Characterizing forbidden pairs for hamiltonian squares

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Abstract. The square of a graph is obtained by adding additional edges joining all pair of vertices of distance two in the original graph. Particularly, if C is a hamiltonian cycle of a graph G, then the square of C is called a hamiltonian square of G. In this paper, we characterize all possible forbidden pairs, which implies the containment of a hamiltonian square, in a 4-connected graph. The connectivity condition is necessary as, except K_3 and K_4, the square of a cycle is always 4-connected.

Keywords. Hamiltonian square; Forbidden pair

1 Introduction

In this paper, we only consider simple and finite graphs. Let G and H be two graphs. We use $G \sqcup H$ to denote the vertex-disjoint union of G and H if G and H are vertex disjoint, use $G \cup H$ to denote the union of G and H, and use $G + H$ to denote the join of G and H, which is the graph on $V(G) \cup V(H)$ with edges including all edges of G and H, and all edges between $V(G)$ and $V(H)$. The notation $\overline{G}$ denotes the complement of G; that is, the graph with vertex set $V(G)$ and edges between all non-adjacent pairs of vertices in G. The *square* of a graph is obtained by adding additional edges joining all pair of vertices of distance two in the original graph. Particularly, if C is a hamiltonian cycle of a graph G, then the square of C is called a *hamiltonian square* of G. If G contains a hamiltonian square, we then say G has an H^2. The earliest problem on hamiltonian square can be traced back to a conjecture proposed by Pósa [4]. The conjecture states that *any n-vertex graph with minimum degree at least $\frac{2n}{3}$ contains a hamiltonian square.* The complete tripartite graph $K_{t,t,t-1}$ has minimum degree $2(3t-1)/3 - 1/3$, but has no H^2. So, if true, the conjecture is best possible. In 1973, Seymour [14] made a more general conjecture, which says that *any n-vertex graph with minimum degree at least $\frac{kn}{k+1}$ contains a kth power of a hamiltonian cycle.* Here, the kth power of a graph is obtained by joining every pair of vertices of distance at most k in the original graph. Pósa's conjecture is almost completely solved. In 1994, Fan and Häggkvist [5] showed Pósa's conjecture for $\delta(G) \geq 5n/7$. Fan and Kierstead [6], in 1996, proved that for *any $\varepsilon > 0$, there is a number m, dependent only on ε, such that if $\delta(G) \geq (2/3 + \varepsilon)n + m$, then G contains the square of a Hamiltonian path between every pair of edges.* This implies that G then also contains the square of a hamiltonian cycle. The same authors in 1996 [7], showed that if $\delta(G) \geq (2n-1)/3$, then G contains the square of a hamiltonian path. For graphs with large orders, Pósa's conjecture was solved by Komlós, Sárközy, and Szemerédi [12] in 1996 using the Regularity Lemma and the Blow-up Lemma. Using the absorbing method in avoiding using the Regularity Lemma, Levitt, Sárközy, and Szemerédi [13] in 2010 improved the bound on the orders. In 2011, Châu, DeBiasio, and Kierstead [2] verified Pósa's conjecture for $n \geq 200,000,000$. The work, in investigating Pósa's conjecture, was trying to find an H^2 in graphs with high minimum degrees. We may ask, what about finding an H^2 in other classes of graphs? One such possible class is the class of graphs forbidding some given small graphs.

Given a family $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$ of graphs, we say that a graph G is *$\mathcal{F}$-free* if G contains no induced subgraph isomorphic to any of $F_i, i = 1, 2, \cdots, k$. Particularly, when $\mathcal{F} = \{F\}$, we simply say that G is *F-free*. If G is $\mathcal{F}$-free, then the graphs in $\mathcal{F}$ are called *forbidden subgraphs*. The use of forbidden subgraphs to obtain classes of graphs possessing special properties has long been a common graphical technique. A pair $\{R, S\}$ of connected graphs is called a *hamiltonian forbidden pair* if every 2-connected $\{R, S\}$-free graph is hamiltonian. The characterizations for hamiltonian forbidden pairs were completely done (for example, see [1], [3], and [8]). Research has also been done on characterizing the forbidden pairs for stronger hamiltonicity properties [8], such as panconectivity (a graph G of order n is said to be panconnected if any two vertices of G, say x and y, are joined by paths of all possible lengths l from $dist(x, y)$ to $n-1$), pancyclicity (an n-vertex graph is pancyclic if it contains cycles of length l, for each $3 \leq l \leq n$). In this paper, we define forbidden pairs for hamiltonian squares (H^2). A pair of connected graphs $\{R, S\}$ is called an *H^2 forbidden pair* if every 4-connected $\{R, S\}$-free graph has an H^2. Further more, we give a full characterization for all the possible H^2 forbidden pairs.

