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# The Hamilton-Waterloo problem for triangle-factors and heptagon-factors

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Abstract Given 2-factors R and S of order n, let r and s be nonnegative integers with  $r+s=\lfloor \frac{n-1}{2}\rfloor$ , the Hamilton-Waterloo problem asks for a 2-factorization of  $K_n$  if n is odd, or of  $K_n-I$  if n is even, in which r of its 2-factors are isomorphic to R and the other s 2-factors are isomorphic to S. In this paper, we solve the problem for the case of triangle-factors and heptagon-factors for odd n with 3 possible exceptions when n=21.

**Keywords** Cycle decomposition · Triangle-factor · Heptagon-factor · 2-factorization

### 1 Introduction

A decomposition of a graph G is a collection of edge-disjoint subgraphs such that every edge of G belongs to exactly one of the subgraphs. A subgraph F of a graph G is a factor if F contains all the vertices of G, if each component of F is isomorphic to a graph H, then F is called an H-factor of G, while if F is a d-regular graph, then we call F a d-factor. A  $C_k$ -factor is a 2-factor consisting entirely of cycles of length k. A factorization of a graph G is a decomposition of G such that each subgraph is a factor, if the factors are all 2-factors then it is called a 2-factorization. An  $\{H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t}\}$ -factorization of a graph G is a factorization of G in which there are precisely  $m_i$   $H_i$ -factors. If such a factorization exists, we say that  $(G; H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t})$  exists.

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Given 2-factors R and S of oder n, let r and s be nonnegative integers with  $r+s=\lfloor \frac{n-1}{2}\rfloor$ , the Hamilton-Waterloo problem asks for a 2-factorization of the complete graph  $K_n$  if n is odd, or  $K_n-I$  if n is even, in which r of its 2-factors are isomorphic to R and the other s 2-factors are isomorphic to S, where I is a 1-factor. The goal of the problem is to determine the spectrum of r (or s) for all possible n, i.e. the set of r (or s) such that the corresponding 2-factorization of  $K_n$  or  $k_n-I$  exists. If R is a  $C_m$ -factor and S is a  $C_k$ -factor, i.e. each 2-factor is uniform, then such a 2-factorization is denoted by HW(n;r,s;m,k).

The uniform cases of the Hamilton-Waterloo problem have attracted much attention in the last decade. The existence of HW(n;r,s;m,k) has been settled when r=0 or s=0 in [2,3,10]. So we only discuss the case  $rs\neq 0$  in this paper.

**Theorem 1** [2,3,10] Let  $n \geq 3$  and  $m \geq 3$ . Let  $G = K_n$  if n is odd,  $G = K_n - I$  if n is even. Then  $(G; C_m^{\lfloor \frac{n-1}{2} \rfloor})$  exists if and only if  $n \equiv 0 \pmod{m}$  and  $(n,m) \notin \{(6,3),(12,3)\}$ .

Adams et al. [1] dealt with the cases  $(m,k) \in \{(4,6), (4,8), (4,16), (8,16), (3,5), (3,15), (5,15)\}$  and completely solved some of them, they also introduced some methods. Danziger et al.[6] almost completely solved the case (m,k)=(3,4) with only 9 possible exceptions. The case (m,k)=(n,3), i.e. R is a Hamilton cycle and S is a triangle-factor, was studied in [7,8,11,13], and is still open. In recent years, remarkable progress has been made on the Hamilton-Waterloo problem when both R and S are consist of even cycles, see [4,5,9,12].

The next two lemmas are useful for our constructions, and have been used in many papers, for example see [9].

**Lemma 1** Suppose  $G_1$  and  $G_2$  are two vertex-disjoint graphs. If  $(G_1; C_m^r, C_k^s)$  and  $(G_2; C_m^r, C_k^s)$  both exist, then  $(G_1 \cup G_2; C_m^r, C_k^s)$  exists.

