# On the full automorphism group of a Hamiltonian cycle system of odd order 

Marco Buratti * Graham J. Lovegrove ${ }^{\dagger}$ Tommaso Traetta ${ }^{\ddagger}$


#### Abstract

It is shown that a necessary condition for an abstract group $G$ to be the full automorphism group of a Hamiltonian cycle system is that $G$ has odd order or it is either binary, or the affine linear group $\operatorname{AGL}(1, p)$ with $p$ prime. We show that this condition is also sufficient except possibly for the class of non-solvable binary groups.


Keywords: Hamiltonian cycle system; automorphism group.

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## 1 Introduction

Denote as usual with $K_{v}$ the complete graph on $v$ vertices. A Hamiltonian cycle system of order $v$ (briefly, an $\operatorname{HCS}(v)$ ) is a set of Hamiltonian cycles of $K_{v}$ whose edges partition the edge-set of $K_{v}$. It is very well known [20] that an $\operatorname{HCS}(v)$ exists if and only if $v$ is odd and $v \geq 3$.

Two HCSs are isomorphic if there exists a bijection (isomorphism) between their set of vertices turning one into the other. An automorphism of a $\operatorname{HCS}(v)$ is an isomorphism of it with itself, i.e., a permutation of the vertices of $K_{v}$ leaving it invariant.

HCSs possessing a non-trivial automorphism group have drawn a certain attention (see [10] for a short recent survey on this topic). Detailed results can be found in: [11, 17] for the cyclic groups; [12] for the dihedral groups; [6] for the doubly transitive groups; 9 for the regular HCS; [1, 8] for the symmetric HCS; [13] for those being both cyclic and symmetric; [4, 15, 3] for the 1-rotational HCS and 2-pyramidal HCS.

Given a particular class of combinatorial designs, to establish whether any abstract finite group is the full automorphism group of an element in the class is in general a quite hard task. Some results in this direction can be found in [21] for the class of Steiner triple and quadruple systems, in [22] for the class of finite projective planes, in $[7$ for the class of non-Hamiltonian 2-factorizations of the complete graph, and very recently in [16, 18, 19] for the class of cycle systems.

This paper deals with the following problem:

> Determining the class $\mathcal{G}$ of finite groups that can be seen as the full automorphism group of a Hamiltonian cycle system of odd order.

As a matter of fact, some partial answers are known. In 15 it is proven that any symmetrically sequenceable group lies in $\mathcal{G}$ (see 3] for HCSs of even order). In particular, any solvable binary group (i.e. with a unique element of order 2) except for the quaternion group $\mathbb{Q}_{8}$, is symmetrically sequenceable [2], hence it can be seen as the full automorphism group of an HCS. In [6] it is shown that the affine linear group $A G L(1, p)$, $p$ prime, is the full automorphism group of the unique doubly transitive $\operatorname{HCS}(p)$.

Here we prove that any finite group $G$ of odd order is the full automorphism group of an HCS. This result will be achieved in Section 3 by means of a new doubling construction described in Section 2. On the other hand, in Section 3 we also prove that if $G \in \mathcal{G}$ has even order, then $G$ is necessarily binary or the affine linear group $\operatorname{AGL}(1, p)$ with $p$ prime; still, we show that $\mathbb{Q}_{8}$ lies in $\mathcal{G}$. We obtain, in this way, the major result of this paper:

Theorem 1.1. If a finite group $G$ is the full automorphism group of a Hamiltonian cycle system of odd order then $G$ has odd order or it is either binary, or the affine linear group $A G L(1, p)$ with $p$ prime. The converse is true except possibly in the case of $G$ binary non-solvable.

We therefore leave open the problem only for non-solvable binary groups.

## 2 A new doubling construction

We describe a new doubling construction that will allow us to constuct an $\operatorname{HCS}(4 n+1)$ starting from three $\operatorname{HCS}(2 n+1)$ not necessarily distinct.

For any even integer $n \geq 1$, take three $\operatorname{HCS}(2 n+1)$, say $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$, on the set $\{\infty\} \cup[2 n]$, where $[2 n]=\{1,2, \ldots, 2 n\}$. Denote by $A_{i}, B_{i}, C_{i}$, for $i \in[n]$, the cycles composing $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$, respectively, let
$A_{i}=\left(\infty, \alpha_{i, 1}, \ldots, \alpha_{i, 2 n}\right) \quad B_{i}=\left(\infty, \beta_{i, 1}, \ldots, \beta_{i, 2 n}\right) \quad C_{i}=\left(\infty, \gamma_{i, 1}, \ldots, \gamma_{i, 2 n}\right)$.
We need these HCS to satisfy the following property:

$$
\begin{equation*}
\alpha_{i 1}=\beta_{i 1}=\gamma_{i 1}, \quad \text { and } \quad \alpha_{i, 2 n}=\beta_{i, 2 n}=\gamma_{i, 2 n}, \quad \text { for } i \in[n] \tag{2.1}
\end{equation*}
$$

We construct an $\operatorname{HCS}(4 n+1) \mathcal{T}$ on the set $\{\infty\} \cup([2 n] \times\{1,-1\})$. For convenience, if $z=(x, y) \in[2 n] \times\{1,-1\}$, then the point $(x,-y)$ will be denoted by $z^{\prime}$.