Theorem 1.1. *A pair $\{R, S\}$ of connected graphs with $R, S \neq P_3$ is an H^2 forbidden pair if and only if $R = K_{1,3}$ and $S = Z_1$, where Z_1, as depicted in Figure 1,* *is obtained from $K_{1,3}$ be adding one edge between two non-adjacent vertices.*

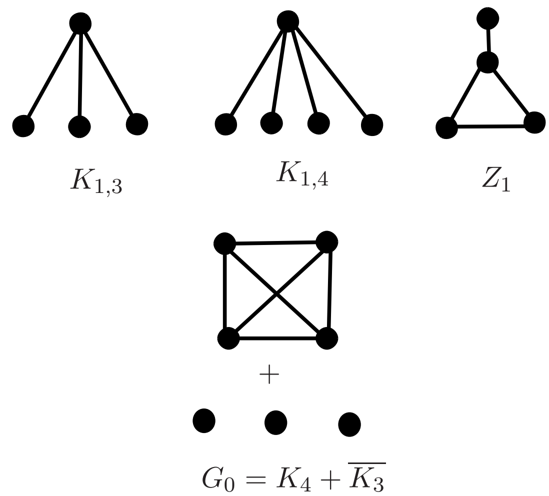

Figure 1: Small subgraphs

To force $R = K_{1,3}$ and $S = Z_1$ in Theorem 1.1, a 4-connected 7-vertex graph with no H^2 is used in the proof. Considering graphs with larger order, we prove a stronger result.

Theorem 1.2. *A pair $\{R, S\}$ of connected graphs with $R, S \neq P_3$ has the property that every 4-connected $\{R, S\}$-free graph with at least 9 vertices has an H^2 if and only if $R \in \{K_{1,3}, K_{1,4}\}$ and $S = Z_1$.*

In the study of forbidden pairs for hamiltonian or related properties, people usually consider pairs $\{K_{1,3}, P_i\}$ for $i \geq 4$. Except 4 classes of graphs, we show that all other 4-connected $\{K_{1,3}, P_4\}$-free graphs have an H^2, as given in the theorem below.

Theorem 1.3. *Every 4-connected $\{K_{1,3}, P_4\}$-free graph G has an H^2 unless G is isomorphic to a graph in one of the following families.*

(i) $(K_1 \sqcup K_3) + (K_m \sqcup K_q)$ *with* $m + q \geq 4$;

(ii) $(K_2 \sqcup K_2) + (K_1 \sqcup K_m)$ *with* $m \geq 3$;

(iii) $(K_2 \sqcup K_3) + (K_1 \sqcup K_m)$ *with* $m \geq 3$;

(iv) $(K_3 \sqcup K_3) + (K_1 \sqcup K_m)$ *with* $m \geq 3$.

It is easy to see that the square of a cycle is pancyclic. This is true for any graphs containing an H^2. Hence, partially, we give an answer to a question asked by Gould at the 2010 SIAM Discrete Math meeting in Austin, TX.

Problem 1. *Characterize the pairs of forbidden subgraphs that imply a 4-connected graph is pancyclic.*

It is worth mentioning that all the known forbidden pairs on Problem 1 include the claw: $K_{1,3}$ (see [10], [9] and [11]). Hence Theorem 1.2 gives a new forbidden pair for pancyclicity.

2 Properties of Some Non-hamiltonian Square Graphs

In this section, we examine some properties of the graphs depicted in Figure 2. These graphs will be used in the following section to characterize the H^2 forbidden pairs. The formal definitions of these graphs are given below.

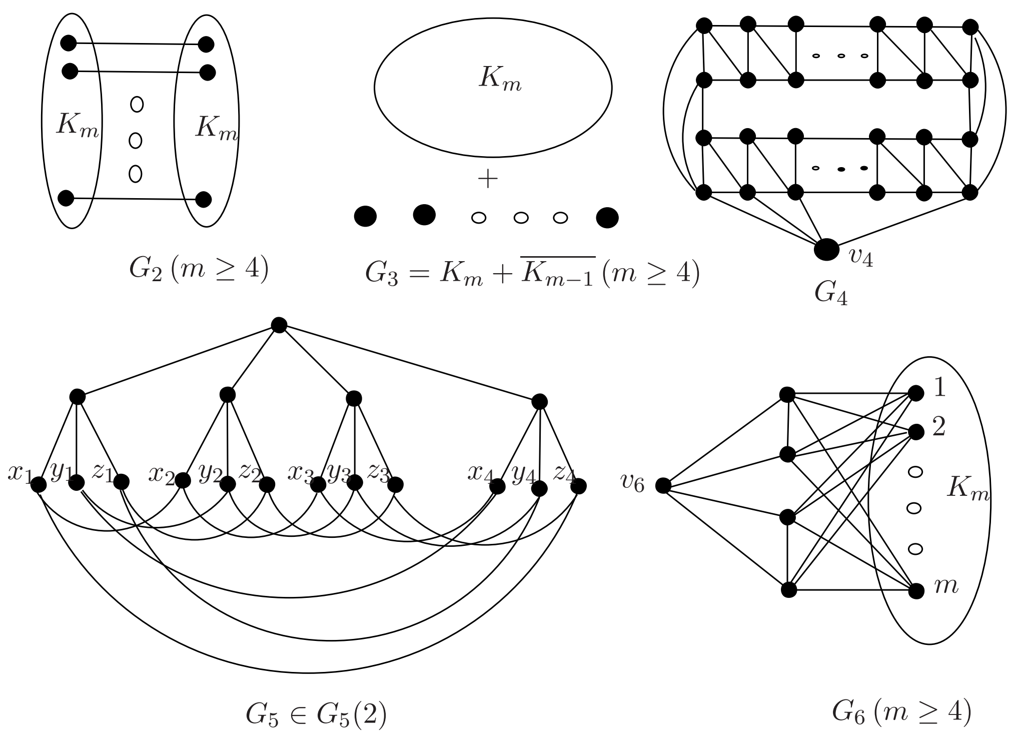

Figure 2: 4-connected no H^2 graphs

G_1: $K_{m,m}$, a complete bipartite graph with m vertices in each bipartite sets, where $m \geq 4$.

