# The Hamilton-Waterloo problem for triangle-factors and heptagon-factors 

Hongchuan Lei • Hung-Lin Fu

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#### Abstract

Given 2-factors $R$ and $S$ of order $n$, let $r$ and $s$ be nonnegative integers with $r+s=\left\lfloor\frac{n-1}{2}\right\rfloor$, the Hamilton-Waterloo problem asks for a 2 factorization of $K_{n}$ if $n$ is odd, or of $K_{n}-I$ if $n$ is even, in which $r$ of its 2 -factors are isomorphic to $R$ and the other $s 2$-factors are isomorphic to $S$. In this paper, we solve the problem for the case of triangle-factors and heptagonfactors for odd $n$ with 3 possible exceptions when $n=21$.


Keywords Cycle decomposition • Triangle-factor • Heptagon-factor •
2-factorization

## 1 Introduction

A decomposition of a graph $G$ is a collection of edge-disjoint subgraphs such that every edge of $G$ belongs to exactly one of the subgraphs. A subgraph $F$ of a graph $G$ is a factor if $F$ contains all the vertices of $G$, if each component of $F$ is isomorphic to a graph $H$, then $F$ is called an $H$-factor of $G$, while if $F$ is a $d$-regular graph, then we call $F$ a $d$-factor. A $C_{k}$-factor is a 2 -factor consisting entirely of cycles of length $k$. A factorization of a graph $G$ is a decomposition of $G$ such that each subgraph is a factor, if the factors are all 2-factors then it is called a 2-factorization. An $\left\{H_{1}^{m_{1}}, H_{2}^{m_{2}}, \ldots, H_{t}^{m_{t}}\right\}$-factorization of a graph $G$ is a factorization of $G$ in which there are precisely $m_{i} H_{i}$-factors. If such a factorization exists, we say that $\left(G ; H_{1}^{m_{1}}, H_{2}^{m_{2}}, \ldots, H_{t}^{m_{t}}\right)$ exists.

Hongchuan Lei
Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan;
Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan
E-mail: hongchuanlei@gmail.com
Hung-Lin Fu
Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan
E-mail: hlfu@math.nctu.edu.tw

Given 2-factors $R$ and $S$ of oder $n$, let $r$ and $s$ be nonnegative integers with $r+s=\left\lfloor\frac{n-1}{2}\right\rfloor$, the Hamilton-Waterloo problem asks for a 2 -factorization of the complete graph $K_{n}$ if $n$ is odd, or $K_{n}-I$ if $n$ is even, in which $r$ of its 2 -factors are isomorphic to $R$ and the other $s 2$-factors are isomorphic to $S$, where $I$ is a 1 -factor. The goal of the problem is to determine the spectrum of $r$ (or $s$ ) for all possible $n$, i.e. the set of $r$ (or $s$ ) such that the corresponding 2-factorization of $K_{n}$ or $k_{n}-I$ exists. If $R$ is a $C_{m}$-factor and $S$ is a $C_{k^{-}}$ factor, i.e. each 2 -factor is uniform, then such a 2 -factorization is denoted by $H W(n ; r, s ; m, k)$.

The uniform cases of the Hamilton-Waterloo problem have attracted much attention in the last decade. The existence of $H W(n ; r, s ; m, k)$ has been settled when $r=0$ or $s=0$ in [2, 3, 10]. So we only discuss the case $r s \neq 0$ in this paper.
Theorem 1 [2, 3, 10] Let $n \geq 3$ and $m \geq 3$. Let $G=K_{n}$ if $n$ is odd, $G=$ $K_{n}-I$ if $n$ is even. Then $\left(G ; C_{m}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\right)$ exists if and only if $n \equiv 0(\bmod m)$ and $(n, m) \notin\{(6,3),(12,3)\}$.

Adams et al. [1] dealt with the cases $(m, k) \in\{(4,6),(4,8),(4,16),(8,16)$, $(3,5),(3,15),(5,15)\}$ and completely solved some of them, they also introduced some methods. Danziger et al. [6] almost completely solved the case $(m, k)=(3,4)$ with only 9 possible exceptions. The case $(m, k)=(n, 3)$, i.e. $R$ is a Hamilton cycle and $S$ is a triangle-factor, was studied in [7,8, 11, 13, and is still open. In recent years, remarkable progress has been made on the Hamilton-Waterloo problem when both $R$ and $S$ are consist of even cycles, see [4, 5, 9, 12].