Let $\mathcal{T}=\left\{T_{i 1}, T_{i 2} \mid i \in[n]\right\}$ be the set of $2 n$ cycles of length $4 n+1$ and vertex-set $\{\infty\} \cup([2 n] \times\{1,-1\})$ obtained from the cycles of $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$ as follows: set $a_{i j}=\left(\alpha_{i j}, 1\right), b_{i j}=\left(\beta_{i j}, 1\right)$, and $c_{i j}=\left(\gamma_{i j}, 1\right)$, for $i \in[n]$ and $j \in[2 n]$ and define the cycles $T_{i 1}, T_{i 2}$, of first and second type respectively, as follows:

$$
\begin{aligned}
& T_{i, 1}=\left(\infty, a_{i, 1}, a_{i, 2}, \ldots, a_{i, 2 n}, b_{i, 2 n}^{\prime}, b_{i, 2 n-1}^{\prime}, \ldots, b_{i 1}^{\prime}\right), \\
& T_{i, 2}=\left(\infty, c_{i, 2 n}, c_{i, 2 n-1}^{\prime}, c_{i, 2 n-2}, c_{i, 2 n-3}^{\prime}, \ldots, c_{i 1}^{\prime}, c_{i, 1}, c_{i, 2}^{\prime}, c_{i, 3}, c_{i, 4}^{\prime}, \ldots, c_{i, 2 n}^{\prime}\right)
\end{aligned}
$$

Remark 2.1. Note that, by construction, the neighbors of $\infty$ and the middle edge of any of the cycles of $\mathcal{T}$ are both pairs of the form $\left(z, z^{\prime}\right)$ for $z \in[2 n] \times\{-1,1\}$. Also, if $T_{1}=\left(\infty, z, \ldots, w, w^{\prime}, \ldots, z^{\prime}\right)$ is a cycle of $\mathcal{T}$, then there is a cycle $T_{2}=\left(\infty, w, \ldots, z^{\prime}, z, \ldots, w^{\prime}\right)$ of $\mathcal{T}$ of alternate type.

We first show that our doubling construction yields a Hamiltonian cycle system.

Lemma 2.1. $\mathcal{T}$ is an $\operatorname{HCS}(4 n+1)$.
Proof. Every unordered pair of form $\left(x_{1}, x_{2}\right)$ or $\left(\infty, x_{1}\right), x_{1}, x_{2} \in[2 n], x_{1} \neq x_{2}$ is contained in a unique cycle of each of $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$. The first type of cycle above contains all pairs of form $\left(\left(x_{1}, y\right),\left(x_{2}, y\right)\right), x_{1}, x_{2} \in[2 n], x_{1} \neq x_{2}, y=-1,1$, and the second cycle type contains all pairs of form $\left(\left(x_{1}, 1\right),\left(x_{2},-1\right)\right), x_{1}, x_{2} \in[2 n]$, $x_{1} \neq x_{2}$. We are left to show that $\mathcal{T}$ contains the following edges: $(\infty, z)$ and $\left(z, z^{\prime}\right)$ for $z \in[2 n] \times\{-1,1\}$.

In view of Property (2.1), we have that $a_{i j}=b_{i j}=c_{i j}$ for $j=1,2 n$. Therefore, the middle edges of the cycles in $\mathcal{T}$ are exactly $\left(a_{i 1}, a_{i 1}^{\prime}\right)$ and $\left(a_{i, 2 n}, a_{i, 2 n}^{\prime}\right)$ for $i \in[n]$. Considering that $\left\{a_{i 1}, a_{i, 2 n} \mid i \in[n]\right\}=[2 n] \times\{1\}$, we conclude that $\mathcal{T}$ covers the edges $\left(z, z^{\prime}\right)$ for $z \in[2 n] \times\{-1,1\}$.

Finally, the edges incident with $\infty$ and covered by $\mathcal{T}$ are the following: $\left(\infty, a_{i 1}\right),\left(\infty, b_{i 1}^{\prime}\right),\left(\infty, c_{i, 2 n}\right),\left(\infty, c_{i, 2 n}^{\prime}\right)$, for $i \in[n]$. With a reasoning similar to the former we easily see that all edges $(\infty, z)$ with $z \in[2 n] \times\{-1,1\}$ are covered by $\mathcal{T}$.

