# On three-color Ramsey number of paths 

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#### Abstract

Let $G_{1}, G_{2}, \ldots, G_{t}$ be graphs. The multicolor Ramsey number $R\left(G_{1}, G_{2}, \ldots, G_{t}\right)$ is the smallest positive integer $n$ such that if the edges of complete graph $K_{n}$ are partitioned into $t$ disjoint color classes giving $t$ graphs $H_{1}, H_{2}, \ldots, H_{t}$, then at least one $H_{i}$ has a subgraph isomorphic to $G_{i}$. In this paper, we prove that if $(n, m) \neq(3,3),(3,4)$ and $m \geq n$, then $R\left(P_{3}, P_{n}, P_{m}\right)=R\left(P_{n}, P_{m}\right)=m+\left\lfloor\frac{n}{2}\right\rfloor-1$. Consequently $R\left(P_{3}, m K_{2}, n K_{2}\right)=2 m+n-1$ for $m \geq n \geq 3$.


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## 1 Introduction

In this paper, we only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a graph $G$, the vertex set, edge set, maximum degree and minimum degree of $G$ are denoted by $V(G), E(G), \Delta(G)$ and $\delta(G)$ (or simply $V, E$, $\Delta, \delta)$, respectively. As usual, the complete graph of order $p$ is denoted by $K_{p}$ and a complete bipartite graph with partite set $(X, Y)$ such that $|X|=m$ and $|Y|=n$ is denoted by $K_{m, n}$. For two disjoint subsets $X$ and $Y$ of the vertices of a graph $G$, we use $E(X, Y)$ to denote the set of all edges with one end point in $X$ and the other in $Y$. For a vertex $v$ and an induced subgraph $H$ of $G$ the set of all neighbors of $v$ in $H$ are denoted by $N_{H}(v)$. Throughout this paper, we denote a cycle and a path on $m$ vertices by $C_{m}$ and $P_{m}$, respectively. Also for a 3-edge coloring (say green, red and blue) of a graph $G$, we denote by $G^{g}$ (resp. $G^{r}$ and $G^{b}$ ) the induced subgraph by the edges of color green (resp. red and blue).

For given graphs $G_{1}, G_{2}, \ldots, G_{t}$ the multicolor Ramsey number $R\left(G_{1}, G_{2}, \ldots, G_{t}\right)$, is the smallest positive integer $n$ such that if the edges of complete graph $K_{n}$ are partitioned into

[^0]$t$ disjoint color classes giving $t$ graphs $H_{1}, H_{2}, \ldots, H_{t}$, then at least one $H_{i}$ has a subgraph isomorphic to $G_{i}$. The existence of such a positive integer is guaranteed by Ramsey's classical result [12]. Since 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. For $t \geq 3$, there is a few results about multicolor Ramsey number $R\left(G_{1}, G_{2}, \ldots, G_{t}\right)$. A survey including some results on Ramsey number of graphs, can be found in [11].

The multicolor Ramsey number $R\left(P_{n_{1}}, P_{n_{2}}, \ldots, P_{n_{t}}\right)$ is not known for $t \geq 3$. In the case $t=2$, a well-known theorem of Gerencsér and Gyárfás [7] states that $R\left(P_{n}, P_{m}\right)=m+\left\lfloor\frac{n}{2}\right\rfloor-1$, where $m \geq n \geq 2$. Faudree and Schelp in [5] determined $R\left(P_{n_{1}}, P_{2 n_{2}+\delta}, \ldots, P_{2 n_{t}}\right)$ where $\delta \in\{0,1\}$ and $n_{1}$ is sufficiently large. As an improvement of this result in 10 the authors determined $R\left(C_{n_{1}}, P_{2 n_{2}+\delta}, \ldots, P_{2 n_{t}}\right)$ where $\delta \in\{0,1\}$ and $n_{1}$ is sufficiently large. In addition, in [5 the authors determined $R\left(P_{n_{1}}, P_{n_{2}}, P_{n_{3}}\right)$ for the case $n_{1} \geq 6\left(n_{2}+n_{3}\right)^{2}$ and they conjectured that

$$
R\left(P_{n}, P_{n}, P_{n}\right)= \begin{cases}2 n-1 & \text { if } n \text { is odd } \\ 2 n-2 & \text { if } n \text { is even }\end{cases}
$$

This conjecture was established by Gyárfás et al. 8 for sufficiently large $n$. In asymptotic form, this was proved by Figaj and Luczak in [6] as a corollary of more general results about the asymptotic results on the Ramsey number for three long even cycles.

It is a natural question to ask whether similar conclusion is true if $K_{R\left(P_{m}, P_{n}\right)}$ is replaced by some weaker structures. One such result was obtained in 9 where it was proved that in every 2 -coloring of the edges of the complete 3 -partite graph $K_{n, n, n}$ there is a monochromatic $P_{(1-o(1)) 2 n}$. The following conjecture involving the minimum degree, was formulated by Schelp [13].

Conjecture 1 Suppose that $n$ is large enough and $G$ is a graph on $R\left(P_{n}, P_{n}\right)$ vertices with minimum degree larger than $\frac{3}{4}|V(G)|$. Then in any 2-coloring of the edges of $G$ there is a monochromatic $P_{n}$.