G_2: $K_m \sqcup K_m \cup M$, a graph obtained from two vertex-disjoint copies of K_m by adding a perfect matching M between them, where $m \geq 4$.

G_3: $K_m + \overline{K_{m-1}}$, the join of K_m and $\overline{K_{m-1}}$, where $m \geq 4$.

G_4: The graph obtained from the square of a cycle, denoted as C^2, by joining a new vertex v_4 to four vertices on C^2 such that the four vertices induces $P_3 \sqcup K_1$ in the C^2.

G_5: Let T_t be a rooted tree of depth t (the length of a longest path from the root to a leaf is t) such that all the leaves are at the same depth and all non-leaves have degree 4 (known as a prefect 4-ary tree). Then $G_5(t)$ ($t \geq 2$) is the graph obtained from T_t by connecting the leaves into a cycle in a way such that the girth of the finally resulted graph is greater than 4. The graph G_5 from the family $G_5(2)$ is depicted in Figure 2. G_5 is obtained as follows: embed a copy of T_2 on the plane, and name the leaves from the left to right, consecutively, as $x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_4, y_4, z_4$; then a cycle $C = x_1x_2x_3x_4y_1 \cdots y_4z_1 \cdots z_4x_1$ is obtained by joining the corresponding edges. The construction can be easily generalized to $G_5(t)$ for $t \geq 3$. (In $G_5(2)$, a cycle using the root vertex contains three non-leaves and at least two leaves; and a cycle not using the root vertex uses at least two non-leaves and 4 leaves. In any case, it indicates that $G_5(2)$ has girth at least 5. Similarly, $G_5(t)$ has girth at least 5.)

G_6: $(K_2 \sqcup K_2) + (K_m \sqcup K_1)$, where $m \geq 4$. Denote the isolated vertex in $K_m \sqcup K_1$ by v_6.

It is not hard to check that all those graphs are 4-connected. Furthermore, we have the following fact.

Lemma 2.1. *None of the graphs in Figire 2 has an* H^2.

Proof. Notice that in an H^2, the neighborhood of any vertex induces a P_4. If G_2 has an H^2, then it must contain one of the edges connecting the two copies of K_m. Let xy be a such edge. Then the neighbors of x on the H^2 consists of y and another three vertices from the copy of K_m containing x. However, those four vertices do not induce a copy of P_4, showing a contradiction. Similarly, neither of the set of neighborhoods of v_4 in G_4 or of v_6 in G_6 induces P_4. Thus, neither G_4 nor G_6 has an H^2. As $G_3 = K_m + \overline{K_{m-1}}$, any hamiltonian cycle of G_3 contains a pair of vertices from $V(\overline{K_{m-1}})$ such that they have distance 2 on the hamiltonian cycle. This in turn implies that G_3 has no H^2. As an H^2 contains triangles, the triangle-free graph $G_5(t)$ has no H^2. ∎

As the graph G_2 will be used more frequently later on, we discuss its properties in more detail here.

Lemma 2.2. *Let $S \notin \{K_3, P_3\}$ be a connected $\{P_4, C_4, K_4\}$-free graph. If G_2 contains S as an induced subgraph, then S is Z_1.*

Proof. Since $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$, $S \notin \{K_1, K_2\}$. Thus $|V(S)| \geq 3$. Since $S \notin \{K_3, P_3\}$ and any connected 3-vertex subgraph of G_2 is either K_3 or P_3, we conclude that $|V(S)| \geq 4$. Furthermore, as S is K_4-free, it contains at most 3 vertices from one of the copies of K_m. Since S is connected and $\{P_4, C_4\}$-free, if it contains at least two vertices from one copy of K_m, then it contains at most one vertex from the other copy of K_m. Hence S contains exactly three vertices from one copy of K_m, and exactly one vertex from the other. The connected graph induced on such four vertices can only be isomorphic to Z_1. ∎

3 Proofs of the Main Results

In this section, we prove Theorem 1.1, Theorem 1.2, and Theorem 1.3. We first characterize the single forbidden subgraph for 4-connected graphs containing an H^2. As any P_3-free graph is complete, we observe that any 4-connected P_3-free graph has an H^2. Conversely, we have the following result.