Example 2.2. Here we show how to construct an $H C S(13)$ by applying the doubling construction to three HCSs or order 7. Let $G=\langle g\rangle$ be the cyclic group of order 6 generated by $g$, and let $\mathcal{H}_{1}, \mathcal{H}_{2}$, and $\mathcal{H}_{3}$ denote the three HCSs of order 7 defined as follows:

1. $\mathcal{H}_{1}=\left\{A_{1}, A_{2}, A_{3}\right\}$ with $A_{1}=\left(\infty, 1, g, g^{5}, g^{2}, g^{4}, g^{3}\right)$ and $A_{i}=A_{1} \cdot g^{i-1}$ for $i=2,3$,
2. $\mathcal{H}_{2}=\left\{B_{1}, B_{2}, B_{3}\right\}$ with $B_{1}=\left(\infty, 1, g^{4}, g^{5}, g^{2}, g, g^{3}\right)$ and $B_{i}=B_{1} \cdot g^{i-1}$ for $i=2,3$,
3. $\mathcal{H}_{3}=\left\{C_{1}, C_{2}, C_{3}\right\}$ and $C_{i}=B_{i}$ for $i=1,2,3$ (hence, $\mathcal{H}_{3}=\mathcal{H}_{2}$ ),
where $A_{1} \cdot g^{i-1}\left(B_{1} \cdot g^{i-1}\right)$ is the cycle that we obtain by replacing each vertex of $A_{1}\left(B_{1}\right)$ different from $\infty$, say $x$, with $x \cdot g^{i-1}$. It is easy to check that $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are $H C S(7)$; also, property 2.1 is satisfied (see Figure 1).


Figure 1: Two HCSs of order 7.
The cycles $T_{i 1}$ and $T_{i 2}$ for $i=1,2,3$ are shown in Figure 2. Each cycle $T_{i 1}$ is basically constructed from the paths that we obtain from $A_{i}$ (continous path) and $B_{i}$ (dashed path) after removing $\infty$, by joining two of their ends with $\infty$ and joining to each other the other two ends (zigzag edges). The construction of each cycle $T_{i 2}$ is only based on $C_{i}=B_{i}$. Consider two copies of the path we obtain from $B_{i}$ (dashed path) after removing $\infty$ and replace the horizontal edges with the diagonal ones. At the end, we add the zigzag edges.

It is easy to check that the set $\mathcal{T}=\left\{T_{i 1}, T_{i 2} \mid i=1,2,3\right\}$ is an $\operatorname{HCS}(13)$.
Remark 2.2. This construction can be used to construct many different HCS from the same base systems by relabelling the vertices of one or more of $\mathcal{H}_{1}, \mathcal{H}_{2}, \mathcal{H}_{3}$ in a way that Property (2.1) is still satisfied.

Lemma 2.3. All automorphisms of $\mathcal{T}$ fix the $\infty$ point for $n>1$.


Figure 2: An HCS(13) resulting from the doubling condstruction.

Proof. Suppose there is an automorphism $\phi$ of $T$ which does not fix the $\infty$ point, and let $u=\phi(\infty)$ and $\phi^{-1}(\infty)=v$. Also denote $\phi\left(v^{\prime}\right)$ by $w$. The cycle $T_{1}$ containing $(\infty, v)$ as an edge has form: $\left(\infty, v, \ldots, z, z^{\prime}, \ldots, v^{\prime}\right)$. This is mapped by $\phi$ to the cycle $\phi\left(T_{1}\right)=\left(u, \infty, \ldots, \phi(z), \phi\left(z^{\prime}\right), \ldots, w\right)$.

Now consider the cycle $T_{2}$ that is of alternate type to $T_{1}$ and that has the middle edge ( $v, v^{\prime}$ ), i.e. $T_{2}=\left(\infty, z, \ldots, v^{\prime}, v, \ldots, z^{\prime}\right)$. Then $T_{2}$ is mapped to the cycle $\phi\left(T_{2}\right)=\left(u, \phi(z), \ldots, w, \infty, \ldots, \phi\left(z^{\prime}\right)\right)$. The middle edge of $\phi\left(C_{2}\right)$ is $(u, \phi(z))$, hence there exists a cycle where the neighbors of $\infty$ are $u$ and $\phi(z)$. Since $u$ is adjacent to $\infty$ in $\phi\left(T_{1}\right)$, then $\phi\left(C_{1}\right)=\left(u, \infty, \phi(z), \phi\left(z^{\prime}\right), w\right)$ and $n=1$.

## 3 HCSs with a prescribed full automorphism group

In this section we prove that any group of odd order lies in the class $\mathcal{G}$ of finite groups than can be seen as the full automorphism group of an HCS of odd order. After that, we prove that whenever a group $G \in \mathcal{G}$ has even order then it is either binary or the affine linear group $\operatorname{AGL}(1, p)$, with $p$ prime; also, we show that the quaternion group $\mathbb{Q}_{8}$ lies in $\mathcal{G}$. As mentioned in the introduction, it is known that any binary solvable group $\neq \mathbb{Q}_{8}$ [15] and $\operatorname{AGL}(1, p)$ ( $p$ prime) [6] lie in $\mathcal{G}$. Therefore, we leave open the problem of determining whether non-solvable binary groups lie in $\mathcal{G}$ as well.

In order to show that any group of odd order is the full automorphism group of a suitable $\mathrm{HCS}(4 n+1)$, we will need some preliminaries on 1-rotational HCSs.

We will use multiplicative notation to denote any abstract group; as usual, the unit will be denoted by 1 .