**Lemma 2** Suppose  $G_1$  and  $G_2$  are two edge-disjoint graphs with the same vertex set. If  $(G_1; C_m^{r_1}, C_k^{s_1})$  and  $(G_2; C_m^{r_2}, C_k^{s_2})$  both exist, then  $(G_1 \cup G_2; C_m^{r_1+r_2}, C_k^{s_1+s_2})$  exists.

In this paper, we deal with the case (m,k)=(3,7) with n odd. Lemma 3.3 in [1] shows that if HW(21;r,s;3,7) exists for all nonnegative integers r and s with r+s=10 then the problem is settled. Unfortunately we can't construct all possible 2-factorizations of this kind for n=21. Instead, using 2-factorizations of  $K_{7,7,7}$ , we will prove the following result.

**Theorem 2** If  $n \equiv 1 \pmod{2}$  and  $rs \neq 0$  with  $r + s = \frac{n-1}{2}$ , then there exists an HW(n; r, s; 3, 7) if and only if  $n \equiv 21 \pmod{42}$  except possibly when n = 21 and r = 2, 4, 6.

In Section 2, we decompose  $K_{7,7,7}$  into  $C_3$ -factors and  $C_7$ -factors. In Section 3, we deal with  $K_{21}$ . In Section 4, we show how to decompose  $K_n$  into  $K_{7,7,7}$ -factors and  $K_{21}$ -factors, then prove Theorem 2.

# 2 Factorizations of $K_{7,7,7}$

Let  $V(K_{7,7,7}) = \{j_i \mid j \in Z_7, i \in Z_3\}$ , and let  $V_i = \{j_i \mid j \in Z_7\}$  for  $i \in Z_3$  be the three partite sets of  $K_{7,7,7}$ . Denote the complete graph on  $V_i$  by  $K_{V_i}$ , the complete bipartite graph on  $V_i$  and  $V_j$  by  $K_{V_i,V_j}$ , and the complete tripartite graph  $K_{7,7,7}$  on  $V_0$ ,  $V_1$  and  $V_2$  by  $K_{V_0,V_1,V_2}$ . Then

$$E(K_{V_0,V_1,V_2}) = E(K_{V_0,V_1}) \cup E(K_{V_1,V_2}) \cup E(K_{V_2,V_0}).$$

For  $i, j \in Z_3$  and  $d \in Z_7$ , let  $E_{ij}(d) = \{\{l_i, (l+d)_j\} \mid l \in Z_7\}$ . It is easy to verify that

$$E(K_{V_i}) = \bigcup_{d=1}^{3} E_{ii}(d),$$

$$E(K_{V_i,V_j}) = \bigcup_{d=0}^{6} E_{ij}(d) \text{ for } i \neq j.$$

Some of the techniques used in the following lemmas are widely used in combinatorial designs, see [14] for example. In the beginning we give a few basic constructions. The first two lemmas are easy to see, so we omit the proofs.

**Lemma 3** Let  $d_0, d_1, d_2 \in \mathbb{Z}_7$ . If  $d_0 + d_1 + d_2 \equiv 0 \pmod{7}$ , then the edges of  $E_{01}(d_0) \cup E_{12}(d_1) \cup E_{20}(d_2)$  form a  $C_3$ -factor of  $K_{7,7,7}$ .

**Lemma 4** If (d,7) = 1, then the edges of  $E_{ii}(d)$  form a Hamilton cycle, i.e. a  $C_7$ -factor of  $K_{V_i}$ .

**Lemma 5** The edges of  $\bigcup_{d \in \{1,6\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$  can be decomposed into 2  $C_7$ -factors of  $K_{7,7,7}$ .

