin G^r$ , all edges between A and B are colored green or blue which clearly G[E(A, B)] contains a blue  $P_m$ . Now consider the case m = 5. By a similar argument, we may assume that  $G^r$  and  $G^b$  don't contain  $C_4$  as subgraph. Since |E(G)| = 15, by Theorem 2.1 we may assume that  $|E(G^g)| = 3$ ,  $|E(G^r)| = 6$  and  $|E(G^b)| = 6$  and so the green edges form a perfect matching. But  $R(P_3, P_4, P_5) = 6$ , by part (ii), and so we may assume that  $G^r$  contains a copy of  $P_4$ , say  $P = v_1 v_2 v_3 v_4$ . Set  $A = V(G) \setminus V(P) = \{v_5, v_6\}$ . Since  $P_5 \notin G^r$ , all edges in  $E(\{v_1, v_4\}, A)$ are colored green or blue. Also since the green edges form a perfect matching, the subgraph of  $G^g$  induced by  $E(\{v_1, v_4\}, A)$  dose not contain a perfect matching. Thus we may assume that  $P' = v_5 v_1 v_6 v_4 \subseteq G^b$  and  $v_4 v_5 \in E(G^g)$ . Now since  $P_5 \notin G^r$ , at least one of  $v_2 v_5$  or  $v_3 v_5$ , say  $v_2 v_5$ , must be blue and so  $v_3 v_5 P' v_4$  form a blue  $P_5$ . This observation completes the proof.

Combining Theorems 2.3 and 2.4, we obtain that  $R(P_3, P_n, P_m) = R(P_n, P_m)$  if  $m \ge n$ ,  $n \in \{3, 4, 5\}$  and  $(n, m) \ne (3, 3), (3, 4)$ . In the rest of this section we prove that  $R(P_3, P_n, P_m) = R(P_n, P_m)$  for  $m \ge n$ ,  $n \in \{6, 7\}$ . But before that we need some lemmas.

**Lemma 2.5** Let G be a graph obtained from the complete bipartite graph  $K_{3,4}$  by removing an edge. If each edge of G is colored red or blue, then  $G^r$  contains  $P_3$  or  $G^b$  contains  $P_7$ .

**Proof.** Let G = (X, Y),  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3, y_4\}$ . Also let  $x_1y_1$  be the edge that removed from  $K_{3,4}$ . If  $G^r$  does not contain  $P_3$ , then  $G^r$  has at most three edges. Let H be a spanning subgraph of G with  $E(H) = E(G^r) \cup \{x_1y_1\}$ . It is clear to see that  $H \subseteq P_3 \cup 2P_2$  or  $H \subseteq P_4 \cup P_2 \cup P_1$  and so the complement of H with respect to  $K_{3,4}$  contains a copy of  $P_7$ . This observation completes the proof.

**Lemma 2.6** Suppose  $m \ge 7$  and the edges of  $K_{m+2}$  are colored with colors green, red and blue such that  $G^b$  contains a copy of  $P_{m-1}$  as a subgraph. Then  $K_{m+2}$  contains a green  $P_3$ , a red  $P_7$  or a blue  $P_m$ .

**Proof.** Assume that  $G = K_{m+2}$  with  $V(G) = \{v_1, v_2, \ldots, v_{m+2}\}$  and  $P = v_1v_2 \ldots v_{m-1}$  is the desired copy of  $P_{m-1}$  in  $G^b$ . We suppose that  $G^b$  contains no copy of  $P_m$ , then we prove that  $K_{m+2}$  contains a green  $P_3$  or a red  $P_7$ . We find two vertices  $v, v' \in P_{m-1}$  such that the bipartite graph with parties  $X = \{v_m, v_{m+1}, v_{m+2}\}$  and  $Y = \{v_1, v, v', v_{m-1}\}$  is a red-green graph with at least 11 edges and then we use Lemma 2.5, which guarantees the existence of a green  $P_3$  or