An $\operatorname{HCS}(2 n+1) \mathcal{H}$ is 1-rotational over a group $\Gamma$ of order $2 n$ if $\Gamma$ is an automorphism group of $\mathcal{H}$ acting sharply transitively on all but one vertex. In this case, it is natural to identify the vertex-set with $\{\infty\} \cup \Gamma$ where $\infty$ is the vertex fixed by any $g \in \Gamma$ and view the action of $\Gamma$ on the vertex-set as the right multiplication where $\infty \cdot g=\infty$ for $g \in \Gamma$.

It is known from [14] (as a special case of a more general result on 1-rotational 2 -factorizations of the complete graph) that $G$ is binary, namely, it has only one element $\lambda$ of order 2 . As usual, we denote by $\Lambda(\Gamma)=\{1, \lambda\}$ the subgroup of $\Gamma$ of order 2 .

In the same paper, the authors also prove that the existence of a 1-rotational $\operatorname{HCS}(2 n+1) \mathcal{H}$ is equivalent to the existence of a cycle $A=\left(\infty, \alpha_{1}, \ldots, \alpha_{2 n}\right)$ with vertex-set $\{\infty\} \cup \Gamma$ such that

$$
\begin{equation*}
A \cdot \lambda=A \quad \text { and } \quad\left\{\alpha_{i} \alpha_{i+1}^{-1}, \alpha_{i+1} \alpha_{i}^{-1} \mid i \in[n-1]\right\}=\Gamma \backslash\{1, \lambda\} \tag{3.1}
\end{equation*}
$$

In this case, $\alpha_{n+1-i}=\alpha_{i} \cdot \lambda$ and $\mathcal{H}=\{A \cdot x \mid x \in X\}$, where $X$ is a complete system of representatives for the cosets of $\Lambda(\Gamma)$ in $\Gamma$. This means that $\mathcal{H}$ is the set of distinct translates of any of its cycles.

We are now ready to prove the following result.
Theorem 3.1. Any group $G$ of odd order $n$ is the full automorphism group of a suitable $H C S(4 n+1)$.

Proof. Consider the set $\mathcal{H}$ of the following 13-cycles:

$$
\begin{aligned}
& \left(\infty, 0_{0}, 1_{0}, 5_{0}, 2_{0}, 4_{0}, 3_{0}, 3_{1}, 1_{1}, 2_{2}, 5_{1}, 4_{1}, 0_{1}\right) \\
& \left(\infty, 1_{0}, 2_{0}, 0_{0}, 3_{0}, 5_{0}, 4_{0}, 4_{1}, 2_{1}, 3_{1}, 0_{1}, 5_{1}, 1_{1}\right) \\
& \left(\infty, 2_{0}, 3_{0}, 1_{0}, 4_{0}, 0_{0}, 5_{0}, 5_{1}, 3_{1}, 4_{1}, 1_{1}, 0_{1}, 2_{1}\right) \\
& \left(0_{0}, 4_{1}, 5_{0}, 2_{1}, 1_{0}, 3_{1}, \infty, 3_{0}, 1_{1}, 2_{0}, 5_{1}, 4_{0}, 0_{1}\right) \\
& \left(1_{0}, 5_{1}, 0_{0}, 3_{1}, 2_{0}, 4_{1}, \infty, 4_{0}, 2_{1}, 3_{0}, 0_{1}, 5_{0}, 1_{1}\right) \\
& \left(2_{0}, 0_{1}, 1_{0}, 4_{1}, 3_{0}, 5_{1}, \infty, 5_{0}, 3_{1}, 4_{0}, 1_{1}, 0_{0}, 2_{1}\right)
\end{aligned}
$$

It is not difficult to check that $\mathcal{H}$ is an $\operatorname{HCS}(13)$ with no non-trivial automorphism.

Now, given a group $G$ of odd order $n \geq 3$, we show that there exists an $\operatorname{HCS}(4 n+1)$ whose full automorphism group is isomorphic to $G$. Let $\mathbb{Z}_{2}=\{1, \lambda\}$, and set $\Gamma=\mathbb{Z}_{2} \times G$. Since $\Gamma$ is a solvable binary group, it is known from [2] that there exists an $\operatorname{HCS}(2 n+1) \mathcal{H}$ with vertices $\{\infty\} \cup \Gamma$ that is 1-rotational under $\Gamma$. Let $G=\left\{g_{1}=1, g_{2}, \ldots, g_{n}\right\}$ and for a given cycle $C=\left(\infty, x_{1}, \ldots, x_{2 n}\right)$ of $\mathcal{H}$ set $C_{i}=C \cdot g_{i}$ for $i \in[n]$; therefore, $\mathcal{H}=\left\{C_{i} \mid i \in[n]\right\}$. We have previously pointed out that $C$ satisfies (3.1). Therefore, we can write $C=$ $\left(\infty, x_{1}, \ldots, x_{n}, \overline{x_{n}}, \ldots, \overline{x_{1}}\right)$, where $\overline{x_{i}}=x_{i} \cdot \lambda$. Also, there exists $k \in[n-1] \backslash\{1\}$ such that $x_{k} x_{k+1}^{-1}=x_{1} x_{2}^{-1} \lambda$ or $x_{k} x_{k+1}^{-1}=x_{2} x_{1}^{-1} \lambda$. We define the $(2 n+1)$-cycle $C^{*}$ as follows:

$$
C^{*}=\left(\infty, x_{1}, \bar{x}_{2}, \ldots, \bar{x}_{k}, x_{k+1}, \ldots, x_{n}, \bar{x}_{n}, \ldots, \bar{x}_{k+1}, x_{k}, \ldots, x_{2}, \bar{x}_{1}\right)
$$

