Long paths and cycles passing through specified vertices under the average degree condition

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Abstract

Let G be a k-connected graph with $k \geq 2$. In this paper we first prove that: For two distinct vertices x and z in G, it contains a path passing through its any $k-2$ specified vertices with length at least the average degree of the vertices other than x and z. Further, with this result, we prove that: If G has n vertices and m edges, then it contains a cycle of length at least $2m/(n-1)$ passing through its any $k-1$ specified vertices. Our results generalize a theorem of Fan on the existence of long paths and a classical theorem of Erdös and Gallai on the existence of long cycles under the average degree condition.

Keywords: Long paths, Long cycles, Average degree

1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple graphs only.

Let G be a graph and H a subgraph of G. We use $V(H)$ and $E(H)$ to denote the set of vertices and edges of H, respectively, and use $e(H)$ for the number of the edges of H. For a vertex $v \in V(G)$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of neighbors of v in H. We call $d_H(v)$ the *degree* of v in H. Let x and z be two distinct vertices of G. A path connecting x and z is called an (x, z)-*path*. For a subset Y of $V(G)$, an (x, z)-path passing through all the vertices in Y is called an (x, Y, z)-*path*, and a cycle passing through all the vertices in Y is called a Y-*cycle*. If Y contains only one vertex y, an $(x, \{y\}, z)$-path and a

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$\{y\}$-cycle are simply denoted by an (x, y, z)-path and a y-cycle, respectively. The *distance* between x and z in H, denoted by $d_H(x, z)$, is the length of a shortest (x, z)-path with all its internal vertices in H. If no such a path exists, we define $d_H(x, z) = \infty$. The *codistance* between x and z in H, denoted by $d^*_H(x, z)$, is the length of a longest (x, z)-path with all its internal vertices in H. If no such a path exists, we define $d^*_H(x, z) = 0$. When no confusion occurs, we use $N(v)$, $d(v)$, $d(x, z)$ and $d^*(x, z)$ instead of $N_G(v)$, $d_G(v)$, $d_G(x, z)$ and $d^*_G(x, z)$, respectively.

Long path and cycle problems are interesting and important in graph theory and have been deeply studied, see [1, 7]. The following Theorem by Erdös and Gallai opened the study on long paths with specified end vertices.

Theorem 1 (Erdös and Gallai [5])**.** *Let G be a 2-connected graph and x and z be two distinct vertices of G. If $d(v) \geq d$ for every vertex $v \in V(G)\backslash\{x, z\}$, then G contains an (x, z)-path of length at least d.*

In fact, Theorem 1 has a stronger extension due to Enotomo.

Theorem 2 (Enotomo [4])**.** *Let G be a 2-connected graph and x and z be two distinct vertices of G. If $d(v) \geq d$ for every vertex in $V(G)\backslash\{x, z\}$, then for every given vertex $y \in V(G)\backslash\{x, z\}$, G contains an (x, y, z)-path of length at least d.*

Another direction of extending Theorem 1 is to weaken the minimum degree condition to an average degree condition. Fan finished this work as follows.

Theorem 3 (Fan [6])**.** *Let G be a 2-connected graph and x and z be two distinct vertices of G. If the average degree of the vertices other than x and z is at least r, then G contains an (x, z)-path of length at least r.*

The following graph shows that one cannot replace the minimum degree condition in Theorem 2 by the average degree condition. Let H be a complete graph on $n - 1$ vertices and $x, z \in V(H)$. Let G be a graph obtained from H by adding a new vertex y and two edges xy, yz. Then the length of the longest (x, y, z)-path in G is 2, less than the average degree of the vertices other than x and z when $n \geq 5$.

In this paper, we first generalize Theorem 3 to k-connected graphs and get the following result.

Theorem 4. *Let G be a k-connected graph with $k \geq 2$, and x and z be two distinct vertices of G. If the average degree of the vertices other than x and z is at least r, then for any subset Y of $V(G)$ with $|Y| = k - 2$, G contains an (x, Y, z)-path of length at least r.*

We postpone the proof of Theorem 4 to Section 3.

Now we consider long cycles passing through specified vertices in graphs. Theorem 5 shows the existence of long cycles in 2-connected graph under the minimum degree condition, and Theorem 6 extends Theorem 5 to k-connected graphs.

Theorem 5 (Locke [8])**.** *Let G be a 2-connected graph. If the minimum degree of G is at least d, then for any two vertices y_1 and y_2 of G, G contains either a $\{y_1, y_2\}$-cycle of length at least $2d$ or a Hamilton cycle.*

Theorem 6 (Egava, Glas and Locke [3])**.** *Let G be a k-connected graph with $k \geq 2$. If the minimum degree of G is at least d, then for any subset Y of $V(G)$ with $|Y| = k$, G contains either a Y-cycle of length at least $2d$ or a Hamilton cycle.*

On the existence of long cycles in graphs with a given number of edges, Erdös and Gallai gave the following result.

Theorem 7 (Erdös and Gallai [5])**.** *Let G be a 2-edge-connected graph on n vertices. Then G contains a cycle of length at least $\frac{2e(G)}{n-1}$.*

In this paper, as an application of Theorems 4, we give the following theorem on long cycles passing through specified vertices of graphs with a given number of edges.