Proposition 3.1. *A connected graph F has the property that every 4-connected F-free graph has an H^2 if and only if $F = P_3$.*

Proof. Since $G_1 = K_{m,m}$ has no H^2, G_1 contains F as an induced subgraph. Hence $F = K_{1,r}$, where $r \geq 2$ or F contains an induced C_4. As the graph G_4 in Figure 2 has no H^2 and is C_4-free, we see that $F = K_{1,r}$. The only induced star contained in all the graphs of family G_2 is $K_{1,2}$; that is, an induced copy of P_3. Hence $F = P_3$. ∎

We study the structure of a connected Z_1-free graph in the following theorem, which will help us in knowing the structure of a $\{K_{1,r}, Z_1\}$-free graph ($r \geq 3$).

Lemma 3.1. *Let G be a connected Z_1-free graph. If there exists a vertex $v \in V(G)$ such that $d(v) \geq 3$ and v is contained in a triangle, then G is isomorphic to a complete multipartite graph $K_{t_1,t_2,\cdots,t_k}$.*

Proof. We use induction on $n = |V(G)|$. When $n = 4$, G is either K_4 or the graph obtained from K_4 by removing one edge, so the result holds. Suppose that $n \geq 5$ and that Lemma 3.1 holds for graphs with less than n vertices. Let $v \in V(G)$ be a vertex such that $d(v) \geq 3$ and v is contained in a triangle. Let $N[v] := N(v) \cup \{v\}$ and $\overline{N}[v] = V(G) - N[v]$. Notice that $\overline{N}[v]$ may be empty. As G is Z_1-free, we know $G[N(v)]$ is $(K_2 \sqcup K_1)$-free. Together with the fact that $G[N(v)]$ contains an edge, we then know $G[N(v)]$ is connected. Before examining the structure of $G[N(v)]$ further, we claim the following.

Claim 3.1. *If $\overline{N}[v] \neq \emptyset$, then for every $w \in \overline{N}[v]$, $N(w) = N(v)$ holds.*

Proof. Let $w \in \overline{N}[v]$. We first claim that if $N(w) \cap N(v) \neq \emptyset$, then $N(v) \subseteq N(w)$. Suppose not, then there exists $v' \in N(v)$ such that $wv' \notin E(G)$. We choose a such v' such that w is adjacent to a neighbor of v', say u', in $N(v)$. However, the graph induced on $\{v, v', u', w\}$ is isomorphic to Z_1, showing a contradiction. Hence $N(v) \subseteq N(w)$. The claim is proved.

We then claim that if $N(w) \cap N(v) \neq \emptyset$, then $N(w) \subseteq N(v)$. Otherwise, assume that w is adjacent to a vertex $w' \in \overline{N}[v]$. If w' is adjacent to a vertex in $N(v)$, then we have $N(v) \subseteq N(w) \cap N(w')$ by the earlier assertion. Let $v' \in N(v) \subseteq N(w) \cap N(w')$. Then $\{v, v', w, w'\}$ induces a Z_1. Hence we assume w' is not adjacent to any vertex in $N(v)$. Let $v', u' \in N(v) \subseteq N(w)$. Then $\{u', v', w, w'\}$ induces a Z_1. Thus w is not adjacent to any vertex in $\overline{N}[v]$.

As G is connected, Claim 3.1 is then implied by the above two assertions. □

We now proceed with the proof according to several cases depending on the structure of $G[N(v)]$. Let $|V(G) - N(v)| = t'$ and $G' = G[N(v)]$. Recall that G' is connected and is $(K_2 \sqcup K_1)$-free.

Case 1. G' has a vertex with degree at least 3 in G' and the vertex is contained in a triangle in G'.

By the induction hypothesis, $G' \cong K_{t_1,t_2,\cdots,t_{k-1}}$. Then we have $G \cong K_{t_1,t_2,\cdots,t_{k-1},t'}$.

So we suppose that the condition in Case 1 is not satisfied by G'. Let $u \in V(G')$ be a vertex of maximum degree in G'.

Case 2. $d_{G'}(u) \leq 2$.

Then G' is the union of vertex disjoint paths and cycles. As G' is connected and is $(K_2 \sqcup K_1)$-free, we know G' is isomorphic to one of the graphs K_3, P_3, or C_4. In any case, G is isomorphic to a complete multipartite graph.

Case 3. $d_{G'}(u) \geq 3$.

As u is not on a triangle in G', $N_{G'}(u)$ is an independent set in G'. If $N_{G'}[u] = V(G') = N(v)$, then it is already seen that G is isomorphic to a complete multiple graph with the size of each parts as $t', 1$, and $d_{G'}(u)$, respectively. Hence, we assume $N(v) - N_{G'}[u] \neq \emptyset$. As G' is connected and is $(K_2 \sqcup K_1)$-free, every vertex in $N(v) - N_{G'}[u]$ is adjacent to every vertex in $N_{G'}(u)$. Again, by the fact that G' is $(K_2 \sqcup K_1)$-free, we know there is no edge with the two ends in $N(v) - N_{G'}[u]$. Hence, $N(v) - N_{G'}[u]$ is an independent set. Let $t_1 = d_{G'}(u)$ and $t_2 = |N(v) - N_{G'}(u)|$. We see that $G \cong K_{t_1,t_2,t'}$.

The proof is complete. ∎

Additionally, if G is a $\{Z_1, K_{1,r}\}$-free graph with a vertex of degree at least r $(r \geq 3)$, then G contains a vertex which is contained in a triangle and is of degree at least 3. Thus by applying Lemma 3.1 and by the fact that G is $K_{1,r}$-free, we have the following result.