Of course, $C^{*} \cdot \lambda=C^{*}$. Also, we have that

$$
\begin{aligned}
& \left\{x_{1} \bar{x}_{2}^{-1}, \bar{x}_{2} x_{1}^{-1}\right\}=\left\{x_{1} x_{2}^{-1} \lambda, x_{2} x_{1}^{-1} \lambda\right\}=\left\{x_{k} x_{k+1}^{-1}, x_{k+1} x_{k}^{-1}\right\} \\
& \left\{\bar{x}_{k} x_{k+1}^{-1}, x_{k+1} \bar{x}_{k}^{-1}\right\}=\left\{x_{k} x_{k+1}^{-1} \lambda, x_{k+1} x_{k}^{-1} \lambda\right\}=\left\{x_{1} x_{2}^{-1}, x_{2} x_{1}^{-1}\right\}, \text { and } \\
& \left\{\bar{x}_{i} \bar{x}_{i+1}^{-1} \mid i \in[n-1] \backslash\{1, k\}\right\}=\left\{x_{i} x_{i+1}^{-1} \mid i \in[n-1] \backslash\{1, k\}\right\} .
\end{aligned}
$$

Therefore, $C^{*}$, as well as $C$, satisfies both conditions in (3.1). It follows that $\mathcal{H}^{*}=\left\{C_{1}^{*}, \ldots, C_{n}^{*}\right\}$, with $C_{i}^{*}=C^{*} \cdot g_{i},\left(\right.$ namely, the $G$-orbit $\mathcal{H}^{*}$ of $\left.C^{*}\right)$ is an $\operatorname{HCS}(2 n+1)$.

Now we apply the doubling construction defined above with $\mathcal{H}_{1}=\mathcal{H}^{*}$ and $\mathcal{H}_{2}=\mathcal{H}_{3}=\mathcal{H}$ and let $\mathcal{T}$ denote the resulting $\operatorname{HCS}(4 n+1)$ with vertex set $\{\infty\} \cup \Gamma \times\{1,-1\}$. Note that Property 2.1 is satisfied, as the vertices adjacent with $\infty$ in $C$ and $C^{*}$ coincide. The starter cycles $T_{1}, T_{2}$ of $\mathcal{T}$ have the following form:

$$
\begin{aligned}
& T_{1}=\left(\infty, a_{1}=b_{1}, a_{2}, \ldots, a_{2 n-1}, a_{2 n}=b_{2 n}, b_{2 n}^{\prime}, \ldots, b_{1}^{\prime}\right), \\
& T_{2}=\left(\infty, b_{2 n}, b_{2 n-1}^{\prime}, \ldots, b_{2}, b_{1}^{\prime}, b_{1}, b_{2}^{\prime}, \ldots, b_{2 n-1}, b_{2 n}^{\prime}\right),
\end{aligned}
$$

where $b_{j}=\left(x_{j}, 1\right), b_{j}^{\prime}=\left(x_{j},-1\right), a_{j}=\left(\alpha_{j}, 1\right)$, and $\left(\alpha_{1}, \ldots, \alpha_{2 n}\right)$ is the ordered sequence $\left(x_{1}, \bar{x}_{2}, \bar{x}_{3}, \ldots\right)$ of the vertices of $C^{*}$.

The cycles of $\mathcal{T}$ are generated from $\mathcal{T}_{1}, \mathcal{T}_{2}$ by the action $(x, i) \mapsto(x g, i)$, $x \in \Gamma, i=1,-1, g \in G$.

For any $g \in G$, this action is an automorphism of $\mathcal{T}$, and we denote the group of such automorphisms by $\tau_{G}$.

We are going to show that $\tau_{G}=\operatorname{Aut}(\mathcal{T})$. Note that $b_{2 n}=b_{1} \cdot(\lambda, 1)$ and that $b_{1}, b_{1}^{\prime}, b_{2 n}, b_{2 n}^{\prime}$ form a complete system of representatives of the $G$-orbit on $\Gamma \times\{1,-1\}$. Therefore, given an automorphism $\varphi$ of $\mathcal{T}$, then there exists $z \in G$ such that $\phi=\tau_{z} \varphi$ maps $b_{1}$ to one of $b_{1}, b_{1}^{\prime}, b_{2 n}, b_{2 n}^{\prime}$. It is enough to prove that it is always $b_{1}$. It will follow that $\phi$ is the identity, because, since the $\infty$ point is always fixed, all the vertices of $T_{1}$ are fixed. Hence, $\varphi=\tau_{z}^{-1} \in \tau_{G}$. Since $\tau_{G}$ is isomorphic to $G$ we get the assertion.