Schelp also noticed that the condition on the minimum degree is sharp. Indeed, suppose that $3 n-1=4 m$ and consider a graph whose vertex set is partitioned into four parts $A_{1}, A_{2}, A_{3}, A_{4}$ with $\left|A_{i}\right|=m$. There are no edges from $A_{1}$ to $A_{2}$ and from $A_{3}$ to $A_{4}$. Edges between $A_{1}, A_{3}$ and $A_{2}, A_{4}$ are red, edges between $A_{1}, A_{4}$ and $A_{2}, A_{3}$ are blue and for $i=1,2,3,4$ the edges of with two end points in $A_{i}$ are colored arbitrary. In this coloring the longest monochromatic path has $2 m$ vertices, much smaller then $2 n$, while the minimum degree is $\frac{3}{4}|V(G)|-1$. Thus, this makes the conjecture surprising, even a minuscule increase in the minimum degree results in a dramatic increase in the length of the longest monochromatic path. Schelp [14] proved that there exists a $c<1$ for which Conjecture 1 holds if the minimum degree is raised to $c|V(G)|$. The main result of this paper is the following.

Theorem 1.1 If $m \geq n$ and $(n, m) \neq(3,3),(3,4)$, then $R\left(P_{3}, P_{n}, P_{m}\right)=m+\left\lfloor\frac{n}{2}\right\rfloor-1$. Moreover, $R\left(P_{3}, P_{3}, P_{3}\right)=R\left(P_{3}, P_{3}, P_{4}\right)=5$.

In other words, $R\left(P_{3}, P_{n}, P_{m}\right)=R\left(P_{n}, P_{m}\right)$ for $m \geq n$ and $(n, m) \neq(3,3),(3,4)$. Clearly $R\left(P_{n}, P_{m}\right)$ is a lower bound for $R\left(P_{3}, P_{n}, P_{m}\right)$ and so we shall always prove just the claimed upper bound for the Ramsey number.
$2 \quad R\left(P_{3}, P_{n}, P_{m}\right)$ for $m \geq n$ and $n \leq 7$

In this section, we provide the exact values of $R\left(P_{3}, P_{n}, P_{m}\right)$ when $3 \leq n \leq 7$ and $m \geq n$. First, we recall a result of Faudree and Schelp.

Theorem 2.1 ([5) If $G$ is a graph with $|V(G)|=n t+r$ where $0 \leq r<n$ and $G$ contains no path on $n+1$ vertices, then $|E(G)| \leq t\binom{n}{2}+\binom{r}{2}$ with equality if and only if either $G \cong t K_{n} \cup K_{r}$ or if $n$ is odd, $t>0$ and $r=(n \pm 1) / 2$

$$
G \cong l K_{n} \cup\left(K_{(n-1) / 2}+\bar{K}_{((n+1) / 2+(t-l-1) n+r)}\right),
$$

for some $0 \leq l<t$.

By Theorem 2.1, it is easy to obtain the following corollary.

Corollary 2.2 For all integer $n \geq 3$,

$$
\begin{aligned}
& e x\left(n, P_{4}\right)= \begin{cases}n & \text { if } n=0(\bmod 3), \\
n-1 & \text { if } n=1,2(\bmod 3) .\end{cases} \\
& e x\left(n, P_{5}\right)= \begin{cases}3 n / 2 & \text { if } n=0(\bmod 4), \\
3 n / 2-2 & \text { if } n=2(\bmod 4), \\
(3 n-3) / 2 & \text { if } n=1,3 \bmod 4 .\end{cases} \\
& e x\left(n, P_{6}\right)= \begin{cases}2 n & \text { if } n=0(\bmod 5), \\
2 n-2 & \text { if } n=1,4(\bmod 5), \\
2 n-3 & \text { if } n=2,3 \bmod 5 .\end{cases}
\end{aligned}
$$

Theorem 2.3 ( 3 , 4]) $R\left(P_{3}, P_{4}, P_{m}\right)=m+1$ for $m \geq 6$ and $R\left(P_{3}, P_{5}, P_{m}\right)=m+1$ for $m \geq 8$.

Theorem 2.4 (i) $R\left(P_{3}, P_{3}, P_{m}\right)=m$ for $m \geq 5$.
(ii) $R\left(P_{3}, P_{4}, P_{m}\right)=m+1$ for $4 \leq m \leq 5$.
(iii) $R\left(P_{3}, P_{5}, P_{m}\right)=m+1$ for $5 \leq m \leq 7$.

Proof. (i) Let $G=K_{m}$ be 3-edge colored green, red and blue such that $G$ does not contain green or red $P_{3}$. It is clear to see that $G^{b}$ is connected and $\delta\left(G^{b}\right) \geq m-3$. Thus $G^{b}$ has a Hamiltonian path(see [1]) and so a $P_{m}$.
(ii) Let $G=K_{m+1}$ be 3-edge colored green, red and blue such that $P_{3} \nsubseteq G^{g}$ and $P_{4} \nsubseteq G^{r}$. First let $m=4$. Using corollary 2.2 we may assume that $\left|E\left(G^{g}\right)\right| \leq 2$ and $\left|E\left(G^{r}\right)\right| \leq 4$. If $\left|E\left(G^{r}\right)\right|=4$, then by Theorem $2.1 G^{r} \cong K_{3} \cup K_{2}$ or $G^{r} \cong K_{1,4}$ which clearly the complement of $G^{r}$ with respect to $G$ is colored green and blue and so it contains a blue copy of $P_{4}$. Thus we may assume that $\left|E\left(G^{r}\right)\right| \leq 3$ and so $\left|E\left(G^{b}\right)\right| \geq 5$. Using corollary $2.2 G^{b}$ contains $P_{4}$. By a similar argument one can show that $R\left(P_{3}, P_{4}, P_{5}\right)=6$.
(iii) Let $G=K_{m+1}$ be 3-edge colored green, red and blue such that $P_{3} \nsubseteq G^{g}$ and $P_{5} \nsubseteq G^{r}$. First let $m \neq 5$. By a result in [11], $R\left(P_{3}, C_{4}, P_{m}\right)=m+1$ for $m \in\{6,7\}$ and so we may assume that $G$ contains a red $C_{4}$. Set $A=V\left(C_{4}\right)$ and $B=V(G) \backslash A$. Since $P_{5} \nsubseteq G^{r}$, all edges between $A$ and $B$ are colored green or blue which clearly $G[E(A, B)]$ contains a blue $P_{m}$. Now consider the case $m=5$. By a similar argument, we may assume that $G^{r}$ and $G^{b}$ don't contain $C_{4}$ as subgraph. Since $|E(G)|=15$, by Theorem 2.1 we may assume that $\left|E\left(G^{g}\right)\right|=3,\left|E\left(G^{r}\right)\right|=6$ and $\left|E\left(G^{b}\right)\right|=6$ and so the green edges form a perfect matching. But $R\left(P_{3}, P_{4}, P_{5}\right)=6$, by part (ii), and so we may assume that $G^{r}$ contains a copy of $P_{4}$, say $P=v_{1} v_{2} v_{3} v_{4}$. Set $A=V(G) \backslash V(P)=\left\{v_{5}, v_{6}\right\}$. Since $P_{5} \nsubseteq G^{r}$, all edges in $E\left(\left\{v_{1}, v_{4}\right\}, A\right)$ are colored green or blue. Also since the green edges form a perfect matching, the subgraph of $G^{g}$ induced by $E\left(\left\{v_{1}, v_{4}\right\}, A\right)$ dose not contain a perfect matching. Thus we may assume that $P^{\prime}=v_{5} v_{1} v_{6} v_{4} \subseteq G^{b}$ and $v_{4} v_{5} \in E\left(G^{g}\right)$. Now since $P_{5} \nsubseteq G^{r}$, at least one of $v_{2} v_{5}$ or $v_{3} v_{5}$, say $v_{2} v_{5}$, must be blue and so $v_{3} v_{5} P^{\prime} v_{4}$ form a blue $P_{5}$. This observation completes the proof.