a red  $P_7$ . Note that we may assume that in  $G^b$ , the vertices  $v_2$  and  $v_{m-2}$  don't have a common neighbor in X. Otherwise, since  $P_m \not\subseteq G^b$ ,  $v_3$  (also  $v_{m-3}$ ) is not adjacent to any vertex of X in  $G^b$  and so  $v_3$  and  $v_{m-3}$  are the desired vertices. Thus we may assume that in  $G^b$  one of  $v_2$  or  $v_{m-2}$ , say  $v_{m-2}$ , has at most one neighbor in X. If  $N_{G^b}(v_{m-2}) \cap X = \emptyset$ , then we may assume that in  $G^b$  each vertex  $v_i \in V(P) \setminus \{v_1, v_{m-2}, v_{m-1}\}$  has at least two neighbors in X, otherwise set  $v = v_i$  and  $v' = v_{m-2}$ . Therefore if  $N_{G^b}(v_{m-2}) \cap X = \emptyset$  we have  $N_{G^b}(v_2) \cap N_{G^b}(v_3) \cap X \neq \emptyset$ , and so  $P_m \subseteq G^b$ , a contradiction. Hence  $|N_{G^b}(v_{m-2}) \cap X| = 1$ ,

Since  $|N_{G^b}(v_{m-2}) \cap X| = 1$ , so we may assume that in  $G^b$  each vertex  $v_i \in V(P) \setminus \{v_1, v_{m-2}, v_{m-1}\}$  has at least one neighbor in X, otherwise set  $v = v_i$  and  $v' = v_{m-2}$ . Since  $P_m \notin G^b$ , one can easily check that  $|N_{G^b}(v_i) \cap X| = 1$ ,  $2 \leq i \leq 5$  and w.l.g  $N_{G^b}(v_2) \cap X = \{v_m\}$ ,  $N_{G^b}(v_3) \cap X = \{v_{m+1}\}$ ,  $N_{G^b}(v_4) \cap X = \{v_{m+2}\}$  and  $N_{G^b}(v_5) \cap X = \{v_{m+1}\}$ . If m = 7 then  $v_1v_2v_3v_8v_5v_4v_9$  is a blue  $P_7$  in  $K_9$ , a contradiction. Now let  $m \geq 8$ . Since  $|N_{G^b}(v_{m-2}) \cap X| = 1$ ,  $v_{m-2}$  must be adjacent to a vertex in X by blue and in any case we have a copy of  $P_m \subseteq G^b$ , a contradiction. This observation completes the proof.

### **Lemma 2.7** $R(P_3, P_6, P_7) = 9.$

**Proof.** Let  $G = K_9$  be 3-edge colored with colors green, red and blue. By a result in [15],  $R(P_3, C_6, C_6) = 9$  and so we may assume that  $G^b$  contains a copy of  $C_6$  as subgraph. Set  $X = V(K_9) \setminus V(C_6)$ . We may assume that all edges between X and  $C_6$  are colored red or green. Therefore by Lemma 2.5,  $K_9$  must contain a green  $P_3$  or a red  $P_6$ , which completes the proof.

Using Lemmas 2.6 and 2.7 we have the following.

### **Theorem 2.8** $R(P_3, P_7, P_m) = R(P_3, P_6, P_m) = m + 2$ for $m \ge 7$ . Moreover $R(P_3, P_6, P_6) = 8$ .

**Proof.** Since  $R(P_3, P_6, P_m) \leq R(P_3, P_7, P_m)$  and  $m + 2 = R(P_6, P_m) \leq R(P_3, P_6, P_m)$ , it is sufficient to show that  $R(P_3, P_7, P_m) \leq m + 2$  for  $m \geq 7$ . Using Lemmas 2.6 and 2.7 we have  $R(P_3, P_7, P_7) = 9$  and again using Lemma 2.6 and induction on m we obtain that  $R(P_3, P_7, P_m) \leq$ m + 2. On the other hand  $8 = R(P_6, P_6) \leq R(P_3, P_6, P_6)$ . To complete the proof it is sufficient to show that  $R(P_3, P_6, P_6) \leq 8$ . Let  $G = K_8$  be 3-edge colored with colors green, red and blue. Suppose G have neither a green  $P_3$  nor a blue  $P_6$ . If G has a red  $P_6$  we are done. So suppose that G does not have any red  $P_6$ . Using (*iii*) of Theorem 2.4 we may assume that G has a red  $P_5$  with vertices  $v_1, v_2, \dots, v_5$  as a subgraph. Then we may assume that  $v_1v_6, v_1v_7, v_5v_7, v_5v_8$ are blue edges. If  $E(\{v_6, v_8\}, \{v_2, v_3, v_4\})$  has a blue edge, combining this edge with the path  $v_6v_1v_7v_5v_8$  gives a blue  $P_6$ , a contradiction. So  $v_8v_2, v_8v_4, v_6v_2, v_6v_4$  are red edges and  $v_3v_8$  and  $v_3v_6$  are green edges and hence G has a green  $P_3$ , a contradiction.