Suppose that $\phi\left(b_{1}\right)=b_{1}^{\prime}$. It follows that the edge $\left(\infty, b_{1}^{\prime}\right)$ of $T_{1}$ is also an edge of $\phi\left(T_{1}\right)$, therefore $T_{1}=\phi\left(T_{1}\right)$. In other words, $\phi$ is the reflection of $T_{1}$ in the axis through $\infty$. In particular, $\phi$ swaps $b_{1}$ and $b_{1}^{\prime}$, and also $a_{2}$ and $b_{2}^{\prime}$. This means that $\phi$ fixes the edge $\left(b_{1}, b_{1}^{\prime}\right)$ and hence it fixes the cycle $T_{2}$ containing this edge. Therefore, $\phi$ is the reflection of $T_{2}$ in the axis through the edge $\left(b_{1}, b_{1}^{\prime}\right)$. It follows that $\phi$ swaps $b_{2}$ and $b_{2}^{\prime}$, but this contradicts the previous conclusion as $b_{2}=\left(x_{2}, 1\right) \neq\left(\bar{x}_{2}, 1\right)=a_{2}$.

Now suppose that $\phi\left(b_{1}\right)=b_{2 n}$. Then $\phi$ maps $T_{1}$ to $T_{2}$. Therefore, $\phi\left(a_{2 n-1}\right)=$ $b_{2}=\left(x_{2}, 1\right)=a_{2 n-1}$, that is, $\phi$ fixes $a_{2 n-1}$. However this implies that $\phi$ is the identity, since then the edge $\left(\infty, a_{2 n-1}\right)$ and all the points in the cycle that contains it are also fixed. A similar argument applies to the possibility that $\phi\left(b_{1}\right)=b_{2 n}^{\prime}$. Thus $\phi$ fixes $b_{1}$.

We point out to the reader that the $\operatorname{HCS}(13) \mathcal{T}$ of Example 2.2 has been constructed following the proof of Theorem 3.1. In fact, $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are two 1-rotational $\operatorname{HCS}(7)$ and if we set $C=B_{1}$, then $C^{*}=A_{1}$. It then follows that $\operatorname{Aut}(\mathcal{T})=\mathbb{Z}_{3}$.

We finally consider the case where a group $G \in \mathcal{G}$ has even order and prove the following:

Theorem 3.2. If $\mathcal{H}$ is an $\operatorname{HCS}(2 n+1)$ whose full isomorphism group has even order, then either $\operatorname{Aut}(\mathcal{H})$ is binary or $2 n+1$ is a prime and $\operatorname{Aut}(\mathcal{H})$ is the affine linear group $A G L(1,2 n+1)$.

Proof. Let $\mathcal{H}$ be an $\operatorname{HCS}(2 n+1)$ and assume that $\operatorname{Aut}(\mathcal{H})$ has even order. We first show that any involution of $\operatorname{Aut}(\mathcal{H})$ has exactly one fixed point and fixes each cycle of $\mathcal{H}$. This means that an involution is uniquely determined once the point it fixes is known. Therefore, distinct involutory automorphisms of $\mathcal{H}$ have distinct fixed points.

Suppose $\alpha$ is an involutory automorphism of $\mathcal{H}$ and let $x$ be any point not fixed by $\alpha$. Then the edge $[x, \alpha(x)]$ occurs in some cycle $C$, and $\alpha$ fixes this edge. It then follows that the entire cycle $C$ is fixed by $\alpha$ which therefore acts on $C$ as the reflection in the axis of $[x, \alpha(x)]$. The point $a$ opposite to this edge is then the only fixed point of $\alpha$. Note that there are $n$ edges of the form $[x, \alpha(x)]$ with $x \neq a$, and that they are partitioned among the $n$ cycles of $\mathcal{H}$. Therefore, reasoning as before, we get that $\alpha$ fixes all cycles of $\mathcal{H}$

Now, assume that $\operatorname{Aut}(\mathcal{H})$ is not binary and let $\beta$ be a second involutory automorphism with fixed point $b$. We denote by $S$ the subgroup of $\operatorname{Aut}(\mathcal{H})$ generated by $\alpha$ and $\beta$. Since both involutions fix all cycles of $\mathcal{H}$, all automorphisms in $S$ fix them. If $C^{\prime}$ is the cycle of $\mathcal{H}$ containing the edge $[a, b]$, then the map $\beta \alpha$ is the rotation of $C^{\prime}$ with step 2 . Since $C^{\prime}$ has odd order, this means that $S$ also contains all rotations of $C^{\prime}$. Hence, $S$ is the dihedral group of order $4 n+2$.

We also have that $2 n+1$ is prime. In fact, let $\varphi \in S$ be an automorphism of prime order $p$; given a point $x$ not fixed by $\varphi$ we denote by $C^{\prime \prime}$ the cycle of $\mathcal{H}$ containing $[x, \varphi(x)]$. Of course, all edges of the form $\left[\varphi^{i}(x), \varphi^{i+1}(x)\right]$ with $i=0, \ldots, p-1$ lies in $C^{\prime \prime}$. It follows that $C^{\prime \prime}$ is the cycle $(x, \varphi(x)$, $\left.\varphi^{2}(x), \ldots, \varphi^{p-2}(x), \varphi^{p-1}(x)\right)$, hence $2 n+1=p$.