Theorem 8. *Let G be a k-connected graph on n vertices with $k \geq 2$. Then for any subset Y of $V(G)$ with $|Y| = k-1$, G contains a Y-cycle of length at least $\frac{2e(G)}{n-1}$.*

In Theorem 8, one cannot expect a cycle passing through k specified vertices of length at least $2e(G)/(n-1)$. Let H be a complete graph on $n-k$ vertices with $n > 3k$ and $u_1, u_2, \ldots, u_k$ be k vertices of H. Let $Y = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices not in $V(H)$. We construct a graph G with $V(G) = V(H) \cup Y$ and $E(G) = E(H) \cup \{u_i v_j : 1 \leq i, j \leq k\}$. Then G is a k-connected graph and the longest Y-cycle has length $2k$, which is less than

$$\frac{2e(G)}{n-1} = \frac{(n-k)(n-k-1) + 2k^2}{n-1}.$$

We postpone the proof of Theorem 8 in Section 4.

2 Preliminaries

Let G be a graph and P, H two disjoint subgraphs of G. We use $E(P, H)$ to denote the set, and $e(P, H)$ the number, of edges with one vertex in P and the other in H. If $E(P, H) \neq \emptyset$, then we call P and H are *joined*. We use $N_P(H)$ to denote the set of vertices in P which are joined to H. If x is a vertex in $G-P$, we say that x is *locally k-connected* to P (in G) if there are k paths connecting x to vertices in P such that any two of them have only the vertex x in common. We say that H is *locally k-connected* to P (in G) if for every vertex $x \in V(H)$, x is locally k-connected to P. Note that if H is locally k-connected to P, then H is locally l-connected to P for all l, $0 \le l \le k$; and, if G is k-connected and $|V(P)| \ge k$, then H is locally k-connected to P in G.

The following propositions on local k-connectedness are proved in [6].

Proposition 1 (Fan [6])**.** *Let H and P be two disjoint subgraphs of a graph G. If H is locally k-connected to P in the subgraph induced by $V(H) \cup V(P)$, then $E(P,H)$ contains an independent set of t edges, where $t \ge \min\{k, |V(H)|\}$.*

Proposition 2 (Fan [6])**.** *Let H and P be two disjoint subgraphs of a graph G. Let $u \in N_P(H)$ and G' be the graph obtained from G by deleting all edges from u to H. If H is locally k-connected to P in G, then H is locally $(k-1)$-connected to P in G'.*

Proposition 3 (Fan [6])**.** *Let H and P be two disjoint subgraphs of a graph G, and B a block of H. Let H' be the subgraph obtained from H by contracting B. If H is locally k-connected to P in G, then H' is also locally k-connected to P in the resulting graph.*

Next we introduce the concept of local maximality for paths.

Let P be a path of a graph G, and $u, v \in V(P)$. We use $P[u,v]$ to denote the segment of P from u to v, and $P(u,v)$ the segment obtained from $P[u,v]$ by deleting the two end vertices u and v. Let H be a component of $G-P$. We say that P is a *locally longest path with respect to H* if we cannot obtain a longer path than P by replacing the segment $P[u,v]$ by a (u,v)-path with all its internal vertices in H. In other words, P is locally longest with respect to H if, for any $u, v \in V(P)$,

$$e(P[u,v]) \ge d_H^*(u,v).$$

If P is an (x,Y,z)-path of G, where $x, z \in V(G)$ and $Y \subset V(G)$, then we say that P is a *locally longest (x,Y,z)-path with respect to H* if we cannot obtain a longer (x,Y,z)-path than P by replacing the segment $P[u,v]$ with $Y \cap V(P(u,v)) = \emptyset$ by a (u,v)-path with all its internal vertices in H. Note that if P is a longest path (longest (x,Y,z)-path) in a graph G, then, of course, P is a locally longest path (locally longest (x,Y,z)-path) with respect to any component of $G-P$. If two vertices u and u' in $V(P)$ are joined to H by two independent edges, then we call $\{u,u'\}$ a *strong attached pair* of H to P. A *strong attachment* of H to P (in G) is a subset $T = \{u_1, u_2, \ldots, u_t\} \subset N_P(H)$, where u_i,

$1 \le i \le t$, are in order along P, such that each ordered pair $\{u_i, u_{i+1}\}$, $1 \le i \le t-1$, is a strong attached pair of H to P. A strong attachment T of H to P is *maximum* if it has maximum cardinality over all strong attachments of H to P.