# **3** $R(P_3, P_n, P_m)$ for $m \ge n \ge 8$

In this section, we compute the value of  $R(P_3, P_n, P_m)$  for  $m \ge n \ge 8$ . Before that we need some lemmas.

**Lemma 3.1** Suppose that  $G = K_{m+\lfloor \frac{n}{2} \rfloor -1}$ ,  $m \ge n \ge 8$ , is 3-edge colored green, red and blue and  $P = v_1 v_2 \cdots v_{m-1}$  is the maximum path in  $G^b$ . Let  $A = V(G) \setminus V(P)$  and H be the subgraph of  $G^r$  induced by the edges in  $E(V(P) \setminus \{v_1, v_{m-1}\}, A)$ . Then either  $P_3 \subseteq G^g$  or  $d_H(v_i) \ge 2$  for some  $i, 2 \le i \le m-2$ .

**Proof.** We suppose that  $G^g$  contains no copy of  $P_3$ . Since  $m \ge n \ge 8$ , we obtain that  $|A| \ge 4$  and so let  $X = \{u_1, u_2, u_3, u_4\} \subseteq A$ . Again since  $m \ge 8$  there is a  $v_j \in V(P) \setminus \{v_1, v_{m-1}\}$  such that all edges in  $E(X, \{v_j\})$  are red and blue. If  $|N_{G^r}(v_j) \cap X| \ge 2$ , we have nothing to prove. Otherwise, we may assume that  $Y = \{u_1, u_2, u_3\} \subseteq N_{G^b}(v_j) \cap X$ . Since  $P_m \not\subseteq G^b$ ,  $G^r$  contains at least two edges in  $E(Y, \{v\})$  for some  $v \in \{v_{j-1}, v_{j+1}\}$  and so  $d_H(v) \ge 2$ .

**Lemma 3.2** Suppose that  $G = K_n$  is 3-edge colored green, red and blue,  $P_3 \notin G^g$  and P is a maximal path in  $G^b$  with endpoints x and y. Then for every two vertices z and w of  $V(G) \setminus V(P)$  either  $xz, yw \in E(G^r)$  or  $xw, yz \in E(G^r)$ .

**Proof.** Since  $P_3 \not\subseteq G^g$  and P is a maximal path in  $G^b$ , each of z and w is adjacent to at least one of x and y in  $G^r$ . With no loss of generality, suppose that  $xz \in E(G^r)$ . If  $yw \in E(G^r)$ , the proof is completed. Otherwise,  $yw \in E(G^g)$  and so  $xw, yz \in E(G^r)$ , which completes the proof.

**Lemma 3.3**  $R(P_3, P_8, P_8) = 11$ 

**Proof.** Let  $G = K_{11}$  be 3-edge colored green, red and blue such that  $P_3 \notin G^g$ . We find monochromatic copy of  $P_8$  in blue or red color. By Theorem 2.8,  $R(P_3, P_7, P_8) = 10$  and so we may assume that  $P_7$  is a maximum path in  $G^r$ . Let  $P = v_1v_2 \ldots v_7 \subseteq G^r$  and  $A = V(G) \setminus V(P) = \{x_1, x_2, x_3, x_4\}$ . Using Lemma 3.1, there exists a  $v_j \in V(P) \setminus \{v_1, v_7\}$  which is adjacent to at least two vertices of A, say  $x_1, x_2$ , in  $G^b$ . By Lemma 3.2, w.l.g we may assume that  $\{x_1v_1, x_2v_7, x_3v_1, x_4v_7\} \subseteq E(G^b)$  and so  $Q_7 = x_3v_1x_1v_jx_2v_7x_4 \subseteq G^b$ . Let  $K = V(P) \setminus \{v_1, v_j, v_7\}$ . Then |K| = 4 and one can easily check that at least one of  $x_3$  or  $x_4$  is adjacent to a vertex of K, say  $v_i$ , in  $G^b$ . Therefore  $Q_7 \cup \{v_i\}$  is a blue  $P_8$ .