Corollary 3.1. *Let G be a connected $\{Z_1, K_{1,r}\}$-free graph with a vertex of degree at least r. Then G is isomorphic to a complete multipartite graph $K_{t_1,t_2,\cdots,t_k}$ such that each $1 \leq t_i \leq r-1$.*

The case of $r = 3$ in the above Corollary has been mentioned in other research papers, for example, in [8]. By Corollary 3.1, we have the following result.

Corollary 3.2. *A connected $\{K_{1,r}, Z_1\}$-free graph with a vertex of degree at least r is $(n-r+1)$-connected.*

By Corollary 3.1, a 4-connected $\{Z_1, K_{1,3}\}$-free graph G is a complete graph missing at most a matching. By finding a hamiltonian cycle of G such that non-adjacent pairs of vertices are of distance at least 3 on the cycle, we can construct an H^2 in G. Hence, we obtain the result below.

Theorem 3.1. *Every 4-connected $\{Z_1, K_{1,3}\}$-free graph contains an H^2.*

For 4-connected $\{Z_1, K_{1,4}\}$-free graphs, we have a similar result.

Theorem 3.2. *Every 4-connected $\{Z_1, K_{1,4}\}$-free graph contains an H^2 provided $|V(G)| \geq 9$.*

Proof. Let $n = |V(G)|$. We use induction on n to show the theorem. By Corollary 3.1, any 4-connected 9-vertex $\{Z_1, K_{1,4}\}$-free graph contains $K_{3,3,3}$ as a spanning subgraph. It is not difficult to verify that $K_{3,3,3}$ contains an H^2. For example, let $\{x_i, y_i, z_i\}\,(i = 1, 2, 3)$ be the three vertices in the i-th tripartition. Then $x_1x_2x_3y_1y_2y_3z_1z_2z_3x_1$ with the additional edges gives an H^2. So we assume $n \geq 10$. Let $v \in V(G)$ be a vertex. We consider the graph $G' = G - v$. Then G' is 6-connected by Corollary 3.2. Additionally, G' has at least 9 vertices and is $\{Z_1, K_{1,4}\}$-free. Hence it contains an H^2, say C_1^2 by the induction hypothesis. Since G is a multipartite graph with each partition of size at most 3, there are at most two vertices on C_1^2 which are not adjacent to v. Thus, there are at least 4 consecutive vertices on C_1^2 such that each of them is adjacent to v. Let v_1, v_2, v_3, v_4 be 4 such consecutive vertices on C_1^2. Then $C_1^2 - \{v_2v_3, v_2v_4, v_1v_3\} \cup \{vv_i \,|\, i = 1, 2, 3, 4\}$ gives an H^2 of G. ■

Notice that the order 9 condition in the above theorem is sharp. The complete tripartite 8-vertex graph $K_{2,3,3}$ is 4-connected and $\{K_{1,4}, Z_1\}$-free, but contains no H^2.

Before proving Theorem 1.1 and Theorem 1.2, we notice that if $\{R, S\}$ is a forbidden pair implying the containment of an H^2 in a 4-connected graph, then neither of R or S is a triangle since an H^2 always contains triangles.

3.1 Proof of Theorem 1.1

The sufficiency follows from Theorem 3.1.

Conversely, we will first show that one of R and S must be a claw. Thus, suppose that $R, S \neq K_{1,3}$. Assume, without loss of generality, that R is an induced subgraph of $G_1 = K_{m,m}$. Then $R = K_{1,r}$, where $r \geq 4$ or R contains an induced C_4. We now consider two cases.

Case 1: $R = K_{1,r}\,(r \geq 4)$.

The graph G_4 has no induced copy of R, so it contains an induced copy of S. As G_4 is $\{K_4, K_{1,3}\}$-free, we see that S contains no K_4 and no induced $K_{1,3}$. Also, R is not an induced subgraph of $G_0 = K_4 + \overline{K_3}$. So G_0 contains S as an induced subgraph. Since $S \notin \{P_3, K_3\}$ and any connected 3-vertex subgraph of G_0 is contained in $\{P_3, K_3\}$, we conclude that S has at least 4 vertices. In G_0, any 4 vertices of G_0 with at most one vertex in $\overline{K_3}$ induces a K_4; and any 4 vertices of G_0 with three vertices in $\overline{K_3}$ induces a $K_{1,3}$. Hence, S contains exactly two vertices from the subgraph K_4 of G_0 and exactly two vertices from the subgraph $\overline{K_3}$ of G_0, as S contains no K_4, and no induced $K_{1,3}$. So S is an induced K_4^- (K_4 with exactly one edge removed). However, G_2 has no induced $R = K_{1,r}$ ($r \geq 4$), and no induced K_4^-. We obtain a contradiction.

Case 2: R contains an induced C_4.