The next two lemmas are useful for our constructions, and have been used in many papers, for example see [9.
Lemma 1 Suppose $G_{1}$ and $G_{2}$ are two vertex-disjoint graphs. If $\left(G_{1} ; C_{m}^{r}, C_{k}^{s}\right)$ and $\left(G_{2} ; C_{m}^{r}, C_{k}^{s}\right)$ both exist, then $\left(G_{1} \cup G_{2} ; C_{m}^{r}, C_{k}^{s}\right)$ exists.
Lemma 2 Suppose $G_{1}$ and $G_{2}$ are two edge-disjoint graphs with the same vertex set. If $\left(G_{1} ; C_{m}^{r_{1}}, C_{k}^{s_{1}}\right)$ and $\left(G_{2} ; C_{m}^{r_{2}}, C_{k}^{s_{2}}\right)$ both exist, then $\left(G_{1} \cup G_{2} ; C_{m}^{r_{1}+r_{2}}\right.$, $\left.C_{k}^{s_{1}+s_{2}}\right)$ exists.

In this paper, we deal with the case $(m, k)=(3,7)$ with $n$ odd. Lemma 3.3 in [1] shows that if $H W(21 ; r, s ; 3,7)$ exists for all nonnegative integers $r$ and $s$ with $r+s=10$ then the problem is settled. Unfortunately we can't construct all possible 2 -factorizations of this kind for $n=21$. Instead, using 2 -factorizations of $K_{7,7,7}$, we will prove the following result.

Theorem 2 If $n \equiv 1(\bmod 2)$ and $r s \neq 0$ with $r+s=\frac{n-1}{2}$, then there exists an $H W(n ; r, s ; 3,7)$ if and only if $n \equiv 21(\bmod 42)$ except possibly when $n=21$ and $r=2,4,6$.
In Section 2, we decompose $K_{7,7,7}$ into $C_{3}$-factors and $C_{7}$-factors. In Section 3 , we deal with $K_{21}$. In Section 4, we show how to decompose $K_{n}$ into $K_{7,7,7^{-}}$ factors and $K_{21}$-factors, then prove Theorem 2,

2 Factorizations of $\boldsymbol{K}_{\mathbf{7 , 7}, \boldsymbol{7}}$
Let $V\left(K_{7,7,7}\right)=\left\{j_{i} \mid j \in Z_{7}, i \in Z_{3}\right\}$, and let $V_{i}=\left\{j_{i} \mid j \in Z_{7}\right\}$ for $i \in Z_{3}$ be the three partite sets of $K_{7,7,7}$. Denote the complete graph on $V_{i}$ by $K_{V_{i}}$, the complete bipartite graph on $V_{i}$ and $V_{j}$ by $K_{V_{i}, V_{j}}$, and the complete tripartite graph $K_{7,7,7}$ on $V_{0}, V_{1}$ and $V_{2}$ by $K_{V_{0}, V_{1}, V_{2}}$. Then

$$
E\left(K_{V_{0}, V_{1}, V_{2}}\right)=E\left(K_{V_{0}, V_{1}}\right) \cup E\left(K_{V_{1}, V_{2}}\right) \cup E\left(K_{V_{2}, V_{0}}\right) .
$$

For $i, j \in Z_{3}$ and $d \in Z_{7}$, let $E_{i j}(d)=\left\{\left\{l_{i},(l+d)_{j}\right\} \mid l \in Z_{7}\right\}$. It is easy to verify that

$$
\begin{gathered}
E\left(K_{V_{i}}\right)=\bigcup_{d=1}^{3} E_{i i}(d) \\
E\left(K_{V_{i}, V_{j}}\right)=\bigcup_{d=0}^{6} E_{i j}(d) \text { for } i \neq j
\end{gathered}
$$

Some of the techniques used in the following lemmas are widely used in combinatorial designs, see [14] for example. In the beginning we give a few basic constructions. The first two lemmas are easy to see, so we omit the proofs.
Lemma 3 Let $d_{0}, d_{1}, d_{2} \in Z_{7}$. If $d_{0}+d_{1}+d_{2} \equiv 0(\bmod 7)$, then the edges of $E_{01}\left(d_{0}\right) \cup E_{12}\left(d_{1}\right) \cup E_{20}\left(d_{2}\right)$ form a $C_{3}$-factor of $K_{7,7,7}$.

Lemma 4 If $(d, 7)=1$, then the edges of $E_{i i}(d)$ form a Hamilton cycle, i.e. a $C_{7}$-factor of $K_{V_{i}}$.