Combining Theorems 2.3 and 2.4, we obtain that $R\left(P_{3}, P_{n}, P_{m}\right)=R\left(P_{n}, P_{m}\right)$ if $m \geq n$, $n \in\{3,4,5\}$ and $(n, m) \neq(3,3),(3,4)$. In the rest of this section we prove that $R\left(P_{3}, P_{n}, P_{m}\right)=$ $R\left(P_{n}, P_{m}\right)$ for $m \geq n, n \in\{6,7\}$. But before that we need some lemmas.

Lemma 2.5 Let $G$ be a graph obtained from the complete bipartite graph $K_{3,4}$ by removing an edge. If each edge of $G$ is colored red or blue, then $G^{r}$ contains $P_{3}$ or $G^{b}$ contains $P_{7}$.

Proof. Let $G=(X, Y), X=\left\{x_{1}, x_{2}, x_{3}\right\}$ and $Y=\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$. Also let $x_{1} y_{1}$ be the edge that removed from $K_{3,4}$. If $G^{r}$ does not contain $P_{3}$, then $G^{r}$ has at most three edges. Let $H$ be a spanning subgraph of $G$ with $E(H)=E\left(G^{r}\right) \cup\left\{x_{1} y_{1}\right\}$. It is clear to see that $H \subseteq P_{3} \cup 2 P_{2}$ or $H \subseteq P_{4} \cup P_{2} \cup P_{1}$ and so the complement of $H$ with respect to $K_{3,4}$ contains a copy of $P_{7}$. This observation completes the proof.

Lemma 2.6 Suppose $m \geq 7$ and the edges of $K_{m+2}$ are colored with colors green, red and blue such that $G^{b}$ contains a copy of $P_{m-1}$ as a subgraph. Then $K_{m+2}$ contains a green $P_{3}$, a red $P_{7}$ or a blue $P_{m}$.

Proof. Assume that $G=K_{m+2}$ with $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{m+2}\right\}$ and $P=v_{1} v_{2} \ldots v_{m-1}$ is the desired copy of $P_{m-1}$ in $G^{b}$. We suppose that $G^{b}$ contains no copy of $P_{m}$, then we prove that $K_{m+2}$ contains a green $P_{3}$ or a red $P_{7}$. We find two vertices $v, v^{\prime} \in P_{m-1}$ such that the bipartite graph with parties $X=\left\{v_{m}, v_{m+1}, v_{m+2}\right\}$ and $Y=\left\{v_{1}, v, v^{\prime}, v_{m-1}\right\}$ is a red-green graph with at least 11 edges and then we use Lemma 2.5, which guarantees the existence of a green $P_{3}$ or
a red $P_{7}$. Note that we may assume that in $G^{b}$, the vertices $v_{2}$ and $v_{m-2}$ don't have a common neighbor in $X$. Otherwise, since $P_{m} \nsubseteq G^{b}, v_{3}$ (also $v_{m-3}$ ) is not adjacent to any vertex of $X$ in $G^{b}$ and so $v_{3}$ and $v_{m-3}$ are the desired vertices. Thus we may assume that in $G^{b}$ one of $v_{2}$ or $v_{m-2}$, say $v_{m-2}$, has at most one neighbor in $X$. If $N_{G^{b}}\left(v_{m-2}\right) \cap X=\emptyset$, then we may assume that in $G^{b}$ each vertex $v_{i} \in V(P) \backslash\left\{v_{1}, v_{m-2}, v_{m-1}\right\}$ has at least two neighbors in $X$, otherwise set $v=v_{i}$ and $v^{\prime}=v_{m-2}$. Therefore if $N_{G^{b}}\left(v_{m-2}\right) \cap X=\emptyset$ we have $N_{G^{b}}\left(v_{2}\right) \cap N_{G^{b}}\left(v_{3}\right) \cap X \neq \emptyset$, and so $P_{m} \subseteq G^{b}$, a contradiction. Hence $\left|N_{G^{b}}\left(v_{m-2}\right) \cap X\right|=1$,