Since G_4 has no induced copy of R, it contains an induced copy of S. As G_4 is $\{K_4, K_{1,3}\}$-free, we see that S contains no K_4 and no induced $K_{1,3}$. Also, R is not an induced subgraph of G_3. So G_3 contains S as an induced subgraph. Since S is connected and $S \notin \{K_1, K_2, K_3, P_3\}$, and any connected 2-vertex, 3-vertex subgraphs of G_3 are contained in $\{K_2, K_3, P_3\}$, we conclude that $|V(S)| \geq 4$. In G_3, any 4 vertices from K_m or any 3 vertices from K_m and one vertex from $\overline{K_{m-1}}$ induce a K_4; and any 4 vertices in which three from $\overline{K_{m-1}}$ induce a $K_{1,3}$. We conclude that S contains exactly two vertices from K_m and exactly two vertices from $\overline{K_{m-1}}$, as S contains no K_4 and no induced $K_{1,3}$. So S is an induced K_4^-. However, each graph in $G_5(t)$ has no H^2, no induced C_4, and no triangle (so no K_4^-). This gives a contradiction.

Thus, one of R and S must be a claw. We assume, without loss of generality, that $R = K_{1,3}$. As R is $K_{1,3}$, S in an induced subgraph of G_2, G_4, and G_6, as none of them contains induced claws. Note that G_4 is $\{C_4, K_4\}$-free, and G_6 is P_4-free, so S is $\{P_4, C_4, K_4\}$-free. Applying Lemma 2.2, we see S is Z_1. ■

3.2 Proof of Theorem 1.2

The sufficiency follows from Theorem 3.2.

Conversely, we will first show that one of R and S must be Z_1. Thus, suppose that $R, S \neq Z_1$. Assume, without loss of generality, that R is an induced subgraph of $G_1 = K_{m,m}$. Then $R = K_{1,r}$, where $r \geq 3$ or R contains an induced C_4. We now consider two cases.

Case 1: $R = K_{1,r}\,(r \geq 3)$.

Then R is not an induced subgraph of G_2. So G_2 contains S as an induced subgraph. Both G_4 and G_6 contains an induced copy of S since neither of them contains an induced copy of R. Since G_4 is $\{C_4, K_4\}$-free and G_6 is P_4-free, we see that S is $\{P_4, C_4, K_4\}$-free. Applying Lemma 2.2, we have $S = Z_1$.

Case 2: R contains an induced C_4.

The graph G_4 has no induced copy of R, so it contains an induced copy of S. As G_4 is $\{K_4, K_{1,3}\}$-free, we see that S contains no K_4 and no induced $K_{1,3}$. Also, R is not an induced subgraph of G_3. So G_3 contains S as an induced subgraph. Since S is connected and $S \notin \{K_1, K_2, K_3, P_3\}$, and any connected 2-vertex, 3-vertex subgraphs of G_3 are contained in $\{K_2, K_3, P_3\}$, we conclude that $|V(S)| \geq 4$. In G_3, any 4 vertices from K_m or any 3 vertices from K_m and one vertex from $\overline{K_{m-1}}$ induce a K_4; and any 4 vertices in which three from $\overline{K_{m-1}}$ induce a $K_{1,3}$. We conclude that S contains exactly two vertices from K_m and exactly two vertices from $\overline{K_{m-1}}$, as S contains no K_4 and no induced $K_{1,3}$. So S is an induced K_4^-. However, G_2 has no induced $R = K_{1,r}\,(r \geq 3)$ and no induced K_4^-. We obtain a contradiction.

Thus one of R and S must be Z_1. Assume, without loss of generality, that $S = Z_1$. As $G_1 = K_{m,m}$ contains no Z_1, G_1 contains an induced copy of R. Hence $R = K_{1,r}$, where $r \geq 3$ or R contains an induced C_4. Since each graph in $G_5(t)\,(t \geq 2)$ is C_4-free, and the only possible stars in it are $K_{1,r}$ for $r \leq 4$, we see that $R = K_{1,r}$ for $r = 3, 4$. ■

3.3 Proof of Theorem 1.3

We now prove Theorem 1.3. Let P be a path. We use P^2 to denote the square of P. In omitting the edges joining distance 2 vertices on the path, we will use the same notation to denote the square of the path. Similar notation for the square of a cycle. Let $P_1^2 = v_1v_2 \cdots v_{s-1}v_s$ and $P_2^2 = u_1u_2 \cdots u_{t-1}u_t$ be two path squares. We denote by $P_1^2P_2^2$ as the concatenation of P_1^2 and P_2^2 by adding edges u_1v_s, u_1v_{s-1} and u_2v_s, where u_1v_{s-1} exists only if $s \geq 2$ and u_2v_s exists only if $t \geq 2$. Also, the notations $v_1P_1^2, P_1^2v_s$, or $v_1P_1^2v_s$ may be used for specifying the end vertices of P_1^2.

We may assume that G is not complete. Let S be a minimum vertex-cut of G. Let $G_i = (V_i, E_i)\,(i = 1, 2, \cdots, k)$ be all the components of $G - S$. Since G is 4-connected, $|S| \geq 4$. As S is a minimum vertex-cut, we have the following claim.

Claim 1: For every vertex $v \in S$, $N(v) \cap V_i \neq \emptyset$, for all $i = 1, 2, \cdots, k$.

Since G is claw-free, from Claim 1 we get Claim 2 below.