Lemma 5 The edges of $\bigcup_{d \in\{1,6\}}\left(E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)\right)$ can be decomposed into $2 C_{7}$-factors of $K_{7,7,7}$.

Proof Let

$$
\begin{aligned}
& F_{1}=\left\{\left(0_{i}, 1_{i+1}, 2_{i+2}, 3_{i}, 4_{i+1}, 5_{i}, 6_{i+1}\right) \mid i \in Z_{3}\right\} \\
& F_{2}=\left\{\left(0_{i}, 1_{i+2}, 2_{i+1}, 3_{i}, 4_{i+2}, 5_{i}, 6_{i+2}\right) \mid i \in Z_{3}\right\}
\end{aligned}
$$

then both $F_{1}$ and $F_{2}$ are $C_{7}$-factors of $K_{7,7,7}$. It is straightforward to verify that

$$
E\left(F_{1}\right) \cup E\left(F_{2}\right)=\bigcup_{d \in\{1,6\}}\left(E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)\right)
$$

Lemma 6 The edges of $\bigcup_{d \in\{2,5\}}\left(E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)\right)$ can be decomposed into $2 C_{7}$-factors of $K_{7,7,7}$.

Proof The proof is similar to Lemma 5, let the $2 C_{7}$-factors be

$$
\begin{aligned}
& F_{1}=\left\{\left(0_{i}, 2_{i+1}, 4_{i+2}, 6_{i}, 1_{i+1}, 3_{i}, 5_{i+1}\right) \mid i \in Z_{3}\right\} \\
& F_{2}=\left\{\left(0_{i}, 2_{i+2}, 4_{i+1}, 6_{i}, 1_{i+2}, 3_{i}, 5_{i+2}\right) \mid i \in Z_{3}\right\}
\end{aligned}
$$

Lemma 7 The edges of $\bigcup_{d \in\{3,4\}}\left(E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)\right)$ can be decomposed into $2 C_{7}$-factors of $K_{7,7,7}$.

Proof Let the $2 C_{7}$-factors be

$$
\begin{aligned}
& F_{1}=\left\{\left(0_{i}, 3_{i+1}, 6_{i+2}, 2_{i}, 5_{i+1}, 1_{i}, 4_{i+1}\right) \mid i \in Z_{3}\right\}, \\
& F_{2}=\left\{\left(0_{i}, 3_{i+2}, 6_{i+1}, 2_{i}, 5_{i+2}, 1_{i}, 4_{i+2}\right) \mid i \in Z_{3}\right\} .
\end{aligned}
$$

Lemma 8 [3] Let $K_{d(m)}$ be the complete multipartite graph with $d$ parts of size $m$, if $d$ and $m$ are both odd integers, then there is a 2-factorization of $K_{d(m)}$, in which each 2-factor is a $C_{m}$-factor.

Now we decompose $K_{7,7,7}$ into $C_{3}$-factors and $C_{7}$-factors.
Lemma $9\left(K_{7,7,7} ; C_{3}^{\alpha}, C_{7}^{\beta}\right)$ exists for $\alpha \in\{0,1,3,5,7\}$ with $\alpha+\beta=7$.
Proof For $\alpha=0,\left(K_{7,7,7} ; C_{7}^{7}\right)$ exists by Lemma 8 ,
For $\alpha=1$, decompose $\left\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d=1,2, \ldots, 6\right\}$ into 6 $C_{7}$-factors by Lemma 5 5 the remaining edges $E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$ form a $C_{3}$-factor by Lemma 3

For $\alpha=3$, decompose $\left\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d=2,3,4,5\right\}$ into $4 C_{7}$-factors by Lemma 6 and Lemma 7. The $3 C_{3}$-factors are $E_{i(i+1)}(0) \cup$ $E_{(i+1)(i+2)}(1) \cup E_{(i+2) i}(6), i \in Z_{3}$ by Lemma 3,

For $\alpha=5$, decompose $\left\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d=3,4\right\}$ into $2 C_{7}$-factors by Lemma 7. By Lemma 3 the $5 C_{3}$-factors are

$$
\begin{aligned}
& E_{01}(0) \cup E_{12}(1) \cup E_{23}(6), E_{01}(2) \cup E_{12}(0) \cup E_{23}(5), \\
& E_{01}(5) \cup E_{12}(2) \cup E_{23}(0), E_{01}(6) \cup E_{12}(6) \cup E_{23}(2), \\
& E_{01}(1) \cup E_{12}(5) \cup E_{23}(1) .
\end{aligned}
$$