**Theorem 3.4** For any  $m \ge n \ge 8$ ,  $R(P_3, P_n, P_m) = m + \lfloor \frac{n}{2} \rfloor - 1$ .

**Proof.** Let  $t = m + \lfloor \frac{n}{2} \rfloor - 1$  and  $G = K_t$  be 3-edge colored green, red and blue such that  $P_3 \notin G^g$  and  $P_m \notin G^b$ . By induction on m + n, we prove that  $P_n \subseteq G^r$ . By Lemma 3.3 theorem is true for m = n = 8. By the induction hypothesis  $R(P_3, P_n, P_{m-1}) \leq m + \lfloor \frac{n}{2} \rfloor - 1$  and so there is a  $P_{m-1} \subseteq G^b$ . Let  $P = P_{m-1} = v_1 v_2 \dots v_{m-1}$ ,  $A = V(G) \setminus V(P)$  and H be the subgraph of  $G^r$  induced by the edges in  $E(V(P) \setminus \{v_1, v_{m-1}\}, A)$ . Suppose Q is a maximal path of H with end points  $u_1$  and  $u_2$  in A, the existence of such a path is guaranteed by Lemma 3.1. Let  $K = (V(P) \setminus \{v_1, v_{m-1}\}) \setminus V(Q)$ . If all vertices in A are covered by Q, then by Lemma 3.2, we may assume that  $u_1v_1, u_2v_{m-1} \in E(G^r)$  and so  $R = v_1u_1Qu_2v_{m-1}$  is a red path on  $2\lfloor \frac{n}{2} \rfloor + 1$  vertices. Thus we may assume that  $A \setminus V(Q) \neq \emptyset$ .

**Case 1.**  $|A \setminus V(Q)| = 1$ .

Let  $A \setminus V(Q) = \{x\}$ . By Lemma 3.2, we may assume that  $v_1u_1, v_{m-1}u_2 \in E(G^r)$ . In the other hand, since P is maximal and  $P_3 \notin G^g$ , x is adjacent to at least one of  $v_1$  and  $v_{m-1}$  in  $G^r$ , say  $v_1$ . Thus  $R = xv_1u_1Qu_2v_{m-1} \subseteq G^r$  form a path on  $2\lfloor \frac{n}{2} \rfloor$  vertices. If n is even, there is nothing to prove and so we may assume that n is odd. Note that  $|K| = m - 3 - (\lfloor \frac{n}{2} \rfloor - 2) \ge \lceil \frac{m}{2} \rceil - 1 > \lfloor \frac{n}{2} \rfloor - 1$  and so by the Pigeonhole principle there exist two consecutive vertices  $v_i, v_{i+1}$  in K. If  $xv_i \in E(G^r)$  (or  $xv_{i+1} \in E(G^r)$ ), then  $\{v_i\} \cup V(R)$  (or  $\{v_{i+1}\} \cup V(R)$ ) form a red  $P_n$ . Otherwise, since both  $xv_i$  and  $xv_{i+1}$  are not in  $E(G^g)$  or  $E(G^b)$ , w.l.g we may assume that  $xv_i \in E(G^b)$ and  $xv_{i+1} \in E(G^g)$  which implies that  $xv_{m-1} \in E(G^r)$ . Therefore  $V(R) \cup \{x\}$  form a copy of  $C_{n-1}$  in  $G^r$ . It is clear to see that at least one of  $v_i$  or  $v_{i+1}$  is adjacent to one of  $u_1$  or  $u_2$  by a red edge Thus, we can find a red  $P_n$ .

### **Case 2.** $|A \setminus V(Q)| = 2$ .

Let  $A \setminus V(Q) = \{x, y\}$ . Using Lemma 3.2 we may assume that  $R = xv_1u_1Qu_2v_{m-1}y$  is a red path on  $2\lfloor \frac{n}{2} \rfloor - 1$  vertices. (Note that in this case,  $|K| = m - 3 - (\lfloor \frac{n}{2} \rfloor - 3) \ge \lceil \frac{m}{2} \rceil \ge \lceil \frac{n}{2} \rceil$ ). We consider the following subcases.

#### Subcase 1. n is even:

By the Pigeonhole principle there exists a pair of vertices  $(v_i, v_{i+1})$  in K. If one of x or y is adjacent to one of  $v_i$  or  $v_{i+1}$ , say  $v_i$ , in  $G^r$ , then  $v_i x R y$  form a red  $P_n$ . Otherwise, green and also blue edges in  $E(\{v_i, v_{i+1}\}, \{x, y\})$  form a matching and so  $yv_1$  is red and w.l.g we may assume that  $u_1v_i$  is red. Thus  $R' = v_i u_1 Q u_2 v_{m-1} y v_1 x \subseteq G^r$  is a path on n vertices.