Claim 2: $k = 2$; that is, $G - S$ has exactly two components.

Also, by the fact that G is P_4-free, we conclude the following claim.

Claim 3: For each $v \in S$, $N_{G_i}(v) = V_i$ for $i = 1, 2$.

As $E(V_1, V_2) = \emptyset$, G is claw-free, and by Claim 3, we obtain Claim 4 as follows.

Claim 4: G_i is a complete subgraph of G for $i = 1, 2$.

We will use induction on $n = |V(G)|$ in some cases of the proof. The smallest 4-connected $\{K_{1,3}, P_4\}$-free graph is K_5, it contains an H^2. So we suppose $n \geq 6$ and suppose that the theorem holds for the described graphs of smaller orders. Let P_i^2 be a hamiltonian path square of G_i $(i = 1, 2)$.

If $G[S]$ is 4-connected and is not isomorphic to any graphs in the exception families, then by the induction hypothesis, $G[S]$ contains an H^2, say C_s^2, which contains at least 4 vertices by the assumption that $G[S]$ is 4-connected. Let x_1, x_2, x_3 and x_4 be 4 consecutive vertices on C_s^2. By Claim 3, $N_{G_i}(x_j) = V_i$ for $j = 1, 2, 3, 4$ and $i = 1, 2$. Hence $C^2 = x_1 x_2 P_1^2 x_3 x_4 P_2^2 C_s^2 x_1$ is an H^2 of G.

So, we assume that $G[S]$ is 4-connected and $G[S]$ is a graph in some of the exception families. In this case, we first show that every graph in the exception families has a hamiltonian path square. Then by concatenating the path square, P_1^2, and P_2^2 together, we can get an H^2 of G.

Let Q be a graph isomorphic to $(K_1 \sqcup K_3) + (K_m \sqcup K_q)$ for some $m + q \geq 4$. We may assume, without loss of generality, that $m \geq 2$. Then we let P_3^2 be a path square of K_3, P_m^2 a path square of K_m, and P_q^2 a path square of K_q. Also, let x be the single vertex from K_1. Then $P_q^2 P_3^2 P_m^2 x$ is a hamiltonian path square of Q. The constructions for a hamiltonian path square for graphs in the families of $(K_2 \sqcup K_2) + (K_1 \sqcup K_m)$, $(K_2 \sqcup K_3) + (K_1 \sqcup K_m)$, and $(K_3 \sqcup K_3) + (K_1 \sqcup K_m)$ are similar, so we omit the details here.

Now let P_s^2 be a hamiltonian path square of $G[S]$, and let x_1, x_2, x_3 and x_4 be 4 consecutive vertices on P_s^2. By Claim 3, for any $v \in S$, $N_{G_i}(v) = V_i\,(i = 1, 2)$. So $C^2 = x_1x_2P_1^2x_3x_4P_2^2P_s^2x_1$ is an H^2 of G.

The remaining proof is divided into two cases according to the connectivity of $G[S]$. Let $G' = G[S]$.

Case 1. Suppose G' is connected but not 4-connected.

If $G' \cong K_4$, let $C_s^2 = x_1x_2x_3x_4$ be an H^2 of it. Then $C^2 = x_1x_2P_1^2x_3x_4P_2^2x_1$ is an H^2 of G. So suppose $G' \ncong K_4$. As $|V(G')| \geq 4$ and G' is not 4-connected, G' is not complete. Let S' be a minimum vertex-cut of G'. Notice that $1 \leq |S'| \leq 3$. Similar discussion as in Claim 1-Claim 4 shows that $G' - S'$ has exactly two components, say, G_1' and G_2' such that each is a complete subgraph, and $G' = G'[S'] + (G_1' \sqcup G_2')$. As G' is also claw-free, we see that S' is $\overline{K_3}$-free. Let P_{1i}^2 be a hamiltonian path square of $G_i'\,(i = 1, 2)$. Suppose, without loss of generality, that $|V(P_{11}^2)| \leq |V(P_{12}^2)|$. We define two new vertex disjoint path squares of G'.

- C1. $|S'| = 1$. Let $S' = \{x_1\}$ and $P_{21}^2 = P_{11}^2x_1$, $P_{22}^2 = P_{12}^2$;
- C2. $|S'| = 2$. Let $S' = \{x_1, x_2\}$ and $P_{21}^2 = P_{11}^2x_1$, $P_{22}^2 = P_{12}^2x_2$;
- C3. $|S'| = 3$. Let $S' = \{x_1, x_2, x_3\}$, and assume that $x_1x_3 \in E(G')$ by the fact that S' is $\overline{K_3}$-free, then let $P_{21}^2 = x_1x_3P_{11}^2$, $P_{22}^2 = P_{12}^2x_2$.