Proof Let

$$\begin{split} F_1 &= \{ (0_i, 1_{i+1}, 2_{i+2}, 3_i, 4_{i+1}, 5_i, 6_{i+1}) \mid i \in Z_3 \}, \\ F_2 &= \{ (0_i, 1_{i+2}, 2_{i+1}, 3_i, 4_{i+2}, 5_i, 6_{i+2}) \mid i \in Z_3 \}, \end{split}$$

then both  $F_1$  and  $F_2$  are  $C_7$ -factors of  $K_{7,7,7}$ . It is straightforward to verify that

$$E(F_1) \cup E(F_2) = \bigcup_{d \in \{1,6\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)).$$

**Lemma 6** The edges of  $\bigcup_{d \in \{2,5\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$  can be decomposed into 2  $C_7$ -factors of  $K_{7,7,7}$ .

*Proof* The proof is similar to Lemma 5, let the 2  $C_7$ -factors be

$$F_1 = \{ (0_i, 2_{i+1}, 4_{i+2}, 6_i, 1_{i+1}, 3_i, 5_{i+1}) \mid i \in Z_3 \},$$
  
$$F_2 = \{ (0_i, 2_{i+2}, 4_{i+1}, 6_i, 1_{i+2}, 3_i, 5_{i+2}) \mid i \in Z_3 \}.$$

**Lemma 7** The edges of  $\bigcup_{d \in \{3,4\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$  can be decomposed into 2  $C_7$ -factors of  $K_{7,7,7}$ .

*Proof* Let the 2  $C_7$ -factors be

$$F_1 = \{ (0_i, 3_{i+1}, 6_{i+2}, 2_i, 5_{i+1}, 1_i, 4_{i+1}) \mid i \in Z_3 \},$$
  
$$F_2 = \{ (0_i, 3_{i+2}, 6_{i+1}, 2_i, 5_{i+2}, 1_i, 4_{i+2}) \mid i \in Z_3 \}.$$

**Lemma 8** [3] Let  $K_{d(m)}$  be the complete multipartite graph with d parts of size m, if d and m are both odd integers, then there is a 2-factorization of  $K_{d(m)}$ , in which each 2-factor is a  $C_m$ -factor.

Now we decompose  $K_{7,7,7}$  into  $C_3$ -factors and  $C_7$ -factors.

**Lemma 9**  $(K_{7,7,7}; C_3^{\alpha}, C_7^{\beta})$  exists for  $\alpha \in \{0, 1, 3, 5, 7\}$  with  $\alpha + \beta = 7$ .

*Proof* For  $\alpha = 0$ ,  $(K_{7,7,7}; C_7^7)$  exists by Lemma 8.

For  $\alpha=1$ , decompose  $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d=1,2,\ldots,6\}$  into 6  $C_7$ -factors by Lemma 5-7, the remaining edges  $E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$  form a  $C_3$ -factor by Lemma 3.

For  $\alpha = 3$ , decompose  $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d = 2, 3, 4, 5\}$  into 4  $C_7$ -factors by Lemma 6 and Lemma 7. The 3  $C_3$ -factors are  $E_{i(i+1)}(0) \cup E_{(i+1)(i+2)}(1) \cup E_{(i+2)i}(6)$ ,  $i \in Z_3$  by Lemma 3.

For  $\alpha = 5$ , decompose  $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d = 3, 4\}$  into 2  $C_7$ -factors by Lemma 7. By Lemma 3 the 5  $C_3$ -factors are

$$E_{01}(0) \cup E_{12}(1) \cup E_{23}(6), \ E_{01}(2) \cup E_{12}(0) \cup E_{23}(5),$$
  
 $E_{01}(5) \cup E_{12}(2) \cup E_{23}(0), \ E_{01}(6) \cup E_{12}(6) \cup E_{23}(2),$   
 $E_{01}(1) \cup E_{12}(5) \cup E_{23}(1).$ 

For  $\alpha = 7$ , by Lemma 3 the 7  $C_3$ -factors are

$$\begin{split} E_{i(i+1)}(1) \cup E_{(i+1)(i+2)}(2) \cup E_{(i+2)i}(4), & i \in Z_3; \\ E_{j(j+1)}(3) \cup E_{(j+1)(j+2)}(5) \cup E_{(j+2)j}(6), & j \in Z_3; \\ E_{01}(0) \cup E_{12}(0) \cup E_{23}(0). \end{split}$$