### Subcase 2. n is odd:

By the Pigeonhole principle there exist two disjoint pairs of vertices  $(v_j, v_{j+1})$  and  $(v_k, v_{k+1})$  in K. It is easy to see that each of x and y is adjacent to a vertex in  $B = \{v_j, v_{j+1}, v_k, v_{k+1}\}$  by red edge. If the mentioned neighbors of x and y are distinct we have a red  $P_n$ , otherwise let  $v_j \in B$  be the only neighbor of x and y. Therefore,  $\{v_j\} \cup V(R)$  form a red  $C_{n-1}$ . It is easy to see that there is an edge in  $G^r$  between  $B \setminus \{v_i\}$  and  $\{u_1, u_2\}$  and so a red  $P_n$  can be found.

Case 3.  $|A \setminus V(Q)| \ge 3$ .

Let  $x, y, z \in A \setminus V(Q)$ .

**Claim 3.5** Let H be the subgraph of  $G^r$  induced by the edges in  $E(A \setminus V(Q), K)$ . There is a vertex  $v \in H \cap K$  such that  $d_H(v) \ge 2$ .

**Proof.** There are at least  $\lceil \frac{n}{2} \rceil + 1$  vertices in K. By the Pigeonhole principle, there are two disjoint pairs of vertices  $(v_i, v_{i+1})$  and  $(v_j, v_{j+1})$  in K. We prove the claim by considering the number of red edges from  $\{v_i, v_{i+1}\}$  to  $\{x, y, z\}$ . If there are more than two such edges, then the claim is proved. Thus we may assume that there are at most two such edges. Since  $P_3 \not\subseteq G^g$  and  $P_m \not\subseteq G^b$ , there is at least one such an edge. Therefore, it is sufficient to consider the following cases.

i) W.l.g,  $G^r$  contains two edges in  $E(\{v_i, v_{i+1}\}, \{x, y, z\})$ 

*ii*) W.l.g,  $G^r$  contains exactly one edge in  $E(\{v_i, v_{i+1}\}, \{x, y, z\})$ .

If (i) occurs, we may assume that there are exactly one edge from each of  $v_i$  and  $v_{i+1}$  to  $\{x, y, z\}$ in  $G^r$ , otherwise we have nothing to prove. Suppose there is no red edge in  $E(\{v_i, v_{i+1}\}, \{z\})$ . Since  $P_3 \not\subseteq G^g$  and  $P_m \not\subseteq G^b$ ,  $G^r$  contains at least one edge in  $E(\{z, u_1, u_2\}, \{v_i, v_{i+1}\})$ . Whereas Q is maximal, this edge has to be in  $E(\{v_i, v_{i+1}\}, \{z\})$ , a contradiction.

If (*ii*) occurs, we may assume that  $xv_i \in E(G^r)$ . Since  $G^r$  contains no edge in  $E(\{v_i, v_{i+1}\}, \{y, z\})$ , green and also blue edges in  $E(\{v_i, v_{i+1}\}, \{y, z\})$  form a matching. Thus, clearly there are two red edges in  $E(\{v_j, v_{j+1}\}, \{y, z\})$ . The reminder of the proof is the same to the case (*i*).

Now, let Q' be a maximal path in the subgraph of  $G^r$  induced by the edges in  $E(A \setminus V(Q), K)$ with endpoints  $w_1$  and  $w_2$  in  $A \setminus V(Q)$  and  $K' = K \setminus V(Q')$ .

**Case 1.**  $|A \setminus (V(Q) \cup V(Q'))| = 0.$ 

Using Lemma 3.2, we may assume that  $G^r$  contains a cycle  $C = w_1 Q' w_2 v_{m-1} u_2 Q u_1 v_1 w_1$  on  $2\lfloor \frac{n}{2} \rfloor$  vertices. If n is even, we are done. Otherwise, since  $|K'| \ge \lceil \frac{n}{2} \rceil - 1$ , there is one pair of vertices  $(v_i, v_{i+1})$  in K'. Since  $G^r$  contains at least one edge in  $E(\{u_1, u_2, w_1, w_2\}, \{v_i, v_{i+1}\})$ , we may suppose that  $v_i u_1 \in E(G^r)$  and so  $R' = v_i u_1 Q u_2 v_{m-1} w_2 Q' w_1 v_1$  is a red  $P_n$ .