Since $\left|N_{G^{b}}\left(v_{m-2}\right) \cap X\right|=1$, so we may assume that in $G^{b}$ each vertex $v_{i} \in V(P) \backslash$ $\left\{v_{1}, v_{m-2}, v_{m-1}\right\}$ has at least one neighbor in $X$, otherwise set $v=v_{i}$ and $v^{\prime}=v_{m-2}$. Since $P_{m} \nsubseteq G^{b}$, one can easily check that $\left|N_{G^{b}}\left(v_{i}\right) \cap X\right|=1,2 \leq i \leq 5$ and w.l.g $N_{G^{b}}\left(v_{2}\right) \cap X=\left\{v_{m}\right\}$, $N_{G^{b}}\left(v_{3}\right) \cap X=\left\{v_{m+1}\right\}, N_{G^{b}}\left(v_{4}\right) \cap X=\left\{v_{m+2}\right\}$ and $N_{G^{b}}\left(v_{5}\right) \cap X=\left\{v_{m+1}\right\}$. If $m=7$ then $v_{1} v_{2} v_{3} v_{8} v_{5} v_{4} v_{9}$ is a blue $P_{7}$ in $K_{9}$, a contradiction. Now let $m \geq 8$. Since $\left|N_{G^{b}}\left(v_{m-2}\right) \cap X\right|=1$, $v_{m-2}$ must be adjacent to a vertex in $X$ by blue and in any case we have a copy of $P_{m} \subseteq G^{b}$, a contradiction. This observation completes the proof.

Lemma $2.7 R\left(P_{3}, P_{6}, P_{7}\right)=9$.
Proof. Let $G=K_{9}$ be 3-edge colored with colors green, red and blue. By a result in [15, $R\left(P_{3}, C_{6}, C_{6}\right)=9$ and so we may assume that $G^{b}$ contains a copy of $C_{6}$ as subgraph. Set $X=V\left(K_{9}\right) \backslash V\left(C_{6}\right)$. We may assume that all edges between $X$ and $C_{6}$ are colored red or green. Therefor by Lemma 2.5, $K_{9}$ must contain a green $P_{3}$ or a red $P_{6}$, which completes the proof.

Using Lemmas 2.6 and 2.7 we have the following.

Theorem 2.8 $R\left(P_{3}, P_{7}, P_{m}\right)=R\left(P_{3}, P_{6}, P_{m}\right)=m+2$ for $m \geq 7$. Moreover $R\left(P_{3}, P_{6}, P_{6}\right)=8$.
Proof. Since $R\left(P_{3}, P_{6}, P_{m}\right) \leq R\left(P_{3}, P_{7}, P_{m}\right)$ and $m+2=R\left(P_{6}, P_{m}\right) \leq R\left(P_{3}, P_{6}, P_{m}\right)$, it is sufficient to show that $R\left(P_{3}, P_{7}, P_{m}\right) \leq m+2$ for $m \geq 7$. Using Lemmas 2.6 and 2.7 we have $R\left(P_{3}, P_{7}, P_{7}\right)=9$ and again using Lemma 2.6 and induction on $m$ we obtain that $R\left(P_{3}, P_{7}, P_{m}\right) \leq$ $m+2$. On the other hand $8=R\left(P_{6}, P_{6}\right) \leq R\left(P_{3}, P_{6}, P_{6}\right)$. To complete the proof it is sufficient to show that $R\left(P_{3}, P_{6}, P_{6}\right) \leq 8$. Let $G=K_{8}$ be 3 -edge colored with colors green, red and blue. Suppose $G$ have neither a green $P_{3}$ nor a blue $P_{6}$. If $G$ has a red $P_{6}$ we are done. So suppose that $G$ does not have any red $P_{6}$. Using (iii) of Theorem 2.4 we may assume that $G$ has a red $P_{5}$ with vertices $v_{1}, v_{2}, \cdots, v_{5}$ as a subgraph. Then we may assume that $v_{1} v_{6}, v_{1} v_{7}, v_{5} v_{7}, v_{5} v_{8}$ are blue edges. If $E\left(\left\{v_{6}, v_{8}\right\},\left\{v_{2}, v_{3}, v_{4}\right\}\right)$ has a blue edge, combining this edge with the path $v_{6} v_{1} v_{7} v_{5} v_{8}$ gives a blue $P_{6}$, a contradiction. So $v_{8} v_{2}, v_{8} v_{4}, v_{6} v_{2}, v_{6} v_{4}$ are red edges and $v_{3} v_{8}$ and $v_{3} v_{6}$ are green edges and hence $G$ has a green $P_{3}$, a contradiction.
$3 \quad R\left(P_{3}, P_{n}, P_{m}\right)$ for $m \geq n \geq 8$
In this section, we compute the value of $R\left(P_{3}, P_{n}, P_{m}\right)$ for $m \geq n \geq 8$. Before that we need some lemmas.

Lemma 3.1 Suppose that $G=K_{m+\left\lfloor\frac{n}{2}\right\rfloor-1}, m \geq n \geq 8$, is 3-edge colored green, red and blue and $P=v_{1} v_{2} \cdots v_{m-1}$ is the maximum path in $G^{b}$. Let $A=V(G) \backslash V(P)$ and $H$ be the subgraph of $G^{r}$ induced by the edges in $E\left(V(P) \backslash\left\{v_{1}, v_{m-1}\right\}, A\right)$. Then either $P_{3} \subseteq G^{g}$ or $d_{H}\left(v_{i}\right) \geq 2$ for some $i, 2 \leq i \leq m-2$.