# 3 Factorizations of $K_{21}$

In this section,  $V_i, K_{V_i}, K_{V_i,V_j}, K_{V_0,V_1,V_2}$ , and  $E_{ij}(d)$  have the same meanings as given in Section 2. Let  $V(K_{21}) = V_0 \cup V_1 \cup V_2$ , then the edge set

$$E(K_{21}) = \left(\bigcup_{i \in Z_3} E(K_{V_i})\right) \cup E(K_{V_0, V_1, V_2}).$$

Now we decompose  $K_{21}$  into  $\gamma$   $C_3$ -factors and  $\delta$   $C_7$ -factors with  $\gamma + \delta = 10$ .

**Lemma 10**  $(K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists for  $\gamma \in \{0, 1, 3, 5, 7, 10\}$  with  $\gamma + \delta = 10$ .

Proof Since  $E(K_{V_i}) = \bigcup_{d=1}^3 E_{ii}(d)$  for  $i \in Z_3$ , by Lemma 4,  $(K_{V_i}; C_7^3)$  exists. Then by Lemma 1,  $(\bigcup_{i \in Z_3} K_{V_i}; C_7^3)$  exists. Hence, it is easy to observe that if  $(K_{V_0,V_1,V_2}; C_3^{\alpha}, C_7^{\beta})$  exists, then  $(K_{V_1 \cup V_2 \cup V_3}; C_3^{\alpha}, C_7^{\beta+3})$  exists by Lemma 2. Thus by Lemma 9,  $(K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists for  $\gamma \in \{0, 1, 3, 5, 7\}$  with  $\gamma + \delta = 10$ . For  $(\gamma, \delta) = (10, 0), (K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists by Theorem 1.

**Lemma 11**  $(K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists for  $(\gamma, \delta) = (8, 2)$ .

Proof Let

$$F_0 = \{(0_0, 1_0, 2_1), (1_1, 4_1, 5_2), (1_2, 6_2, 3_0), (2_0, 6_1, 4_2), (4_0, 0_1, 3_2), (5_0, 3_1, 2_2), (6_0, 5_1, 0_2)\},\$$

then  $F_0$  is a  $C_3$ -factor of  $K_{21}$ . Six additional  $C_3$ -factors, denoted by  $F_1, F_2, \ldots$ ,  $F_6$ , are formed by developing  $F_0 \mod(7, -)$ . Let  $F_7 = E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$ , then by Lemma 3  $F_7$  is a  $C_3$ -factor. Let  $F_8 = E_{00}(2) \cup E_{11}(1) \cup E_{22}(1)$ ,  $F_9 = E_{00}(3) \cup E_{11}(2) \cup E_{22}(3)$ , then  $F_8$  and  $F_9$  are both  $C_7$ -factors of  $K_{21}$  by Lemma 1 and Lemma 4. Finally, one can check that each edge of  $K_{21}$  is used exactly once.

**Lemma 12**  $(K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists for  $(\gamma, \delta) = (9, 1)$ .

Proof Let

$$F_0 = \{(0_0, 1_0, 6_1), (0_1, 1_1, 4_2), (0_2, 2_2, 3_0), (2_0, 4_0, 4_1), (3_1, 5_1, 3_2), (5_2, 6_2, 5_0), (6_0, 2_1, 1_2)\},\$$

then  $F_0$  is a  $C_3$ -factor of  $K_{21}$ . Six additional  $C_3$ -factors, denoted by  $F_1, F_2, \ldots$ ,  $F_6$ , are formed by developing  $F_0 \mod(7, -)$ . Let  $F_7 = E_{01}(1) \cup E_{12}(2) \cup E_{20}(4)$ ,  $F_8 = E_{01}(4) \cup E_{12}(1) \cup E_{20}(2)$ , then by Lemma 3  $F_7$  and  $F_8$  are both  $C_3$ -factors. Let  $F_9 = E_{00}(3) \cup E_{11}(3) \cup E_{22}(3)$ , then  $F_9$  is a  $C_7$ -factor of  $K_{21}$  by Lemma 1 and Lemma 4. Again, one can check that each edge of  $K_{21}$  is used exactly once.