Lemma 1 (Fan [6]). *Let G be a graph and P a path of G. Suppose that H is a component of $G-P$ and $T=\{u_1,u_2,\ldots,u_t\}$ is a maximum strong attachment of H to P. Set $S=N_P(H)\backslash T$ and $s=|S|$. Then the following statements are true:*
(1) Every vertex in S is joined to exactly one vertex in H.
(2) For each segment $P[u_i,u_{i+1}]$, $1 \le i \le t-1$, suppose that

$$N_P(H)\cap V(P[u_i,u_{i+1}])=\{a_0,a_1,\ldots,a_q,a_{q+1}\},$$

where $a_0=u_i$, $a_{q+1}=u_{i+1}$ and a_j, $0\le j\le q+1$, are in order along P. Then there is a subscript m, $0\le m\le q$, such that

$$N_H(a_j)=N_H(a_0),\ \textit{for } 0\le j\le m,$$

and

$$N_H(a_j)=N_H(a_{q+1}),\ \textit{for } m+1\le j\le q+1.$$

Besides, if

$$N_P(H)\cap V(P[x,u_1])=\{a_1,\ldots,a_q,a_{q+1}\},$$

where, $a_{q+1}=u_1$, then

$$N_H(a_j)=N_H(a_{q+1}),\ \textit{for } 1\le j\le q+1;$$

and if

$$N_P(H)\cap V(P[u_t,z])=\{a_0,a_1,\ldots,a_q\},$$

where, $a_0=u_t$, then

$$N_H(a_j)=N_H(a_0),\ \textit{for } 0\le j\le q.$$

(3) If H is locally k-connected to P in G, then

$$t\ge\min\{k,h+d_2\},$$

where $h=|V(H)|$ and d_2 is the number of vertices in $N_P(H)$ which has at least two neighbors in H.

Lemma 1 (2) is somewhat different from that in [6], but the proofs of them are similar.

For a path P, we use $l(P)$ to denote the length of P.

Lemma 2. *Let G be a graph, P an (x, Y, z)-path of G, where $x, z \in V(G)$ and $Y \subset V(G)$, H a component of $G - P$ and $T = \{u_1, u_2, \ldots, u_t\}$ a maximum strong attachment of H to P. Set $S = N_P(H) \backslash T$ and $s = |S|$. Suppose that P is a locally longest (x, Y, z)-path with respect to H, and $\theta = |\{x, z\} \cap N_P(H)|$. Set*

$$T_r = \{u_i \in T \backslash \{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\} \text{ and } t_r = |T_r|.$$

Then

$$l(P) \geq \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta.$$

Proof. If $t = 0$, then $s = 0$ and the statement is trivially true. Suppose now that $t \geq 1$.

Consider a segment $P[u_i, u_{i+1}]$, $1 \leq i \leq t - 1$. Suppose that

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \ldots, a_q, a_{q+1}\},$$

where $q = |S \cap V(P[u_i, u_{i+1}])|$, $a_0 = u_i$, $a_{q+1} = u_{i+1}$, and a_j, $0 \leq j \leq q + 1$, are in order along P.

If $Y \cap V(P(u_i, u_{i+1})) = \emptyset$, then by Lemma 1 (2), there is a subscript m, $0 \leq m \leq q$, such that

$$N_H(a_0) = N_H(a_m) \text{ and } N_H(a_{q+1}) = N_H(a_{m+1}).$$

Therefore

$$d_H^*(a_m, a_{m+1}) = d_H^*(a_0, a_{q+1}) = d_H^*(u_i, u_{i+1}).$$

Since P is a locally longest (x, Y, z)-path with respect to H, we have

$$\begin{aligned} l(P[u_i, u_{i+1}]) \geq \sum_{j=0}^{q} d_H^*(a_j, a_{j+1}) &= d_H^*(a_m, a_{m+1}) + \sum_{\substack{j=0 \\ j \neq m}}^{q} d_H^*(a_j, a_{j+1}) \\ &= d_H^*(u_i, u_{i+1}) + \sum_{\substack{j=0 \\ j \neq m}}^{q} d_H^*(a_j, a_{j+1}). \end{aligned}$$

Note that $d_H^*(a_j, a_{j+1}) \geq 2$, for every j, $0 \leq j \leq q$, we have

$$l(P[u_i, u_{i+1}]) \geq d_H^*(u_i, u_{i+1}) + 2q.$$

If $Y \cap V(P(u_i, u_{i+1})) \neq \emptyset$, then noting that $l(P[a_j, a_{j+1}]) \geq 2$, we have

$$l(P[u_i, u_{i+1}]) = \sum_{j=0}^{q} l(P[a_j, a_{j+1}]) \geq 2q + 2.$$

Besides, consider the two segments $P[x, u_1]$ and $P[u_t, z]$. Suppose that

$$N_P(H) \cap V(P[x, u_1]) = \{a_0, a_1, \ldots, a_m\}$$

and

$$N_P(H) \cap V(P[u_t, z]) = \{a_{m+1}, a_{m+2}, \ldots, a_{q+1}\},$$

where $m = |S \cap V(P[x, u_1])|$, $q - m = |S \cap V(P[u_t, z])|$, $a_m = u_1$, $a_{m+1} = u_t$, and a_j, $0 \leq j \leq q + 1$ are in order along P. Note that $l(P[x, a_0]) + l(P[a_{q+1}, z]) \geq 2 - \theta$ and $l(P[a_j, a_{j+1}]) \geq 2$, for every $0 \leq j \leq q$, and $j \neq m$, we have

$$l(P[x, u_1]) + l(P[u_t, z]) \geq 2q + 2 - \theta.$$

Thus summing over the lengths of all the segments, yields

$$\begin{aligned} l(P) &= l(P[x, u_1]) + \sum_{i=1}^{t-1} l(P[u_i, u_{i+1}]) + l(P[u_t, z]) \\ &\geq 2(|S \cap V(P[x, u_1])| + |S \cap V(P[u_t, z])|) + 2 - \theta \\ &+ \sum_{\substack{i=1 \\ u_i \in T_r}}^{t-1} (d^*_H(u_i, u_{i+1}) + 2|S \cap V(P[u_i, u_{i+1}])|) + \sum_{\substack{i=1 \\ u_i \notin T_r}}^{t-1} (2|S \cap V(P[u_i, u_{i+1}])| + 2) \\ &= \sum_{u_i \in T_r} d^*_H(u_i, u_{i+1}) + 2(s + t - t_r) - \theta. \end{aligned}$$

This ends the proof. $\square$

In the following, we call a strong attached pair $\{u_j, u_{j+1}\}$ of H to P in G *transitive* if $Y \cap V(P(u_j, u_{j+1})) = \emptyset$.