Proof. We suppose that $G^{g}$ contains no copy of $P_{3}$. Since $m \geq n \geq 8$, we obtain that $|A| \geq 4$ and so let $X=\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\} \subseteq A$. Again since $m \geq 8$ there is a $v_{j} \in V(P) \backslash\left\{v_{1}, v_{m-1}\right\}$ such that all edges in $E\left(X,\left\{v_{j}\right\}\right)$ are red and blue. If $\left|N_{G^{r}}\left(v_{j}\right) \cap X\right| \geq 2$, we have nothing to prove. Otherwise, we may assume that $Y=\left\{u_{1}, u_{2}, u_{3}\right\} \subseteq N_{G^{b}}\left(v_{j}\right) \cap X$. Since $P_{m} \nsubseteq G^{b}, G^{r}$ contains at least two edges in $E(Y,\{v\})$ for some $v \in\left\{v_{j-1}, v_{j+1}\right\}$ and so $d_{H}(v) \geq 2$.

Lemma 3.2 Suppose that $G=K_{n}$ is 3-edge colored green, red and blue, $P_{3} \nsubseteq G^{g}$ and $P$ is a maximal path in $G^{b}$ with endpoints $x$ and $y$. Then for every two vertices $z$ and $w$ of $V(G) \backslash V(P)$ either $x z, y w \in E\left(G^{r}\right)$ or $x w, y z \in E\left(G^{r}\right)$.

Proof. Since $P_{3} \nsubseteq G^{g}$ and $P$ is a maximal path in $G^{b}$, each of $z$ and $w$ is adjacent to at least one of $x$ and $y$ in $G^{r}$. With no loss of generality, suppose that $x z \in E\left(G^{r}\right)$. If $y w \in E\left(G^{r}\right)$, the proof is completed. Otherwise, $y w \in E\left(G^{g}\right)$ and so $x w, y z \in E\left(G^{r}\right)$, which completes the proof.

Lemma 3.3 $R\left(P_{3}, P_{8}, P_{8}\right)=11$

Proof. Let $G=K_{11}$ be 3 -edge colored green, red and blue such that $P_{3} \nsubseteq G^{g}$. We find monochromatic copy of $P_{8}$ in blue or red color. By Theorem [2.8, $R\left(P_{3}, P_{7}, P_{8}\right)=10$ and so we may assume that $P_{7}$ is a maximum path in $G^{r}$. Let $P=v_{1} v_{2} \ldots v_{7} \subseteq G^{r}$ and $A=$ $V(G) \backslash V(P)=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$. Using Lemma 3.1, there exists a $v_{j} \in V(P) \backslash\left\{v_{1}, v_{7}\right\}$ which is adjacent to at least two vertices of $A$, say $x_{1}, x_{2}$, in $G^{b}$. By Lemma3.2, w.l.g we may assume that $\left\{x_{1} v_{1}, x_{2} v_{7}, x_{3} v_{1}, x_{4} v_{7}\right\} \subseteq E\left(G^{b}\right)$ and so $Q_{7}=x_{3} v_{1} x_{1} v_{j} x_{2} v_{7} x_{4} \subseteq G^{b}$. Let $K=V(P) \backslash\left\{v_{1}, v_{j}, v_{7}\right\}$. Then $|K|=4$ and one can easily check that at least one of $x_{3}$ or $x_{4}$ is adjacent to a vertex of $K$, say $v_{i}$, in $G^{b}$. Therefore $Q_{7} \cup\left\{v_{i}\right\}$ is a blue $P_{8}$.

Theorem 3.4 For any $m \geq n \geq 8, R\left(P_{3}, P_{n}, P_{m}\right)=m+\left\lfloor\frac{n}{2}\right\rfloor-1$.
Proof. Let $t=m+\left\lfloor\frac{n}{2}\right\rfloor-1$ and $G=K_{t}$ be 3 -edge colored green, red and blue such that $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}$. By induction on $m+n$, we prove that $P_{n} \subseteq G^{r}$. By Lemma 3.3 theorem is true for $m=n=8$. By the induction hypothesis $R\left(P_{3}, P_{n}, P_{m-1}\right) \leq m+\left\lfloor\frac{n}{2}\right\rfloor-1$ and so there is a $P_{m-1} \subseteq G^{b}$. Let $P=P_{m-1}=v_{1} v_{2} \ldots v_{m-1}, A=V(G) \backslash V(P)$ and $H$ be the subgraph of $G^{r}$ induced by the edges in $E\left(V(P) \backslash\left\{v_{1}, v_{m-1}\right\}, A\right)$. Suppose $Q$ is a maximal path of $H$ with end points $u_{1}$ and $u_{2}$ in $A$, the existence of such a path is guaranteed by Lemma 3.1. Let $K=\left(V(P) \backslash\left\{v_{1}, v_{m-1}\right\}\right) \backslash V(Q)$. If all vertices in $A$ are covered by $Q$, then by Lemma 3.2, we may assume that $u_{1} v_{1}, u_{2} v_{m-1} \in E\left(G^{r}\right)$ and so $R=v_{1} u_{1} Q u_{2} v_{m-1}$ is a red path on $2\left\lfloor\frac{n}{2}\right\rfloor+1$ vertices. Thus we may assume that $A \backslash V(Q) \neq \emptyset$.

Case 1. $|A \backslash V(Q)|=1$.