For $\alpha=7$, by Lemma 3 the $7 C_{3}$-factors are

$$
\begin{aligned}
& E_{i(i+1)}(1) \cup E_{(i+1)(i+2)}(2) \cup E_{(i+2) i}(4), i \in Z_{3} \\
& E_{j(j+1)}(3) \cup E_{(j+1)(j+2)}(5) \cup E_{(j+2) j}(6), j \in Z_{3} ; \\
& E_{01}(0) \cup E_{12}(0) \cup E_{23}(0)
\end{aligned}
$$

## 3 Factorizations of $\boldsymbol{K}_{\mathbf{2 1}}$

In this section, $V_{i}, K_{V_{i}}, K_{V_{i}, V_{j}}, K_{V_{0}, V_{1}, V_{2}}$, and $E_{i j}(d)$ have the same meanings as given in Section 2. Let $V\left(K_{21}\right)=V_{0} \cup V_{1} \cup V_{2}$, then the edge set

$$
E\left(K_{21}\right)=\left(\bigcup_{i \in Z_{3}} E\left(K_{V_{i}}\right)\right) \cup E\left(K_{V_{0}, V_{1}, V_{2}}\right)
$$

Now we decompose $K_{21}$ into $\gamma C_{3}$-factors and $\delta C_{7}$-factors with $\gamma+\delta=10$.
Lemma $10\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists for $\gamma \in\{0,1,3,5,7,10\}$ with $\gamma+\delta=10$.

Proof Since $E\left(K_{V_{i}}\right)=\bigcup_{d=1}^{3} E_{i i}(d)$ for $i \in Z_{3}$, by Lemma 目 $\left(K_{V_{i}} ; C_{7}^{3}\right)$ exists. Then by Lemma $1\left(\bigcup_{i \in Z_{3}}^{d=1} K_{V_{i}} ; C_{7}^{3}\right)$ exists. Hence, it is easy to observe that if $\left(K_{V_{0}, V_{1}, V_{2}} ; C_{3}^{\alpha}, C_{7}^{\beta}\right)$ exists, then $\left(K_{V_{1} \cup V_{2} \cup V_{3}} ; C_{3}^{\alpha}, C_{7}^{\beta+3}\right)$ exists by Lemma 2, Thus by Lemma 9 , $\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists for $\gamma \in\{0,1,3,5,7\}$ with $\gamma+\delta=10$.

For $(\gamma, \delta)=(10,0),\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists by Theorem 1 .
Lemma $11\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists for $(\gamma, \delta)=(8,2)$.
Proof Let

$$
\begin{aligned}
F_{0}=\{ & \left(0_{0}, 1_{0}, 2_{1}\right),\left(1_{1}, 4_{1}, 5_{2}\right),\left(1_{2}, 6_{2}, 3_{0}\right),\left(2_{0}, 6_{1}, 4_{2}\right), \\
& \left.\left(4_{0}, 0_{1}, 3_{2}\right),\left(5_{0}, 3_{1}, 2_{2}\right),\left(6_{0}, 5_{1}, 0_{2}\right)\right\},
\end{aligned}
$$

then $F_{0}$ is a $C_{3}$-factor of $K_{21}$. Six additional $C_{3}$-factors, denoted by $F_{1}, F_{2}, \ldots$, $F_{6}$, are formed by developing $F_{0} \bmod (7,-)$. Let $F_{7}=E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$, then by Lemma $3 F_{7}$ is a $C_{3}$-factor. Let $F_{8}=E_{00}(2) \cup E_{11}(1) \cup E_{22}(1), F_{9}=$ $E_{00}(3) \cup E_{11}(2) \cup E_{22}(3)$, then $F_{8}$ and $F_{9}$ are both $C_{7}$-factors of $K_{21}$ by Lemma 1 and Lemma 4 Finally, one can check that each edge of $K_{21}$ is used exactly once.