Lemma 3. *Let G be a graph and P a path of G. Suppose that H is a separable component of $G - P$, B is an endblock of H, b is the cut vertex of H contained in B, $M = B - b$. Let $T = \{u_1, u_2, \ldots, u_t\}$ be a maximum strong attachment of H to P. If H is locally k-connected to P, then*

(1) $|N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\}$; and

(2) there exist at least $\min\{k - 1, m + d'_2\}$ strong attached pairs which are joined to M, where $m = |V(M)|$ and d'_2 is the number of vertices in $N_P(M)$ which has at least two neighbors in H.

Proof. Since H is locally k-connected to P, $|V(P)| \geq k$. It is easy to know that M is locally $(k-1)$-connected to P in the subgraph induced by $V(P) \cup V(M)$. By Proposition 1, there are $\min\{k-1, m\}$ independent edges in $E(P, M)$. Let v_iw_i, $1 \leq i \leq \min\{k-1, m\}$ be such edges, where $v_i \in V(P)$ and $w_i \in V(M)$.

If v_i has at least two neighbors in H, then by Lemma 1 (1), $v_i \in T$. If v_i has only one neighbor w_i in H, then by Lemma 1 (2), there exists a vertex v'_i (maybe $= v_i$) in T which also has only one neighbor w_i in H. This implies that $|N_P(M) \cap T| \geq \min\{k - 1, m\}$.

Now, we prove (1) by induction on d_2'. If $d_2' = 0$, then by the analysis above, the assertion is true. Thus we assume that $d_2' \geq 1$.

Let u_j be a vertex in $N_P(M)$ which has at least two neighbors in H (u_j is of course in T by Lemma 1 (1)). Let G' be the graph obtained from G by deleting all edges from u_j to H. By Proposition 2, H is locally $(k-1)$-connected to P in G'.

If $u_j = u_1$ or u_t, or $\{u_{j-1}, u_{j+1}\}$ are joined to H by two independent edges, then $T' = T \backslash \{u_j\}$ is a strong attachment of H to P in G'. Since u_j is joined to at least two vertices of H in G, any strong attachment of H to P in G' together with u_j is a strong attachment of H to P in G. Since $|T'| = t-1$, we see that T' is a maximum strong attachment of H to P in G'. By the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k-2, m + d_2' - 1\}.$$

Therefore

$$|N_P(M) \cap T| \geq \min\{k-1, m + d_2'\},$$

as required.

If $u_j \in \{u_2, \ldots, u_{t-1}\}$, and $\{u_{j-1}, u_{j+1}\}$ are not joined to H by two independent edges, i.e.,

$$N_H(u_{j-1}) = N_H(u_{j+1}) = \{w\},$$

for some $w \in V(H)$, then

$$T' = T \backslash \{u_j, u_{j+1}\} = \{u_1, \ldots, u_{j-1}, u_{j+2}, \ldots, u_t\}$$

is a strong attachment of H to P in G'. We prove now that T' is maximum by showing that any strong attachment of H to G' has cardinality at most $t - 2 = |T'|$.

Let v_1, v_2 ($\neq u_j$) be the two vertices in $N_P(H)$ which are closest to u_j on P, say v_1 preceding, and v_2 following, u_j on P (but not necessarily adjacent to u_j on P). Since $|N_H(u_j)| \geq 2$ and by Lemma 1 (2),

$$N_H(v_1) = N_H(u_{j-1}) = \{w\} = N_H(u_{j+1}) = N_H(v_2).$$

By the choice of v_1 and v_2, for any maximum strong attachment $\{a_1, a_2, \ldots, a_p\}$ of H to P in G', there is an integer l, $0 \leq l \leq p$, such that $v_1, v_2 \in V(P[a_l, a_{l+1}])$, where $a_0 = x$ and $a_{p+1} = z$. Since $N_H(v_1) = \{w\} = N_H(v_2)$, it follows from Lemma 1 (2) that either $N_H(a_l)$ or $N_H(a_{l+1}) = \{w\}$. The former implies a strong attachment $\{a_1, \ldots, a_l, u_j, v_2, a_{l+1}, \ldots, a_p\}$, the latter a strong attachment $\{a_1, \ldots, a_l, v_1, u_j, a_{l+1}, \ldots, a_p\}$, of H to P in G; in either case we have that $p + 2 \leq t$, that is, $p \leq t - 2 = |T'|$. This shows that T' is a maximum strong attachment of H to P in G', as claimed. As before, by the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k-2, m+d_2'-1\}.$$

Consequently

$$|N_P(M) \cap T| \geq \min\{k-1, m+d_2'\},$$

which completes the proof of (1).