One can easily see that $\mathcal{H}$ is then the unique 2-transitive $\operatorname{HCS}(p)$ whose full automorphism group is $\operatorname{AGL}(1, p)$ [6] and this completes the proof.

The following theorem provides a sufficient condition for a binary group $H$ of order $4 m$ to be the full automorphism group of infinitely many HCSs of odd order.

Theorem 3.3. Let $H$ be a binary group of order $4 m$ and let $d \geq 3$ be an odd integer. If there exists a 1 -rotational $H C S(4 m d+1)$ under $H \times \mathbb{Z}_{d}$, then there exists an HCS $(4 m d+1)$ whose full automorphism group is $H$.
Proof. Let $\mathbb{Z}_{d}=\langle z\rangle$ denote the cyclic group of order $d$ generated by $z$ and let $\lambda$ be the unique element of order 2 in $H$. It is straightforward that $G=H \times \mathbb{Z}_{d}$ is a binary group and its element of order 2 is $\lambda$.

Now, let $\mathcal{H}$ be a 1 -rotational $\operatorname{HCS}(4 m d+1)$ under $G$, and let $A=\left(\infty, a_{1}\right.$, $\left.a_{2}, \ldots, a_{2 m d}, a_{2 m d} \lambda, \ldots, a_{2} \lambda, a_{1} \lambda\right)$ denote its starter cycle. Also, let $\operatorname{Orb}_{\mathbb{Z}_{d}}(A)=$ $\left\{A_{0}, A_{1}, \ldots, A_{d-1}\right\}$ be the $\mathbb{Z}_{d}$-orbit of $A$ where $A_{i}=A \cdot z^{i}$ for $i=0,1, \ldots, d-1$. We can then see $\mathcal{H}$ as the union of the $H$-orbits of each cycle in $\operatorname{Orb}_{\mathbb{Z}_{d}}(A)$, that is, $\mathcal{H}=\cup_{i=0}^{d-1}\left(\operatorname{Orb}_{H}\left(A_{i}\right)\right)$.

We now construct a new $\operatorname{HCS}(4 m d+1) \mathcal{H}^{*}$ through a slight modification of the cycle $A_{0}$ :

1. Since the binary group $H$ has order $4 m$, there is at least an element $x \in H$ of order 4. Also $\Delta A$ covers all non-zero elements of $H \times \mathbb{Z}_{d}$. It follows that there exists $j \in[2 m d-1]$ such that $x=a_{j+1} a_{j}^{-1}$. Now,
let $A_{0}^{*}$ denote the graph we get from $A_{0}$ by replacing the edges in $\mathcal{E}=$ $\left\{\left[a_{j}, a_{j+1}\right],\left[a_{j} \lambda, a_{j+1} \lambda\right]\right\}$ with those in $\mathcal{E}^{*}=\left\{\left[a_{j}, a_{j+1} \lambda\right],\left[a_{j} \lambda, a_{j+1}\right]\right\}$.
2. Set $\mathcal{H}^{*}=\cup_{i=1}^{d-1} \operatorname{Orb}_{H}\left(A_{i}\right) \cup \operatorname{Orb}_{H}\left(A_{0}^{*}\right)$.

It is easy to see that $A_{0}^{*}$ is a $(4 m d+1)$-cycle. Also, $E\left(A_{0}\right) \backslash \mathcal{E}=E\left(A_{0}^{*}\right) \backslash \mathcal{E}^{*}$. We show that $\mathcal{E}^{*}=\mathcal{E} \cdot y$, where $y=a_{j}^{-1} x a_{j}$. We first point out that $x^{2}=\lambda$, since $x^{2}$ has order 2 and $H$ is binary. Also, note that $a_{j+1}=x a_{j}$ and recall that $\lambda$ commutes with every element in $G$. Therefore,

$$
\begin{aligned}
& {\left[a_{j}, a_{j+1}\right] y=\left[x a_{j}, x^{2} a_{j}\right]=\left[a_{j+1}, a_{j} \lambda\right], \quad \text { and }} \\
& {\left[a_{j} \lambda, a_{j+1} \lambda\right] y=\left[a_{j}, a_{j+1}\right] y \lambda=\left[a_{j+1}, a_{j} \lambda\right] \lambda=\left[a_{j+1} \lambda, a_{j}\right]}
\end{aligned}
$$

Since $y \in H$, we have that $\operatorname{Orb}_{H}(\mathcal{E})=\operatorname{Orb}_{H}\left(\mathcal{E}^{*}\right)$. It follows that $\operatorname{Orb}_{H}\left(A_{0}\right)$ and $\operatorname{Orb}_{H}\left(A_{0}^{*}\right)$ cover the same set of edges. Since $\mathcal{H} \backslash \operatorname{Orb}_{H}\left(A_{0}\right)=\mathcal{H}^{*} \backslash \operatorname{Orb}_{H}\left(A_{0}^{*}\right)$, we have that $\mathcal{H}^{*}$ covers the same set of edges covered by $\mathcal{H}$, that is, $\mathcal{H}^{*}$ is an $\operatorname{HCS}(4 m d+1)$.