**Case 2.**  $|A \setminus (V(Q) \cup V(Q'))| = 1.$ 

Let  $A \setminus (V(Q) \cup V(Q')) = \{x\}$ . Using Lemma 3.2 we may assume that  $u_1v_1, u_2v_{m-1}, w_1v_{m-1}$ and  $w_2v_1$  are red edges. Since  $P_3 \notin G^g$ ,  $G^r$  contains at least one edge in  $E(\{v_1, v_{m-1}\}, \{x\})$ , say  $xv_1$ . Thus  $R = xv_1u_1Qu_2v_{m-1}w_1Q'w_2$  is a red  $P_{2|\frac{n}{2}|-1}$ . We consider the following subcases.

Subcase 1. n is even:

Since  $|K'| \geq \lceil \frac{n}{2} \rceil$ , there is at least one pair of vertices  $(v_i, v_{i+1})$  in K'. If  $xv_i$  (or  $xv_{i+1}$ ) is red, then  $v_i x R w_2$  (or  $v_{i+1} x R w_2$ ) form a red  $P_n$ . Otherwise, we may assume that  $xv_i \in E(G^b)$  and  $xv_{i+1} \in E(G^g)$ . Therefore  $xv_{m-1} \in E(G^r)$  and  $R' = u_1 Q u_2 v_{m-1} x v_1 w_2 Q' w_1$  is a red  $P_{n-1}$ . Whereas  $G^r$  contains at least one edge of  $E(\{v_i, v_{i+1}\}, \{u_1, w_1\})$ , we can extend R' to a red  $P_n$ .

Subcase 2. n is odd:

Since  $|K'| \ge \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ , there are at least two disjoint pairs of vertices  $(v_j, v_{j+1})$  and  $(v_k, v_{k+1})$ in K'. Clearly, each of x and  $w_2$  in  $G^r$  has at least one neighbor in  $B = \{v_j, v_{j+1}, v_k, v_{k+1}\}$ , say  $s_1$  and  $s_2$  respectively. If  $s_1 \ne s_2$ ,  $s_1 x R w_2 s_2$  is a red  $P_n$ , else  $s_1 x R w_2 s_1$  is a red  $C_{n-1}$ . One can easily check that  $G^r$  contains at least one edge of  $E(B \setminus \{s_1\}, \{u_1, u_2, w_1\})$ , and so adding this edge to  $C_{n-1}$  yields a  $P_n \subseteq G^r$ .

Case 3.  $|A \setminus (V(Q) \cup V(Q'))| \ge 2$ .

Let  $x, y \in A \setminus (V(Q) \cup V(Q'))$ . We show that this case is impossible. Since  $|K'| \ge \lfloor \frac{n}{2} \rfloor + 1$  and at most  $\lfloor \frac{n}{2} \rfloor - 4$  vertices of  $V(P) \setminus \{v_1, v_{m-1}\}$  are covered by Q and Q', by the Pigeonhole principle we have one of the following cases.

i) K' contains four disjoint pairs of vertices  $(v_k, v_{k+1}), (v_i, v_{i+1}), (v_j, v_{j+1})$  and  $(v_l, v_{l+1})$ .

ii) K' contains three consecutive vertices  $v_k, v_{k+1}, v_{k+2}$ .

If (i) occurs, since  $P_3 \not\subseteq G^g$  and  $P_m \not\subseteq G^b$  there is a red edge between x and any two pairs of vertices and so w.l.g we may assume that  $xv_{k+1}$ ,  $xv_{l+1}$ ,  $xv_{i+1} \in E(G^r)$ . Since Q and Q'are maximal,  $G^r$  contains no edge in  $E(\{u_1, u_2, w_1, w_2\}, \{v_{k+1}, v_{l+1}, v_{i+1}\})$ . If there is a red edge in  $E(\{u_1, u_2\}, \{v_t, v_{t+1}\})$  (resp. in  $E(\{w_1, w_2\}, \{v_t, v_{t+1}\})$ ) for some  $t \in \{i, k, l\}$ , then the maximality of Q and Q' implies that green and also blue edges in  $E(\{w_1, w_2\}, \{v_t, v_{t+1}\})$ (resp. in  $E(\{u_1, u_2\}, \{v_t, v_{t+1}\})$ ) form perfect matchings on four vertices. Now since there is at least one red edge in  $E(\{u_1, u_2\}, \{v_t, v_{t+1}\})$  (resp. in  $E(\{w_1, w_2\}, \{v_t, v_{t+1}\})$ ) for some  $t \in \{i, k, l\}$ , w.l.g we may assume that green and also blue edges in both  $E(\{u_1, u_2\}, \{v_k, v_{k+1}\})$ and  $E(\{w_1, w_2\}, \{v_l, v_{l+1}\})$  form matchings. Therefore  $\{u_1v_{i+1}, u_2v_{i+1}, w_1v_{i+1}, w_2v_{i+1}\} \subseteq E(G^b)$ and consequently  $\{u_1v_i, u_2v_i, w_1v_i, w_2v_i\} \subseteq E(G^r)$  which is a contradiction.