Now we prove (2). Clearly for every vertex $u_j \in N_P(M) \cap T \backslash \{u_t\}$, the strong attached pair $\{u_j, u_{j+1}\}$ is joined to M. If $|N_P(M) \cap T \backslash \{u_t\}| \geq \min\{k-1, m+d_2'\}$, then the assertion is true. By (1), we assume that $|N_P(M) \cap T| = \min\{k-1, m+d_2'\}$ and $u_t \in N_P(M) \cap T$.

By Lemma 1 (3), $t \geq \min\{k, h+d_2\} \geq \min\{k-1, m+d_2'\}+1$. This implies that there exists at least one vertex in $T \backslash N_P(M)$. We chose a vertex $u_i \in T \backslash N_P(M)$ such that $u_{i+1} \in N_P(M) \cap T$. Then $\{u_i, u_{i+1}\}$ together with $\{u_j, u_{j+1}\}$ for $u_j \in N_P(M) \cup T \backslash \{u_t\}$ are $\min\{k-1, m+d_2'\}$ strong attached pairs joined to M. □

In the following, we call a strong attached pair which is joined to M a *good pair* (with respect to M). Let $\{u_j, u_{j+1}\}$ be a strong attached pair. If one of the vertices in $\{u_j, u_{j+1}\}$ is joined to M, and the other to $H-M$, then we call it a *better pair* (with respect to M); and if one of the vertices in $\{u_j, u_{j+1}\}$ is joined to M, and the other to $H-B$, then we call it a *best pair* (with respect to M).

3 Proof of Theorem 4

In order to prove the theorem, we chose a longest (x, Y, z)-path P in G. Clearly $|V(P)| \geq k$. Moreover, by the k-connectedness of G, for each component H of $G-P$, H is locally k-connected to P, and P is a locally longest (x, Y, z)-path with respect to H. So it is sufficient to prove that:

Proposition 4. *Let G be a graph, P an (x, Y, z)-path of G, where $x, z \in V(G)$, $Y \subset V(G)$, and $|Y| = k-2$. Suppose that the average degree of vertices in $V(G) \backslash \{x, z\}$ is r. If for each component H of $G-P$, H is locally k-connected to P, and P is a locally longest (x, Y, z)-path with respect to H, then $l(P) \geq r$.*

Proof. We prove this proposition by induction on $|V(G-P)|$. If $V(G-P) = \emptyset$, note that $r \leq |V(G)|-1$, the result is trivially true. So we assume that $V(G-P) \neq \emptyset$. Let H be a component of $G-P$.

Let $d = |N_P(H)|$, $\theta = |\{x,z\} \cap N_P(H)|$ and $N_P(H) = \{v_1, v_2, \ldots, v_d\}$, where v_i, $1 \le i \le d$, are in order along P. Then, we have

$$l(P) = l(P[x, v_1]) + \sum_{i=1}^{d-1} l(P[v_i, v_{i+1}]) + l(P[v_d, z]).$$

It is easy to know that $l(P[x,v_1]) + l(P[v_d,z]) \ge 2-\theta$ and $l(P[v_i, v_{i+1}]) \ge 2$ for $1 \le i \le d-1$. Thus, we have

$$l(P) \ge 2d - \theta.$$

Note that $d \ge k$ by the local k-connectedness of H to P and clearly $\theta \le 2$. If $r \le 2k-2$, then we have $l(P) \ge 2k-2 \ge r$, and the proof is complete. Thus we assume that

$$r > 2k - 2. \tag{1}$$

Besides, if $d \ge (r+\theta)/2$, then $l(P) \ge r$, and we complete the proof. Thus, we assume that

$$d < (r + \theta)/2. \tag{2}$$

Let $T = \{u_1, u_2, \ldots, u_t\}$ be a maximum strong attachment of H to P. Set $S = N_P(H) \backslash T$ and $s = |S|$ (note that $s + t = d$). Let $T_r = \{u_i \in T \backslash \{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\}$ and $t_r = |T_r|$.

Clearly, for every transitive strong attached pair $\{u_j, u_{j+1}\}$, where $u_j \in T_r$, we have

$$d^*_H(u_j, u_{j+1}) \ge 2. \tag{3}$$

We distinguish two cases:

Case 1. H is nonseparable.

Let $h = |V(H)|$ and r' the average degree of vertices in $V(H)$. If $r'h + e(P - \{x,z\}, H) \le rh$, then we consider the graph G' obtained from G by deleting the component H. Note that

$$\begin{aligned}\sum_{v \in V(G') \backslash \{x,z\}} d_{G'}(v) &= r(|V(G)|-2) - r'h - e(P - \{x,z\}, H) \\ &\ge r(|V(G)|-2) - rh \\ &= r(|V(G')|-2).\end{aligned}$$

By the induction hypothesis, we have $l(P) \ge r$, and the proof is complete. Thus we assume that

$$r'h + e(P - \{x,z\}, H) > rh \tag{4}$$

We use d_1 to denote the number of vertices in $N_P(H)$ which have only one neighbor in $V(H)$, $d_2 = d - d_1$, θ_1 to denote the number of vertices in $\{x, z\}$ which have only one neighbor in $V(H)$ and $\theta_2 = \theta - \theta_1$.