Let $A \backslash V(Q)=\{x\}$. By Lemma 3.2, we may assume that $v_{1} u_{1}, v_{m-1} u_{2} \in E\left(G^{r}\right)$. In the other hand, since $P$ is maximal and $P_{3} \nsubseteq G^{g}, x$ is adjacent to at least one of $v_{1}$ and $v_{m-1}$ in $G^{r}$, say $v_{1}$. Thus $R=x v_{1} u_{1} Q u_{2} v_{m-1} \subseteq G^{r}$ form a path on $2\left\lfloor\frac{n}{2}\right\rfloor$ vertices. If $n$ is even, there is nothing to prove and so we may assume that $n$ is odd. Note that $|K|=m-3-\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right) \geq\left\lceil\frac{m}{2}\right\rceil-1>$ $\left\lfloor\frac{n}{2}\right\rfloor-1$ and so by the Pigeonhole principle there exist two consecutive vertices $v_{i}, v_{i+1}$ in $K$. If $x v_{i} \in E\left(G^{r}\right)\left(\right.$ or $x v_{i+1} \in E\left(G^{r}\right)$ ), then $\left\{v_{i}\right\} \cup V(R)$ (or $\left.\left\{v_{i+1}\right\} \cup V(R)\right)$ form a red $P_{n}$. Otherwise, since both $x v_{i}$ and $x v_{i+1}$ are not in $E\left(G^{g}\right)$ or $E\left(G^{b}\right)$, w.l.g we may assume that $x v_{i} \in E\left(G^{b}\right)$ and $x v_{i+1} \in E\left(G^{g}\right)$ which implies that $x v_{m-1} \in E\left(G^{r}\right)$. Therefore $V(R) \cup\{x\}$ form a copy of $C_{n-1}$ in $G^{r}$. It is clear to see that at least one of $v_{i}$ or $v_{i+1}$ is adjacent to one of $u_{1}$ or $u_{2}$ by a red edge Thus, we can find a red $P_{n}$.

Case 2. $|A \backslash V(Q)|=2$.
Let $A \backslash V(Q)=\{x, y\}$. Using Lemma 3.2 we may assume that $R=x v_{1} u_{1} Q u_{2} v_{m-1} y$ is a red path on $2\left\lfloor\frac{n}{2}\right\rfloor-1$ vertices. (Note that in this case, $\left.|K|=m-3-\left(\left\lfloor\frac{n}{2}\right\rfloor-3\right) \geq\left\lceil\frac{m}{2}\right\rceil \geq\left\lceil\frac{n}{2}\right\rceil\right)$. We consider the following subcases.

Subcase 1. $n$ is even:
By the Pigeonhole principle there exists a pair of vertices $\left(v_{i}, v_{i+1}\right)$ in $K$. If one of $x$ or $y$ is adjacent to one of $v_{i}$ or $v_{i+1}$, say $v_{i}$, in $G^{r}$, then $v_{i} x R y$ form a red $P_{n}$. Otherwise, green and also blue edges in $E\left(\left\{v_{i}, v_{i+1}\right\},\{x, y\}\right)$ form a matching and so $y v_{1}$ is red and w.l.g we may assume that $u_{1} v_{i}$ is red. Thus $R^{\prime}=v_{i} u_{1} Q u_{2} v_{m-1} y v_{1} x \subseteq G^{r}$ is a path on $n$ vertices.

Subcase 2. $n$ is odd:
By the Pigeonhole principle there exist two disjoint pairs of vertices $\left(v_{j}, v_{j+1}\right)$ and $\left(v_{k}, v_{k+1}\right)$ in $K$. It is easy to see that each of $x$ and $y$ is adjacent to a vertex in $B=\left\{v_{j}, v_{j+1}, v_{k}, v_{k+1}\right\}$ by red edge. If the mentioned neighbors of $x$ and $y$ are distinct we have a red $P_{n}$, otherwise let $v_{j} \in B$ be the only neighbor of $x$ and $y$. Therefore, $\left\{v_{j}\right\} \cup V(R)$ form a red $C_{n-1}$. It is easy to see that there is an edge in $G^{r}$ between $B \backslash\left\{v_{j}\right\}$ and $\left\{u_{1}, u_{2}\right\}$ and so a red $P_{n}$ can be found.

Case 3. $|A \backslash V(Q)| \geq 3$.
Let $x, y, z \in A \backslash V(Q)$.
Claim 3.5 Let $H$ be the subgraph of $G^{r}$ induced by the edges in $E(A \backslash V(Q), K)$. There is a vertex $v \in H \cap K$ such that $d_{H}(v) \geq 2$.

Proof. There are at least $\left\lceil\frac{n}{2}\right\rceil+1$ vertices in $K$. By the Pigeonhole principle, there are two disjoint pairs of vertices $\left(v_{i}, v_{i+1}\right)$ and $\left(v_{j}, v_{j+1}\right)$ in $K$. We prove the claim by considering the number of red edges from $\left\{v_{i}, v_{i+1}\right\}$ to $\{x, y, z\}$. If there are more than two such edges, then the claim is proved. Thus we may assume that there are at most two such edges. Since $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}$, there is at least one such an edge. Therefore, it is sufficient to consider the following cases.
i) W.l.g, $G^{r}$ contains two edges in $E\left(\left\{v_{i}, v_{i+1}\right\},\{x, y, z\}\right)$
ii) W.l.g, $G^{r}$ contains exactly one edge in $E\left(\left\{v_{i}, v_{i+1}\right\},\{x, y, z\}\right)$.

If $(i)$ occurs, we may assume that there are exactly one edge from each of $v_{i}$ and $v_{i+1}$ to $\{x, y, z\}$ in $G^{r}$, otherwise we have nothing to prove. Suppose there is no red edge in $E\left(\left\{v_{i}, v_{i+1}\right\},\{z\}\right)$. Since $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}, G^{r}$ contains at least one edge in $E\left(\left\{z, u_{1}, u_{2}\right\},\left\{v_{i}, v_{i+1}\right\}\right)$. Whereas $Q$ is maximal, this edge has to be in $E\left(\left\{v_{i}, v_{i+1}\right\},\{z\}\right)$, a contradiction.

If (ii) occurs, we may assume that $x v_{i} \in E\left(G^{r}\right)$. Since $G^{r}$ contains no edge in $E\left(\left\{v_{i}, v_{i+1}\right\},\{y, z\}\right)$, green and also blue edges in $E\left(\left\{v_{i}, v_{i+1}\right\},\{y, z\}\right)$ form a matching. Thus, clearly there are two red edges in $E\left(\left\{v_{j}, v_{j+1}\right\},\{y, z\}\right)$. The reminder of the proof is the same to the case $(i)$.