If C1 is true, then $\max\{|V_1|, |V_2|\} \geq 2$. Otherwise, $S' \cup V_1 \cup V_2$, a 3-set, separates G_1' and G_2', contradicting the 4-connectedness assumption of G. Assume, without loss of generality, that $|V_1| \geq 2$. To specify the end vertices, we denote $P_{21}^2 = x_1P_{21}^2x$ and $P_{22}^2 = zP_{22}^2x_2$, where $x_1 \in S'$ and $z \in V(G') - S'$. Clearly, $x_1z \in E(G)$. As $|V(G')| \geq 4$ and $|S'| = 1$, $|V(P_{12}^2)| \geq 2$ by the assumption that $|V(P_{11}^2)| \leq |V(P_{12}^2)|$. Hence, both P_{21}^2 and P_{22}^2 have at least 2 vertices. In specifying one end of the hamiltonian path square P_2^2 of G_2, let $P_2^2 = P_2^2w$. Then $x_1P_{21}^2xP_1^2x_2P_{22}^2zP_2^2wx_1$ is an H^2 of G even if $|V(P_2^2)| = 1$.

For cases C2 and C3, to specify the end vertices, we denote $P_{21}^2 = x_1P_{21}^2x$ and $P_{22}^2 = zP_{22}^2x_2$, where $x_1, x_2 \in S'$ and $x, z \in V(G') - S'$. Since each of P_{11}^2 and P_{12}^2 has at least one vertex, each of the P_{21}^2 and P_{22}^2 defined in C2 and C3 has at least two vertices. By the fact that $G' = G'[S'] + (G_1' \sqcup G_2')$ and the assumption that $x_1x_3 \in E(G)$, we see both P_{21}^2 and P_{22}^2 are path squares satisfying $x_1z, xx_2 \in E(G)$.

In specifying one end of the hamiltonian path square P_2^2 of G_2, let $P_2^2 = P_2^2 w$. Then $x_1 P_{21}^2 x P_1^2 x_2 P_{22}^2 z P_2^2 w x_1$ is an H^2 of G even if $|V(P_1^2)| = 1$ or $|V(P_2^2)| = 1$.

Case 2. Suppose G' is disconnected.

As G is claw-free and $G = G' + (G_1 \sqcup G_2)$, we see that G' consists of exactly two complete components, say G'_1 and G'_2. So $G = (G'_1 \sqcup G'_2) + (G_1 \sqcup G_2)$ and $V_1 \cup V_2$ is also a vertex-cut of G. For $i = 1, 2$, let $|V(G'_i)| = |V'_i|$. So $|V_1 \cup V_2| \geq |V'_1 \cup V'_2| = |S|$, by the minimality of $|S|$. Recall that G is not isomorphic to any of the graphs in the following families:

(i) $(K_1 \sqcup K_3) + (K_m \sqcup K_q)$ with $m + q \geq 4$;

(ii) $(K_2 \sqcup K_2) + (K_1 \sqcup K_m)$ with $m \geq 3$;

(iii) $(K_2 \sqcup K_3) + (K_1 \sqcup K_m)$ with $m \geq 3$;

(iv) $(K_3 \sqcup K_3) + (K_1 \sqcup K_m)$ with $m \geq 3$.

Assume first that $\min\{|V_1|, |V_2|, |V'_1|, |V'_2|\} \geq 2$. In specifying the end vertices, we let $P_1^2 = x_1 P_1^2 y_1$, $P_2^2 = x_2 P_2^2 y_2$, $P_{11}^2 = x_{11} P_{11}^2 y_{11}$, and $P_{12}^2 = x_{21} P_{12}^2 y_{21}$ be the hamiltonian path square of G_1, G_2, G'_1 and G'_2, respectively. Then as $G = (G'_1 \sqcup G'_2) + (G_1 \sqcup G_2)$, we know $x_1 P_1^2 y_1 x_{11} P_{11}^2 y_{11} x_2 P_2^2 y_2 x_{21} P_{12}^2 y_{21} x_1$ is an H^2 of G. So assume, without loss of generality, that $|V_1| = 1$. Then as G is not isomorphic to any graphs in (i) and $|V_1 \cup V_2| \geq |V'_1 \cup V'_2| = |S| \geq 4$, we have that $|V_2| \geq 4$. So, $G_1 \sqcup G_2 \cong K_1 \sqcup K_m$ for some $m \geq 4$. Also, as G is not isomorphic to any graphs in (i)-(iv), $G'_1 \sqcup G'_2 \ncong K_1 \sqcup K_3, K_2 \sqcup K_2, K_2 \sqcup K_3, K_3 \sqcup K_3$. This indicates that $\max\{|V'_1|, |V'_2|\} \geq 4$. We may assume, without loss of generality, that $|V'_1| \geq 4$. Let $P_2^2 = x_{21} x_{22} \cdots x_{2,s-1} x_{2s}\,(s \geq 4)$ be the hamiltonian path square of G_2 specified earlier, $P_{11}^2 = x_{11} x_{12} \cdots x_{1,t-1} x_{1t}\,(t \geq 4)$ be a hamiltonian path square of G'_1, and let P_{12}^2 be a hamiltonian path square of G'_2. Then $x_{11} x_{12} P_1^2 x_{13} x_{14} P_{11}^2 x_{1,t-1} x_{1t} x_{21} x_{22} P_{12}^2 x_{23} x_{24} P_2^2 x_{2,s-1} x_{2s} x_{11}$ is an H^2 of G.

The proof of Theorem 1.3 is then complete. ∎

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