On the balanced decomposition number

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Abstract

A *balanced coloring* of a graph G means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of the vertex-set $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. A *balanced decomposition* associated with the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some r) such that, for every $i \in \{1, \cdots, r\}$, the subgraph $G[V_i]$ of G is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Then the *balanced decomposition number* of a graph G is defined as the minimum integer s such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G, there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \cdots, r)$ has at most s vertices. S. Fujita and H. Liu [SIAM J. Discrete Math. 24, (2010), pp. 1597–1616] proved a nice theorem which states that the balanced decomposition number of a graph G is at most 3 if and only if G is $\lfloor \frac{|V(G)|}{2} \rfloor$-connected. Unfortunately, their proof is lengthy (about 10 pages) and complicated. Here we give an immediate proof of the theorem. This proof makes clear a relationship between balanced decomposition number and graph matching.

keywords: graph decomposition, coloring, connectivity, bipartite matching

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1 Introduction

Throughout this paper, we only consider finite undirected graphs with no multiple edges or loops. For a graph G, let $V(G)$ and $E(G)$ denote the vertex-set of G and the edge-set of G, respectively. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X, and $N_G(X)$ denotes the set $\{y \in V(G) \setminus X | \exists x \in X, \{x, y\} \in E(G)\}$. This set $N_G(X)$ is called the *open neighborhood* of X in G. A subset $Y \subseteq V(G)$ is called a *vertex-cut* of G if there is a partition $V(G) \setminus Y = X_1 \uplus X_2$ such that $|X_i| \geqq 1$ and $N_{G[V(G)\setminus Y]}(X_i) = \emptyset$ $(i = 1, 2)$. For other basic definitions in graph theory, please consult [2].

In 2008, S. Fujita and T. Nakamigawa [4] introduced a new graph invariant, namely the *balanced decomposition number* of a graph, which was motivated by the estimation of the number of steps for pebble motion on graphs. A *balanced coloring* of a graph G means a triple $\{P_1, P_2, X\}$ of mutually disjoint subsets of $V(G)$ such that $V(G) = P_1 \uplus P_2 \uplus X$ and $|P_1| = |P_2|$. Then a *balanced decomposition* of G associated with its balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ is defined as a partition of $V(G) = V_1 \uplus \cdots \uplus V_r$ (for some r) such that, for every $i \in \{1, \cdots, r\}$, $G[V_i]$ is connected and $|V_i \cap P_1| = |V_i \cap P_2|$. Note that every disconnected graph has a balanced coloring which admits no balanced decompositions. Now the *balanced decomposition number* of a connected graph G is defined as the minimum integer s such that, for every balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G, there exists a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \cdots, r)$ has at most s vertices.

The set of the starting and the target arrangements of mutually indistinguishable pebbles on a graph G can be modeled as a balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G. Then, as is pointed out in [4], the balanced decomposition number of G gives us an upper-bound for the minimum number of necessary steps to the pebble motion problem, and, for several graph-classes, this upper bound is sharp.

In addition to the initial motivations and their applications in [4], this newcomer graph invariant turns out to have deep connections to some essential graph theoretical concepts. For example, the following conjecture in [4] indicates a relationship between this invariant and the vertex-connectivity of graphs:

Conjecture 1. (S. Fujita and T. Nakamigawa (2008)) *The balanced decomposition number of G is at most $\lfloor \frac{|V(G)|}{2} \rfloor + 1$ if G is 2-connected.*

Recently, G. J. Chang and N. Narayanan [1] announced a solution to this conjecture.

Then especially, S. Fujita and H. Liu [3] proved the affirmation of the "high"-connectivity counterpart of the above conjecture, as follows:

Theorem 1. (S. Fujita and H. Liu (2010)) *Let G be a connected graph with at least 3 vertices. Then the balanced decomposition number of G is at most 3 if and only if G is $\lfloor \frac{|V(G)|}{2} \rfloor$-connected.*

Thus, there may be a trade-off between the vertex-connectivity and the balanced decomposition number. This interesting relationship should be investigated for its own sake.

Unfortunately, the proof of Theorem 1 in [3] is lengthy (about 10 pages) and complicated.

In this note, we give a new proof of the theorem 1. The advantages of our proof is that it is immediate and makes clear a relationship between balanced decomposition number and graph matching.

2 A quick proof of Theorem 1

We show our proof of the theorem 1 here.

Proof of Theorem 1. In order to prove the **if part**, let us define the following new bipartite graph H from a given balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of a graph G:

1. The partite sets of H are $V_1(H) := P_1 \uplus X_1$ and $V_2(H) := P_2 \uplus X_2$, where each $X_i := \{(x, i) \mid x \in X\}$ $(i = 1, 2)$ is a copy of the set $X (\subseteq V(G))$.

2. The edge set $E(H)$ of H is defined as follows:

$$\begin{aligned} E(H) := & \big\{\{p_1, p_2\} \mid p_1 \in P_1, p_2 \in P_2, \{p_1, p_2\} \in E(G)\big\} \\ & \cup \big\{\{p_1, (x, 2)\} \mid p_1 \in P_1, x \in X, \{p_1, x\} \in E(G)\big\} \\ & \cup \big\{\{(x, 1), p_2\} \mid x \in X, p_2 \in P_2, \{x, p_2\} \in E(G)\big\} \\ & \cup \big\{\{(x, 1), (x, 2)\} \mid x \in X\big\}. \end{aligned}$$

Then clearly, the balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G has a balanced decomposition $V(G) = V_1 \uplus \cdots \uplus V_r$ whose every element $V_i (i = 1, \ldots, r)$ consists of at most 3 vertices, if and only if the graph H has a perfect matching. Then we use here the famous "Hall's Marriage Theorem"[5], as follows.