Lemma $12\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists for $(\gamma, \delta)=(9,1)$.
Proof Let

$$
\begin{aligned}
F_{0}=\{ & \left(0_{0}, 1_{0}, 6_{1}\right),\left(0_{1}, 1_{1}, 4_{2}\right),\left(0_{2}, 2_{2}, 3_{0}\right),\left(2_{0}, 4_{0}, 4_{1}\right), \\
& \left.\left(3_{1}, 5_{1}, 3_{2}\right),\left(5_{2}, 6_{2}, 5_{0}\right),\left(6_{0}, 2_{1}, 1_{2}\right)\right\},
\end{aligned}
$$

then $F_{0}$ is a $C_{3}$-factor of $K_{21}$. Six additional $C_{3}$-factors, denoted by $F_{1}, F_{2}, \ldots$, $F_{6}$, are formed by developing $F_{0} \bmod (7,-)$. Let $F_{7}=E_{01}(1) \cup E_{12}(2) \cup E_{20}(4)$, $F_{8}=E_{01}(4) \cup E_{12}(1) \cup E_{20}(2)$, then by Lemma $3 F_{7}$ and $F_{8}$ are both $C_{3}$-factors. Let $F_{9}=E_{00}(3) \cup E_{11}(3) \cup E_{22}(3)$, then $F_{9}$ is a $C_{7}$-factor of $K_{21}$ by Lemma 1 and Lemma 4 Again, one can check that each edge of $K_{21}$ is used exactly once.

Combining Lemmas $10+12$ we have the following result.
Lemma $13\left(K_{21} ; C_{3}^{\gamma}, C_{7}^{\delta}\right)$ exists for $\gamma \in\{0,1,3,5,7,8,9,10\}$ with $\gamma+\delta=10$.

## 4 Main Results

Let $n$ be an odd integer. Let $r$ and $s$ be positive integers with $r+s=\frac{n-1}{2}$. It is easy to see that a necessary condition for the existence of an $H W(n ; r, s ; 3,7)$ is $n \equiv 21(\bmod 42)$. Let $n=42 t+21, t \geq 0$. Let the vertex set of $K_{n}$ be $V\left(K_{n}\right)=\left\{j_{i} \mid j \in Z_{7}, i \in Z_{6 t+3}\right\}$, denote $V_{i}=Z_{7} \times\{i\}$ for $i \in Z_{6 t+3}$. The next lemma is based on a construction given in the paper [1].

Lemma 14 For $n=42 t+21$ and $t \geq 0,\left(K_{n} ; K_{7,7,7}^{3 t}, K_{21}\right)$ exists.

Proof By Theorem $11\left(K_{6 t+3} ; C_{3}^{3 t+1}\right)$ exists for $t \geq 0$, it is actually the well known Kirkman triple system of order $6 t+3$. Let the vertex set of $K_{6 t+3}$ be $\left\{V_{i} \mid i \in Z_{6 t+3}\right\}$, replace each 3-cycle $\left(V_{i}, V_{j}, V_{k}\right)$ with the complete tripartite graph $K_{7,7,7}$ on vertex sets $V_{i}, V_{j}$ and $V_{k}$, then each $C_{3}$-factor of $K_{6 t+3}$ corresponds to a $K_{7,7,7}$-factor of $K_{n}$, also these $K_{7,7,7}$-factors form the complete multipartite graph $K_{(6 t+3)(7)}$ on vertex sets $V_{0}, V_{1}, \ldots, V_{6 t+2}$, i.e. $\left(K_{(6 t+3)(7)} ; K_{7,7,7}^{3 t+1}\right)$ exists. Hence $\left(K_{n} ; K_{7,7,7}^{3 t+1}, K_{7}\right)$ exists and the union of any $K_{7,7,7}$-factor and the $K_{7}$-factor of $K_{n}$ is actually a $K_{21}$-factor. Therefore, $\left(K_{n} ; K_{7,7,7}^{3 t}, K_{21}\right)$ exists.
Lemma 15 Let $\alpha_{i} \in\{0,1,3,5,7\}$ with $\alpha_{i}+\beta_{i}=7$ for $i=1,2, \ldots, 3 t$, and $\gamma \in\{0,1,3,5,7,8,9,10\}$ with $\gamma+\delta=10$, then there exists an $H W\left(n ; \sum_{i=1}^{3 t} \alpha_{i}+\right.$ $\left.\gamma, \sum_{i=1}^{3 t} \beta_{i}+\delta ; 3,7\right)$.
Proof By Lemma 14, we decompose $K_{n}$ into $3 t K_{7,7,7}$-factors and a $K_{21}$-factor.
For the $i$ th $K_{7,7,7}$-factor, let $\alpha_{i} \in\{0,1,3,5,7\}$ and $\alpha_{i}+\beta_{i}=7$. Then decompose each $K_{7,7,7}$ of this $K_{7,7,7}$-factor into $\alpha_{i} C_{3}$-factors and $\beta_{i} C_{7}$-factors by Lemma 9 by Lemma 1 these 2 -factors of $K_{7,7,7}$ form $\alpha_{i} C_{3}$-factors and $\beta_{i}$ $C_{7}$-factors of $K_{n}$.