Combining Lemmas 10-12, we have the following result.

**Lemma 13**  $(K_{21}; C_3^{\gamma}, C_7^{\delta})$  exists for  $\gamma \in \{0, 1, 3, 5, 7, 8, 9, 10\}$  with  $\gamma + \delta = 10$ .

#### 4 Main Results

Let n be an odd integer. Let r and s be positive integers with  $r+s=\frac{n-1}{2}$ . It is easy to see that a necessary condition for the existence of an HW(n;r,s;3,7) is  $n \equiv 21 \pmod{42}$ . Let n=42t+21,  $t \geq 0$ . Let the vertex set of  $K_n$  be  $V(K_n)=\{j_i\mid j\in Z_7, i\in Z_{6t+3}\}$ , denote  $V_i=Z_7\times\{i\}$  for  $i\in Z_{6t+3}$ . The next lemma is based on a construction given in the paper [1].

**Lemma 14** For n = 42t + 21 and  $t \ge 0$ ,  $(K_n; K_{7,7,7}^{3t}, K_{21})$  exists.

*Proof* By Theorem 1,  $(K_{6t+3}; C_3^{3t+1})$  exists for  $t \geq 0$ , it is actually the well known Kirkman triple system of order 6t + 3. Let the vertex set of  $K_{6t+3}$ be  $\{V_i \mid i \in Z_{6t+3}\}$ , replace each 3-cycle  $(V_i, V_j, V_k)$  with the complete tripartite graph  $K_{7,7,7}$  on vertex sets  $V_i, V_j$  and  $V_k$ , then each  $C_3$ -factor of  $K_{6t+3}$  corresponds to a  $K_{7,7,7}$ -factor of  $K_n$ , also these  $K_{7,7,7}$ -factors form the complete multipartite graph  $K_{(6t+3)(7)}$  on vertex sets  $V_0, V_1, \ldots, V_{6t+2}$ , i.e.  $(K_{(6t+3)(7)}; K_{7,7,7}^{3t+1})$  exists. Hence  $(K_n; K_{7,7,7}^{3t+1}, K_7)$  exists and the union of any  $K_{7,7,7}$ -factor and the  $K_7$ -factor of  $K_n$  is actually a  $K_{21}$ -factor. Therefore,  $(K_n; K_{7,7,7}^{3t}, K_{21})$  exists.

**Lemma 15** Let  $\alpha_i \in \{0, 1, 3, 5, 7\}$  with  $\alpha_i + \beta_i = 7$  for i = 1, 2, ..., 3t, and  $\gamma \in \{0, 1, 3, 5, 7, 8, 9, 10\}$  with  $\gamma + \delta = 10$ , then there exists an  $HW(n; \sum_{i=1}^{3t} \alpha_i + 1)$  $\gamma, \sum_{i=1}^{3t} \beta_i + \delta; 3, 7).$ 

*Proof* By Lemma 14, we decompose  $K_n$  into  $3t K_{7,7,7}$ -factors and a  $K_{21}$ -factor. For the ith  $K_{7,7,7}$ -factor, let  $\alpha_i \in \{0,1,3,5,7\}$  and  $\alpha_i + \beta_i = 7$ . Then decompose each  $K_{7,7,7}$  of this  $K_{7,7,7}$ -factor into  $\alpha_i$   $C_3$ -factors and  $\beta_i$   $C_7$ -factors by Lemma 9, by Lemma 1 these 2-factors of  $K_{7,7,7}$  form  $\alpha_i$   $C_3$ -factors and  $\beta_i$  $C_7$ -factors of  $K_n$ .