Lemma 2. (P. Hall(1935)) *Let G be a bipartite graph whose partite sets are $V_1(G)$ and $V_2(G)$. Suppose that $|V_1(G)| = |V_2(G)|$. Then G has a perfect matching if and only if every subset U of $V_1(G)$ satisfies $|U| \leqq |N_G(U)|$.*

Now, suppose that H does not have any perfect matching. Then, from lemma 2, $\exists A \subseteq P_1, \exists B \subseteq X_1, |N_H(A \cup B)| \leqq |A| + |B| - 1$. Let $C := P_2 \setminus N_H(A \cup B)$ and $D := X_2 \setminus N_H(A \cup B)$. Then, by symmetry, $|N_H(C \cup D)| \leqq |C| + |D| - 1$ also holds. Furthermore, by the definition of H, $|B| \leqq |X_2 \setminus D|$

and $|D| \leqq |X_1 \setminus B|$ hold, and hence $0 \leqq |X|-|B|-|D| \leqq |A|+|C|-|P_1|-1 = |A| + |C| - |P_2| - 1$ satisfies. Please see Figure 1 which shows this situation. The vertex-cut of $V(G)$ corresponding to the set $(P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_1 \setminus B)$

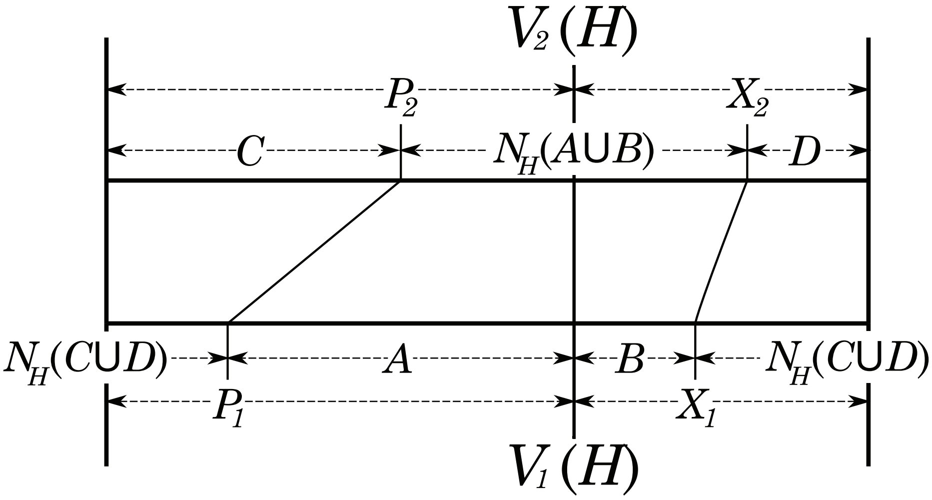

Figure 1: The bipartite graph H which has no perfect matching.

separates $G[C]$ from its remainder. By symmetry, the vertex-cut of $V(G)$ corresponding to the set $(P_1 \setminus A) \cup (P_2 \setminus C) \cup (X_2 \setminus D)$ separates $G[A]$ from its remainder. Hence if G is $\lfloor \frac{|V(G)|}{2} \rfloor$-connected, $|V(G)| - 1 \leqq 2(|P_1| - |A| + |P_2| - |C|) + (|X| - |B|) + (|X| - |D|) = (|P_1| + |P_2| + |X|) - 2\big((|A| + |C| - |P_1|) - (|X| - |B| - |D|)\big) - (|X| - |B| - |D|) \leqq |V(G)| - 2$, a contradiction.

The proof of the **only if part** is given by a construction of special balanced colorings, which is the same as the original one in [3]. We will transcribe the construction only for the convenience of readers.

Suppose that G is not $\lfloor \frac{|V(G)|}{2} \rfloor$-connected. And let Y denote a minimum vertex-cut of G. Note that $2|Y| \leqq |V(G)| - 2$. Then $G[V(G) \setminus Y]$ is divided into two graphs G_1 and G_2 such that $|V(G_i)| \geqq 1$ and $N_{G[V(G)\setminus Y]}(V(G_i)) = \emptyset$ $(i = 1, 2)$. Without loss of generality, we assume that $|V(G_1)| \leqq |V(G_2)|$. Let l denote the number $\min\{|Y|, |V(G_1)| - 1\}$. Suppose an arbitrary balanced coloring $V(G) = P_1 \uplus P_2 \uplus X$ of G such that $|Y \cap P_1| = l$ and $|Y \cap P_2| = |Y| - l$ and $|V(G_1) \cap P_2| = l + 1$ and $V(G_1) \cap P_1 = \emptyset$. Then, it is easy to see that every balanced decomposition associated with such a balanced coloring has at least one component whose vertex-size is at least 4, that is, the balanced decomposition number of G is at least 4. ■

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