Similarly, the $K_{21}$-factor of $K_{n}$ can be decomposed into $\gamma C_{3}$-factors and $\delta C_{7}$-factors for $\gamma \in\{0,1,3,5,7,8,9,10\}$ with $\gamma+\delta=10$ by Lemma 1 and 13 ,

Then by Lemma 2 $\left(K_{n} ; C_{3}^{\sum_{i=1}^{3 t} \alpha_{i}+\gamma}, C_{7}^{\sum_{i=1}^{3 t} \beta_{i}+\delta}\right)$ exists, i.e. there exists an $H W\left(n ; \sum_{i=1}^{3 t} \alpha_{i}+\gamma, \sum_{i=1}^{3 t} \beta_{i}+\delta ; 3,7\right)$.

We are now ready to prove the main theorem of this paper.
Proof of Theorem 2 As noted earlier, the condition $n \equiv 21(\bmod 42)$ is necessary, we now prove sufficiency. Let $n=42 t+21$, the case $t=0$ (i.e. $n=21$ ) is solved by Lemma 13 ,

For the case $t>0$, let $r=7 a+b$, where $0 \leq b<7$. For the existence of an $H W(n ; r, s ; 3,7)$, we only need to assign a proper value to each of $\left\{\gamma, \alpha_{i} \mid i=\right.$ $1,2, \ldots, 3 t\}$ in Lemma 15. Note that if $a=3 t+1$, then $b<3$ (the case $b=3$ is the case $s=0$, which is covered by Theorem (1).

If $b=0$ and $a<3 t+1$, then let $\gamma=0$ and $\alpha_{i}= \begin{cases}7, & \text { for } 1 \leq i \leq a, \\ 0, & \text { for } a<i \leq 3 t .\end{cases}$
If $b=0$ and $a=3 t+1$, then let $\gamma=7$ and $\alpha_{i}=7$ for $i=1,2, \ldots, 3 t$.
If $b=1$ and $a<3 t+1$, then let $\gamma=1$ and $\alpha_{i}= \begin{cases}7, & \text { for } 1 \leq i \leq a, \\ 0, & \text { for } a<i \leq 3 t .\end{cases}$
If $b=1$ and $a=3 t+1$, then let $\gamma=8$ and $\alpha_{i}=7$ for $i=1,2, \ldots, 3 t$.
If $b=2$ and $a<3 t$, then let $\gamma=1$ and $\alpha_{i}= \begin{cases}1, & \text { for } i=1, \\ 7, & \text { for } 2 \leq i \leq a+1, \\ 0, & \text { for } a+1<i \leq 3 t .\end{cases}$
If $b=2$ and $a=3 t$, then let $\gamma=9$ and $\alpha_{i}= \begin{cases}1, & \text { for } i=1, \\ 7, & \text { for } 2 \leq i \leq 3 t .\end{cases}$
If $b=2$ and $a=3 t+1$, then let $\gamma=8$ and $\alpha_{i}=7$ for $i=1,2, \ldots, 3 t$.

If $b=3$, then let $\gamma=3$ and $\alpha_{i}= \begin{cases}7, & \text { for } 1 \leq i \leq a, \\ 0, & \text { for } a<i \leq 3 t .\end{cases}$
If $b=4$ and $a<3 t$, then let $\gamma=3$ and $\alpha_{i}= \begin{cases}1, & \text { for } i=1, \\ 7, & \text { for } 2 \leq i \leq a+1, \\ 0, & \text { for } a+1<i \leq 3 t .\end{cases}$
If $b=4$ and $a=3 t$, then let $\gamma=8$ and $\alpha_{i}= \begin{cases}3, & \text { for } i=1, \\ 7, & \text { for } 1<i \leq 3 t .\end{cases}$
If $b=5$, then let $\gamma=5$ and $\alpha_{i}= \begin{cases}7, & \text { for } 1 \leq i \leq a, \\ 0, & \text { for } a<i \leq 3 t .\end{cases}$
If $b=6$ and $a<3 t$, then let $\gamma=1$ and $\alpha_{i}= \begin{cases}5, & \text { for } i=1, \\ 7, & \text { for } 2 \leq i \leq a+1, \\ 0, & \text { for } a+1<i \leq 3 t .\end{cases}$
If $b=6$ and $a=3 t$, then let $\gamma=8$ and $\alpha_{i}= \begin{cases}5, & \text { for } i=1, \\ 7, & \text { for } 1<i \leq 3 t .\end{cases}$

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