Similarly, the  $K_{21}$ -factor of  $K_n$  can be decomposed into  $\gamma$   $C_3$ -factors and

 $\delta$   $C_7$ -factors for  $\gamma \in \{0, 1, 3, 5, 7, 8, 9, 10\}$  with  $\gamma + \delta = 10$  by Lemma 1 and 13. Then by Lemma 2,  $(K_n; C_3^{\sum_{i=1}^{3t} \alpha_i + \gamma}, C_7^{\sum_{i=1}^{3t} \beta_i + \delta})$  exists, i.e. there exists an  $HW(n; \sum_{i=1}^{3t} \alpha_i + \gamma, \sum_{i=1}^{3t} \beta_i + \delta; 3, 7)$ .

We are now ready to prove the main theorem of this paper.

*Proof of Theorem 2* As noted earlier, the condition  $n \equiv 21 \pmod{42}$  is necessary, we now prove sufficiency. Let n = 42t + 21, the case t = 0 (i.e. n = 21) is solved by Lemma 13.

For the case t > 0, let r = 7a + b, where  $0 \le b < 7$ . For the existence of an HW(n;r,s;3,7), we only need to assign a proper value to each of  $\{\gamma,\alpha_i\mid i=1\}$  $1, 2, \ldots, 3t$  in Lemma 15. Note that if a = 3t + 1, then b < 3 (the case b = 3is the case s = 0, which is covered by Theorem 1).

the case 
$$s = 0$$
, which is covered by Theorem 1).

If  $b = 0$  and  $a < 3t + 1$ , then let  $\gamma = 0$  and  $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$ 

If  $b = 0$  and  $a = 3t + 1$ , then let  $\gamma = 7$  and  $\alpha_i = 7$  for  $i = 1, 2, ..., 3t$ .

If  $b = 1$  and  $a < 3t + 1$ , then let  $\gamma = 1$  and  $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$ 

If  $b = 1$  and  $a = 3t + 1$ , then let  $\gamma = 8$  and  $\alpha_i = 7$  for  $i = 1, 2, ..., 3t$ .

If  $b = 2$  and  $a < 3t$ , then let  $\gamma = 1$  and  $\alpha_i = \begin{cases} 1, & \text{for } i = 1, \\ 7, & \text{for } 2 \leq i \leq a + 1, \\ 0, & \text{for } a + 1 < i \leq 3t. \end{cases}$ 

If  $b = 2$  and  $a = 3t$ , then let  $\gamma = 9$  and  $\alpha_i = \begin{cases} 1, & \text{for } i = 1, \\ 7, & \text{for } 2 \leq i \leq a + 1, \\ 7, & \text{for } 2 \leq i \leq 3t. \end{cases}$ 

If  $b = 2$  and  $a = 3t + 1$ , then let  $\gamma = 8$  and  $\alpha_i = 7$  for  $i = 1, 2, ..., 3t$ .

If 
$$b=3$$
, then let  $\gamma=3$  and  $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$ 

If  $b=4$  and  $a < 3t$ , then let  $\gamma=3$  and  $\alpha_i = \begin{cases} 1, & \text{for } i=1, \\ 7, & \text{for } 2 \leq i \leq a+1, \\ 0, & \text{for } a+1 < i \leq 3t. \end{cases}$ 

If  $b=4$  and  $a=3t$ , then let  $\gamma=8$  and  $\alpha_i = \begin{cases} 3, & \text{for } i=1, \\ 7, & \text{for } 1 < i \leq 3t. \end{cases}$ 

If  $b=5$ , then let  $\gamma=5$  and  $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$ 

If  $b=6$  and  $a < 3t$ , then let  $\gamma=1$  and  $\alpha_i = \begin{cases} 5, & \text{for } i=1, \\ 7, & \text{for } 2 \leq i \leq a+1, \\ 0, & \text{for } a+1 < i \leq 3t. \end{cases}$ 

If  $b=6$  and  $a=3t$ , then let  $\gamma=8$  and  $\alpha_i = \begin{cases} 5, & \text{for } i=1, \\ 7, & \text{for } 1 < i \leq 3t. \end{cases}$ 

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