Clearly,

$$r'h \le h(h - 1 + d_2) + d_1 \text{ and } e(P - \{x, z\}, H) \le h(d_2 - \theta_2) + d_1 - \theta_1.$$

Thus, by (4), we have

$$h(h - 1 + 2d_2 - \theta_2) + 2d_1 - \theta_1 \ge r'h + e(P - \{x, z\}, H) > rh.$$

Note that $d_1 = d - d_2$ and $\theta_1 = \theta - \theta_2$, we have

$$h(h - 1 + 2d_2 - \theta_2) + 2d - 2d_2 - \theta + \theta_2 \ge rh.$$

By (2), we have

$$h(h - 1 + 2d_2 - \theta_2) + (r + \theta) - 2d_2 - \theta + \theta_2 > rh.$$

Thus

$$(h - 1)(h + 2d_2 - r - \theta_2) > 0.$$

This implies that $h \ge 2$ and $h + 2d_2 > r + \theta_2 \ge r$, and then $2h + 2d_2 > r + 2$. By (1), we have $2h + 2d_2 > 2k$, that is

$$h + d_2 > k. \tag{5}$$

By (5) and Lemma 1 (3), $t \ge k$. Since $|Y| \le k - 2$, there exists at least one transitive strong attached pair (u_p, u_{p+1}) in T, where $u_p \in T_r$.

Let G' be the subgraph induced by $V(H) \cup \{u_p, u_{p+1}\}$. If $u_p u_{p+1} \notin E(G)$, we add the edge $u_p u_{p+1}$ in G'. Thus G' is 2-connected and

$$\begin{aligned}
\sum_{v \in V(G') \backslash \{u_p, u_{p+1}\}} d_{G'}(v) &= \sum_{v \in V(H)} d(v) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H) \\
&= r'h - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H) \\
&\ge rh - e(P - \{x, z\}, H) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H).
\end{aligned}$$

Note that

$$\begin{aligned}
&e(P - \{x, z\}, H) \le (s + t - \theta)h, \text{ and} \\
&e(N_P(H) \backslash \{u_p, u_{p+1}\}, H) \le (s + t - 2)h,
\end{aligned}$$

we have

$$\sum_{v\in V(G')\backslash\{u_p,u_{p+1}\}} d_{G'}(v) \geq rh-(s+t-\theta)h-(s+t-2)h$$
$$=(r-2s-2t+\theta+2)h.$$

By Theorem 3, G' contains a (u_p,u_{p+1})-path of length at least $r-2s-2t+\theta+2$, which implies that

$$d^*_H(u_p,u_{p+1}) \geq r-2s-2t+\theta+2. \tag{6}$$

Substituting (6) for $d^*_H(u_p,u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P) \geq (r-2s-2t+\theta+2)+2(t_r-1)+2(s+t-t_r)-\theta \geq r.$$

Case 2. H is separable.

Let B be an endblock of H, b the cut vertex of H contained in B, $M=B-b$, $m=|V(M)|$, and r'' the average degree of the vertices in $V(M)$.

If $r''m+e(P-\{x,z\},M)+d_M(b)\leq rm$, then we consider the graph G' obtained from G by contracting B. Let H' be the component of $G'-P$ obtained from H by contracting B. By Proposition 3, H' is locally k-connected to P. Clearly P is a locally longest (x,Y,z)-path with respect to H', and

$$\sum_{v\in V(G')\backslash\{x,z\}} d_{G'}(v) \geq \sum_{v\in V(G)\backslash\{x,z\}} d(v)-r''m-e(P-\{x,z\},M)-d_M(b)$$
$$\geq r(|V(G)|-2)-rm$$
$$=r(|V(G')|-2).$$

By the induction hypothesis, $l(P)\geq r$, and the proof is complete. Thus we assume that

$$r''m+e(P-\{x,z\},M)+d_M(b)>rm. \tag{7}$$

Let $d'_0=|N_P(H)\backslash N_P(M)|$, d'_1 be the number of vertices in $N_P(M)$ which have only one neighbor in $V(H)$, $d'_2=d-d'_0-d'_1$; $\theta'_0=|\{x,z\}\cap N_P(H)\backslash N_P(M)|$, θ'_1 be the number of vertices in $\{x,z\}\cap N_P(M)$ which have only one neighbor in $V(H)$ and $\theta'_2=\theta-\theta'_0-\theta'_1$.

Now we prove that

$$m+d'_2\geq k-1. \tag{8}$$

Let B' be an endblock of H other than B, b' the cut vertex of H contained in B', $M'=B'-b'$ and $m'=|V(M')|$.