Now, let $Q^{\prime}$ be a maximal path in the subgraph of $G^{r}$ induced by the edges in $E(A \backslash V(Q), K)$ with endpoints $w_{1}$ and $w_{2}$ in $A \backslash V(Q)$ and $K^{\prime}=K \backslash V\left(Q^{\prime}\right)$.

Case 1. $\left|A \backslash\left(V(Q) \cup V\left(Q^{\prime}\right)\right)\right|=0$.
Using Lemma 3.2, we may assume that $G^{r}$ contains a cycle $C=w_{1} Q^{\prime} w_{2} v_{m-1} u_{2} Q u_{1} v_{1} w_{1}$ on $2\left\lfloor\frac{n}{2}\right\rfloor$ vertices. If $n$ is even, we are done. Otherwise, since $\left|K^{\prime}\right| \geq\left\lceil\frac{n}{2}\right\rceil-1$, there is one pair of vertices $\left(v_{i}, v_{i+1}\right)$ in $K^{\prime}$. Since $G^{r}$ contains at least one edge in $E\left(\left\{u_{1}, u_{2}, w_{1}, w_{2}\right\},\left\{v_{i}, v_{i+1}\right\}\right)$, we may suppose that $v_{i} u_{1} \in E\left(G^{r}\right)$ and so $R^{\prime}=v_{i} u_{1} Q u_{2} v_{m-1} w_{2} Q^{\prime} w_{1} v_{1}$ is a red $P_{n}$.

Case 2. $\left|A \backslash\left(V(Q) \cup V\left(Q^{\prime}\right)\right)\right|=1$.
Let $A \backslash\left(V(Q) \cup V\left(Q^{\prime}\right)\right)=\{x\}$. Using Lemma 3.2 we may assume that $u_{1} v_{1}, u_{2} v_{m-1}, w_{1} v_{m-1}$ and $w_{2} v_{1}$ are red edges. Since $P_{3} \nsubseteq G^{g}, G^{r}$ contains at least one edge in $E\left(\left\{v_{1}, v_{m-1}\right\},\{x\}\right)$, say $x v_{1}$. Thus $R=x v_{1} u_{1} Q u_{2} v_{m-1} w_{1} Q^{\prime} w_{2}$ is a red $P_{2\left\lfloor\frac{n}{2}\right\rfloor-1}$. We consider the following subcases.

Subcase 1. $n$ is even:
Since $\left|K^{\prime}\right| \geq\left\lceil\frac{n}{2}\right\rceil$, there is at least one pair of vertices $\left(v_{i}, v_{i+1}\right)$ in $K^{\prime}$. If $x v_{i}$ (or $x v_{i+1}$ ) is red, then $v_{i} x R w_{2}$ (or $v_{i+1} x R w_{2}$ ) form a red $P_{n}$. Otherwise, we may assume that $x v_{i} \in E\left(G^{b}\right)$ and $x v_{i+1} \in E\left(G^{g}\right)$. Therefore $x v_{m-1} \in E\left(G^{r}\right)$ and $R^{\prime}=u_{1} Q u_{2} v_{m-1} x v_{1} w_{2} Q^{\prime} w_{1}$ is a red $P_{n-1}$. Whereas $G^{r}$ contains at least one edge of $E\left(\left\{v_{i}, v_{i+1}\right\},\left\{u_{1}, w_{1}\right\}\right)$, we can extend $R^{\prime}$ to a red $P_{n}$.

Subcase 2. $n$ is odd:
Since $\left|K^{\prime}\right| \geq\left\lceil\frac{n}{2}\right\rceil=\frac{n+1}{2}$, there are at least two disjoint pairs of vertices $\left(v_{j}, v_{j+1}\right)$ and $\left(v_{k}, v_{k+1}\right)$ in $K^{\prime}$. Clearly, each of $x$ and $w_{2}$ in $G^{r}$ has at least one neighbor in $B=\left\{v_{j}, v_{j+1}, v_{k}, v_{k+1}\right\}$, say $s_{1}$ and $s_{2}$ respectively. If $s_{1} \neq s_{2}, s_{1} x R w_{2} s_{2}$ is a red $P_{n}$, else $s_{1} x R w_{2} s_{1}$ is a red $C_{n-1}$. One can easily check that $G^{r}$ contains at least one edge of $E\left(B \backslash\left\{s_{1}\right\},\left\{u_{1}, u_{2}, w_{1}\right\}\right)$, and so adding this edge to $C_{n-1}$ yields a $P_{n} \subseteq G^{r}$.

Case 3. $\left|A \backslash\left(V(Q) \cup V\left(Q^{\prime}\right)\right)\right| \geq 2$.
Let $x, y \in A \backslash\left(V(Q) \cup V\left(Q^{\prime}\right)\right)$. We show that this case is impossible. Since $\left|K^{\prime}\right| \geq\left\lceil\frac{n}{2}\right\rceil+1$ and at most $\left\lfloor\frac{n}{2}\right\rfloor-4$ vertices of $V(P) \backslash\left\{v_{1}, v_{m-1}\right\}$ are covered by $Q$ and $Q^{\prime}$, by the Pigeonhole principle we have one of the following cases.
i) $K^{\prime}$ contains four disjoint pairs of vertices $\left(v_{k}, v_{k+1}\right),\left(v_{i}, v_{i+1}\right),\left(v_{j}, v_{j+1}\right)$ and $\left(v_{l}, v_{l+1}\right)$.
ii) $K^{\prime}$ contains three consecutive vertices $v_{k}, v_{k+1}, v_{k+2}$.