If (*ii*) occurs, at least five vertices of  $\{u_1, u_2, w_1, w_2, x, y\}$  are adjacent to some vertices of  $\{v_k, v_{k+1}, v_{k+2}\}$  in  $G^r$ , since  $P_3 \notin G^g$  and  $P_m \notin G^b$ . Let *B* be the set of the vertices in  $\{u_1, u_2, w_1, w_2, x, y\}$  that are adjacent to a vertex in  $\{v_k, v_{k+1}, v_{k+2}\}$  by a red edge. Since  $P_3 \notin G^g$  and  $P_m \notin G^b$  then every vertex of *B* has exactly one red neighbor in  $\{v_k, v_{k+1}, v_{k+2}\}$ . Now, we have the following subcases.

Subcase 1.  $\{x, y\} \subseteq B$ :

By the maximality of Q and Q', we may suppose that the edges  $xv_t, yv_t, w_1v_{t'}, w_2v_{t'}$  are red for some  $t, t' \in \{k, k + 1, k + 2\}, t < t'$  and  $u_1v_r \in E(G^r)$  where  $r \neq t, t'$ . If  $t, t' \in \{k, k + 1\}$ (resp.  $t, t' \in \{k + 1, k + 2\}$ ) then green and also blue edges in  $E(\{x, y\}, \{v_r, v_{t'}\})$  (resp.  $E(\{w_1, w_2\}, \{v_r, v_t\})$ ) form matchings and so there is a red edge in  $E(\{u_1, u_2\}, \{v_t, v_{t'}\})$  (resp.  $E(\{u_1, u_2\}, \{v_t, v_{t'}\})$ ), and this contradicts the maximality of Q and Q'. Finally if  $t, t' \in \{k, k+2\}$ then green and also blue edges in  $E(\{w_1, w_2\}, \{v_r, v_t\})$  form matchings and so there is a red edge in  $E(\{x, y\}, \{v_r, v_{t'}\})$ ) and again this contradicts the maximality of Q and Q'.

Subcase 2.  $\{x, y\} \cap B = \{x\}$ :

By a similar argument as in subcase 1, we have a contradiction which completes the proof of the theorem.  $\hfill\blacksquare$ 

**Proof of Theorem 1.1.** It is clear that  $5 \le R(P_3, P_3, P_3) \le R(P_3, P_3, P_4)$ . On the other hand by corollary 2.2,  $R(P_3, P_3, P_4) \le 5$ . Then  $R(P_3, P_3, P_3) = R(P_3, P_3, P_4) = 5$ . Combining Theorems 2.3, 2.4, 2.8 and 3.4 give a proof for Theorem 1.1.

**Corollary 3.6**  $R(P_3, nK_2, mK_2) = 2m + n - 1$  for every  $m \ge n \ge 3$ .

**Proof.** To see  $2m+n-1 \leq R(P_3, nK_2, mK_2)$ , let  $H = K_{n-1} + \bar{K}_{2m-1}$  and  $\bar{H}$  be the complement of H with respect to  $K_{2m+n-2}$ . Clearly coloring H by red and  $\bar{H}$  by blue yields a 2-edge coloring of  $K_{2m+n-2}$  such that  $nK_2 \notin G^r$  and  $mK_2 \notin G^b$ . This means that  $2m+n-1 \leq R(P_3, nK_2, mK_2)$ . Now we prove the upper bound. It is easy to see that  $R(P_3, nK_2, mK_2) \leq R(P_3, P_{2n}, P_{2m})$  and by Theorem 1.1,  $R(P_3, P_{2n}, P_{2m}) = 2m + n - 1$ . This observation completes the proof.

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