By the local k-connectedness of H to P, $|N_P(M')| \geq k-1$. If $|N_P(M')\backslash N_P(M)| \leq m$, then $d_2' \geq |N_P(M) \cap N_P(M')| \geq k-1-m$, and $m + d_2' \geq k-1$, and (8) holds. Thus we assume that $|N_P(M')\backslash N_P(M)| \geq m+1$. So we have

$$d_0' \geq m+1. \tag{9}$$

Clearly,

$$\begin{aligned}
&r''m \leq m(m+d_2') + d_1',\\
&e(P - \{x,z\}, M) \leq m(d_2' - \theta_2') + d_1' - \theta_1', \text{ and}\\
&d_M(b) \leq m.
\end{aligned}$$

Thus, by (7),

$$m(m + 2d_2' + 1 - \theta_2') + 2d_1' - \theta_1' \geq r''m + e(P - \{x,z\}, M) + d_M(b) > rm.$$

Note that $d_1' = d - d_0' - d_2'$ and $\theta_1' = \theta - \theta_0' - \theta_2'$, we have

$$m(m + 2d_2' + 1 - \theta_2') + 2d - 2d_0' - 2d_2' - \theta + \theta_0' + \theta_2' > rm.$$

By (2) and (9), we have

$$m(m + 2d_2' + 1 - \theta_2') + (r+\theta) - 2(m+1) - 2d_2' - \theta + \theta_0' + \theta_2' > rm.$$

Thus

$$(m-1)(m + 2d_2' - r - \theta_2') > 2 - \theta_0' \geq 0.$$

This implies that $m \geq 2$ and $m + 2d_2' > r + \theta_2' \geq r$, and then $2m + 2d_2' > r+2$. By (1), $2m + 2d_2' > 2k$, that is $m + d_2' > k$, and (8) holds.

By Lemma 3 (2), there exist at least $k-1$ good pairs with respect to M. Since $|Y| = k-2$, there exists at least one transitive good pair $\{u_p, u_{p+1}\}$ with respect to M. Similarly there exists at least one transitive good pair $\{u_q, u_{q+1}\}$ with respect to M'.

First we assume that there is a transitive best pair with respect to M or M'. Without loss of generality, we assume that $\{u_p, u_{p+1}\}$ is a best pair, where $u_p \in N_P(M)$ and $u_{p+1} \in N_P(H-B)$. Consider the subgraph G' induced by $V(B) \cup \{u_p\}$. If $u_pb \notin E(G)$, we add the edge u_pb in G'. Thus G' is 2-connected and

$$\begin{aligned}
\sum_{v \in V(G')\backslash\{u_p,b\}} d_{G'}(v) &= \sum_{v \in V(M)} d(v) - e(N_P(H)\backslash\{u_p\}, M)\\
&= r''m - e(N_P(H)\backslash\{u_p\}, M)\\
&\geq rm - e(P - \{x,z\}, M) - d_M(b) - e(N_P(H)\backslash\{u_p\}, M).
\end{aligned}$$

Note that

$$e(P - \{x,z\}, M) \le (s+t-\theta)m,$$
$$d_M(b) \le m, \text{ and}$$
$$e(N_P(H)\backslash\{u_p\}, M) \le (s+t-1)m,$$

we have

$$\sum_{v \in V(G')\backslash\{u_p,b\}} d_{G'}(v) \ge rm - (s+t-\theta)m - m - (s+t-1)m$$
$$= (r - 2s - 2t + \theta)m.$$

By Theorem 3, G' contains a (u_p, b)-path of length at least $r - 2s - 2t + \theta$. It is clear that there is a (b, u_{p+1})-path in $H - B$ of length at least 2, which implies that

$$d^*_H(u_p, u_{p+1}) \ge r - 2s - 2t + \theta + 2. \tag{10}$$

Substituting (10) for $d^*_H(u_p, u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P) \ge (r - 2s - 2t + \theta + 2) + 2(t_r - 1) + 2(s + t - t_r) - \theta \ge r,$$

as required.

So, we assume that there are no transitive best pairs with respect to M or M'.

Now we assume that there is a transitive better pair (but not best pair) with respect to M or M'. Without loss of generality, we assume that $\{u_p, u_{p+1}\}$ is a better pair, where $u_p \in N_P(M)$ and $u_{p+1} \in N_P(b)$. Consider the subgraph G' induced by $V(B) \cup \{u_p\}$. If $u_p b \notin E(G)$, we add the edge $u_p b$ in G'. Thus G' is 2-connected and

$$\sum_{v \in V(G')\backslash\{u_p,b\}} d_{G'}(v) \ge rm - e(P - \{x,z\}, M) - d_M(b) - e(N_P(H)\backslash\{u_p\}, M).$$

Note that

$$e(P - \{x,z\}, M) \le (s+t-\theta)m, \text{ and}$$
$$d_M(b) \le m,$$

and since at least one vertex of u_q and u_{q+1} is not joined to M (otherwise, $\{u_q, u_{q+1}\}$ will be a best pair), we have

$$e(N_P(H)\backslash\{u_p\}, M) \le (s+t-2)m.$$

Thus we have

$$\sum_{v\in V(G')\backslash\{u_p,b\}} d_{G'}(v) \geq rm-(s+t-\theta)m-m-(s+t-2)m$$
$$=(r-2s-2t+\theta+1)m.$$

By Theorem 3, G' contains a (u_p,b)-path of length at least $r-2s-2t+\theta+1$, and then, by $bu_{p+1}\in E(G)$,

$$d_H^*(u_p,u_{p+1})\geq r-2s-2t+\theta+2.$$

Thus we also have $l(P)\geq r$.