If ( $i$ ) occurs, since $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}$ there is a red edge between $x$ and any two pairs of vertices and so w.l.g we may assume that $x v_{k+1}, x v_{l+1}, x v_{i+1} \in E\left(G^{r}\right)$. Since $Q$ and $Q^{\prime}$ are maximal, $G^{r}$ contains no edge in $E\left(\left\{u_{1}, u_{2}, w_{1}, w_{2}\right\},\left\{v_{k+1}, v_{l+1}, v_{i+1}\right\}\right)$. If there is a red edge in $E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)$ (resp. in $E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)$ ) for some $t \in\{i, k, l\}$, then the maximality of $Q$ and $Q^{\prime}$ implies that green and also blue edges in $E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)$ (resp. in $\left.E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)\right)$ form perfect matchings on four vertices. Now since there is at least one red edge in $E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)$ (resp. in $E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{t}, v_{t+1}\right\}\right)$ ) for some $t \in\{i, k, l\}$, w.l.g we may assume that green and also blue edges in both $E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{k}, v_{k+1}\right\}\right)$ and $E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{l}, v_{l+1}\right\}\right)$ form matchings. Therefore $\left\{u_{1} v_{i+1}, u_{2} v_{i+1}, w_{1} v_{i+1}, w_{2} v_{i+1}\right\} \subseteq E\left(G^{b}\right)$ and consequently $\left\{u_{1} v_{i}, u_{2} v_{i}, w_{1} v_{i}, w_{2} v_{i}\right\} \subseteq E\left(G^{r}\right)$ which is a contradiction.

If (ii) occurs, at least five vertices of $\left\{u_{1}, u_{2}, w_{1}, w_{2}, x, y\right\}$ are adjacent to some vertices of $\left\{v_{k}, v_{k+1}, v_{k+2}\right\}$ in $G^{r}$, since $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}$. Let $B$ be the set of the vertices in $\left\{u_{1}, u_{2}, w_{1}, w_{2}, x, y\right\}$ that are adjacent to a vertex in $\left\{v_{k}, v_{k+1}, v_{k+2}\right\}$ by a red edge. Since $P_{3} \nsubseteq G^{g}$ and $P_{m} \nsubseteq G^{b}$ then every vertex of $B$ has exactly one red neighbor in $\left\{v_{k}, v_{k+1}, v_{k+2}\right\}$. Now, we have the following subcases.

Subcase 1. $\{x, y\} \subseteq B$ :
By the maximality of $Q$ and $Q^{\prime}$, we may suppose that the edges $x v_{t}, y v_{t}, w_{1} v_{t^{\prime}}, w_{2} v_{t^{\prime}}$ are red for some $t, t^{\prime} \in\{k, k+1, k+2\}, t<t^{\prime}$ and $u_{1} v_{r} \in E\left(G^{r}\right)$ where $r \neq t, t^{\prime}$. If $t, t^{\prime} \in\{k, k+1\}$ (resp. $\left.t, t^{\prime} \in\{k+1, k+2\}\right)$ then green and also blue edges in $E\left(\{x, y\},\left\{v_{r}, v_{t^{\prime}}\right\}\right)$ (resp. $\left.E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{r}, v_{t}\right\}\right)\right)$ form matchings and so there is a red edge in $E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{t}, v_{t^{\prime}}\right\}\right)$ (resp. $\left.E\left(\left\{u_{1}, u_{2}\right\},\left\{v_{t}, v_{t^{\prime}}\right\}\right)\right)$, and this contradicts the maximality of $Q$ and $Q^{\prime}$. Finally if $t, t^{\prime} \in\{k, k+2\}$ then green and also blue edges in $E\left(\left\{w_{1}, w_{2}\right\},\left\{v_{r}, v_{t}\right\}\right)$ form matchings and so there is a red edge in $E\left(\{x, y\},\left\{v_{r}, v_{t^{\prime}}\right\}\right)$ and again this contradicts the maximality of $Q$ and $Q^{\prime}$.

Subcase 2. $\{x, y\} \cap B=\{x\}$ :
By a similar argument as in subcase 1, we have a contradiction which completes the proof of the theorem.

Proof of Theorem 1.1. It is clear that $5 \leq R\left(P_{3}, P_{3}, P_{3}\right) \leq R\left(P_{3}, P_{3}, P_{4}\right)$. On the other hand by corollary [2.2, $R\left(P_{3}, P_{3}, P_{4}\right) \leq 5$. Then $R\left(P_{3}, P_{3}, P_{3}\right)=R\left(P_{3}, P_{3}, P_{4}\right)=5$. Combining Theorems 2.3, 2.4, 2.8 and 3.4 give a proof for Theorem 1.1,

Corollary 3.6 $R\left(P_{3}, n K_{2}, m K_{2}\right)=2 m+n-1$ for every $m \geq n \geq 3$.
Proof. To see $2 m+n-1 \leq R\left(P_{3}, n K_{2}, m K_{2}\right)$, let $H=K_{n-1}+\bar{K}_{2 m-1}$ and $\bar{H}$ be the complement of $H$ with respect to $K_{2 m+n-2}$. Clearly coloring $H$ by red and $\bar{H}$ by blue yields a 2-edge coloring of $K_{2 m+n-2}$ such that $n K_{2} \nsubseteq G^{r}$ and $m K_{2} \nsubseteq G^{b}$. This means that $2 m+n-1 \leq R\left(P_{3}, n K_{2}, m K_{2}\right)$. Now we prove the upper bound. It is easy to see that $R\left(P_{3}, n K_{2}, m K_{2}\right) \leq R\left(P_{3}, P_{2 n}, P_{2 m}\right)$ and by Theorem [1.1, $R\left(P_{3}, P_{2 n}, P_{2 m}\right)=2 m+n-1$. This observation completes the proof.

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