So, we assume that there are no transitive better pairs with respect to M or M'. Thus $\{u_p,u_{p+1}\}$ and $\{u_q,u_{q+1}\}$ are two distinct strong attached pairs.

If $m=1$, then $\{u_p,u_{p+1}\}$ will be a better pair with respect to M. Thus we assume that $m\geq 2$.

If $m=2$, then B is a triangle, and $d_H^*(u_p,u_{p+1})=4$. Since $\{u_p,u_{p+1}\}$ is not a better pair, we have that $u_p\in N_P(M)$. Similar to the analysis above, we have $d_H^*(u_p,b)\geq r-2s-2t+\theta+1$. But $d_H^*(u_p,b)=3$, we have

$$d_H^*(u_p,u_{p+1})=4\geq r-2s-2t+\theta+2.$$

Then $l(P)\geq r$.

So we assume that

$$m\geq 3,\ \text{and similarly, } m'\geq 3. \tag{11}$$

It is easy to know that $d_H^*(u_p,u_{p+1})\geq 4$. Thus if $r-2s-2t+\theta\leq 2$, we will have

$$d_H^*(u_p,u_{p+1})\geq r-2s-2t+\theta+2,$$

and then $l(P)\geq r$. So we assume that

$$r-2s-2t+\theta\geq 2. \tag{12}$$

Note that u_p and u_{p+1} are joined to B by two independent edges. Consider the subgraph G' induced by $V(B)\cup\{u_p,u_{p+1}\}$. If $u_pu_{p+1}\notin E(G)$, we add the edge u_pu_{p+1} in

G'. Thus G' is 2-connected and

$$\begin{aligned}
&\sum_{v\in V(G')\backslash\{u_p,u_{p+1}\}} d_{G'}(v)\\
&= \sum_{v\in V(M)} d(v) - e(N_P(H)\backslash\{u_p,u_{p+1}\},M) + d_M(b) + |\{u_p,u_{p+1}\}\cap N(b)|\\
&= r''m + d_M(b) - e(N_P(H)\backslash\{u_p,u_{p+1}\},M) + |\{u_p,u_{p+1}\}\cap N(b)|\\
&\geq rm - e(P-\{x,z\},M) - e(N_P(H)\backslash\{u_p,u_{p+1}\},M).
\end{aligned}$$

Note that

$$\begin{aligned}
&e(P-\{x,z\},M) \leq (s+t-\theta)m, \text{ and}\\
&e(N_P(H)\backslash\{u_p,u_{p+1}\},M) \leq (s+t-2)m,
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{v\in V(G')\backslash\{u_p,u_{p+1}\}} d_{G'}(v) &\geq rm - (s+t-\theta)m - (s+t-2)m\\
&= (r-2s-2t+\theta+2)m.
\end{aligned}$$

By Theorem 3, G' contains a (u_p,u_{p+1})-path of length at least $(r-2s-2t+\theta+2)m/(1+m)$, which implies that

$$\begin{aligned}
d_H^*(u_p,u_{p+1}) &\geq (r-2s-2t+\theta+2)\frac{m}{1+m}\\
&\geq \frac{3}{4}(r-2s-2t+\theta+2).
\end{aligned}$$

(note that $m\geq 3$), and similarly,

$$d_H^*(u_q,u_{q+1}) \geq \frac{3}{4}(r-2s-2t+\theta+2).$$

Then by (12),

$$\begin{aligned}
&d_H^*(u_p,u_{p+1}) + d_H^*(u_q,u_{q+1})\\
&\qquad\geq \frac{3}{2}(r-2s-2t+\theta+2)\\
&\qquad= (r-2s-2t+\theta+2) + \frac{1}{2}(r-2s-2t+\theta+2)\\
&\qquad\geq r-2s-2t+\theta+4.
\end{aligned}$$

Thus, by Lemma 2, we have

$$l(P) \geq (r-2s-2t+\theta+4) + 2(t_r-2) + 2(s+t-t_r) - \theta \geq r.$$

The proof is complete. $\square$

4 Proof of Theorem 8

By the k-connectedness of G, it contains a Y-cycle. If $2e(G)/(n-1) \leq 3$, then the result is trivially true. Thus we assume that $2e(G)/(n-1) > 3$.

We chose a vertex $y \in Y$, and construct a graph G' such that $V(G') = V(G) \cup \{y'\}$, where $y' \notin V(G)$ and $E(G') = E(G) \cup \{vy' : v \in N_G(y)\}$. Clearly, G' is k-connected. Besides, we have that

$$e(G') = e(G) + d_G(y) \text{ and } d_{G'}(y) = d_{G'}(y') = d_G(y),$$

and the order of G' is $n+1$. Now, by Theorem 4, there exists a $(y, Y \backslash \{y\}, y')$-path P of length at least

$$\frac{2e(G') - d_{G'}(y) - d_{G'}(y')}{(n+1)-2} = \frac{2(e(G) + d_G(y)) - 2d_G(y)}{n-1} = \frac{2e(G)}{n-1}.$$

Let uy' be the last edge of P, then $uy \in E(G)$ and $C = P[y,u]uy$ is a cycle of G passing through all the vertices in Y of length at least $2e(G)/(n-1)$, which completes the proof. $\square$

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