We now show that the full automorphism groups of $\mathcal{H}^{*}$ is isomorphic to $H$. We set $\mathbb{A}=\operatorname{Aut}\left(\mathcal{H}^{*}\right)$ and denote by $\mathbb{A}_{\infty}$ the $\mathbb{A}$-stabilizer of $\infty$. Also, let $\tau_{g}$ denote the translation by the element $g \in G$, that is, the permutation on $G \cup\{\infty\}$ fixing $\infty$ and mapping $x \in G$ to $x g$ for any $x, g \in G$. Finally, let $\tau_{G}$ and $\tau_{H}$ denote the group of all translations by the elements of $G$ and $H$, respectively. It is easy to check that, by construction, $\tau_{H}$ is an automorphism group of $\mathcal{H}^{*}$ fixing $\infty$, that is, $\tau_{H} \subseteq \mathbb{A}_{\infty}$; on the other hand, the replacement of $\operatorname{Orb}_{H}\left(A_{0}\right)$ with $\operatorname{Orb}_{H}\left(A_{0}^{*}\right)$ ensures that $\tau_{g}$ is not an automorphism of $\mathcal{H}^{*}$ whenever $g \in G \backslash H$. In other words, for any $g \in G$ we have that

$$
\begin{equation*}
\tau_{g} \in \mathbb{A} \text { if and only if } g \in H \tag{3.2}
\end{equation*}
$$

We are going to show that $\mathbb{A}_{\infty}=\tau_{H}$. Let $\varphi \in \mathbb{A}_{\infty}$ and note that $\left|\operatorname{Orb}_{H}\left(A_{0}^{*}\right)\right|<$ $\left|\mathcal{H}^{*} \backslash \operatorname{Orb}_{H}\left(A_{0}^{*}\right)\right|$. Therefore, there exists a cycle $C=\left(\infty, g_{1}, g_{2}, \ldots, g_{4 m p}\right) \in$ $\mathcal{H}^{*} \backslash \operatorname{Orb}_{H}\left(A_{0}^{*}\right)$ such that $\varphi(C)=\mathcal{H}^{*} \backslash \operatorname{Orb}_{H}\left(A_{0}^{*}\right)$. Since $\varphi$ fixes $\infty$, then $\varphi(C)=\left(\infty, \varphi\left(g_{1}\right), \varphi\left(g_{2}\right), \ldots, \varphi\left(g_{4 m p}\right)\right)$. Note that $\mathcal{H}^{*} \backslash \operatorname{Orb}_{H}\left(A_{0}^{*}\right) \subseteq \mathcal{H}$, therefore $\varphi(C)$ is a translate of $C$, namely, there exists $x \in G$ such that $\varphi\left(g_{i}\right)=g_{i} \cdot x$. This means that $\varphi=\tau_{x}$, and in view of $(\sqrt[3.2]{ }), x \in H$. It follows that $\varphi \in \tau_{H}$, hence $\mathbb{A}_{\infty}=\tau_{H}$.

Since $\tau_{H}$ is isomorphic to $H$, we are left to show $\mathbb{A}=\mathbb{A}_{\infty}$. Assume that there exists $w \in \operatorname{Orb}_{\mathbb{A}}(\infty) \backslash\{\infty\}$. From a classic result on permutations groups, we have that $\mathbb{A}_{w}$ and $\mathbb{A}_{\infty}$ are conjugate hence, in particular, they are isomorphic. This means that $\mathbb{A}_{w}$ contains an involution $\psi$. Since $\mathbb{A}_{\infty} \cap \mathbb{A}_{w}=\{i d\}$ (otherwise there would be a non-trivial automorphism fixing two vertices), we have that $\tau_{\lambda}$ and $\psi$ are distinct. Therefore, in view of Theorem 3.2, we have that $4 m d+1$ is a prime and $\mathbb{A}=A G L(1,4 m d+1)$; in particular, $|\mathbb{A}|=4 m d(4 m d+1)$. But this leads to a contradiction since $|\mathbb{A}|=\left|\mathbb{A}_{\infty}\right|\left|\operatorname{Orb}_{\mathbb{A}}(\infty)\right|$ and $\left|\operatorname{Orb}_{\mathbb{A}}(\infty)\right| \leq 4 m d+1$, that is, $|\mathbb{A}| \leq 4 m(4 m d+1)$ where $d \geq 3$. Therefore, $\operatorname{Orb}_{\mathbb{A}}(\infty)=\{\infty\}$, namely, $\mathbb{A}=\mathbb{A}_{\infty}$.

As a consequence we obtain the following:
Theorem 3.4. The quaternion group $\mathbb{Q}_{8}$ is the full automorphism group of an $H C S(8 d+1)$, for any $d \geq 3$.

Proof. It is enough to observe that $\mathbb{Q}_{8} \times \mathbb{Z}_{d}, d \geq 3$, is a binary solvable group $\neq \mathbb{Q}_{8}$. As mentioned earlier in this paper, any binary solvable group is the full automorphism group of an HCS of odd order. Then, the conclusion immediately follows by Theorem 3.3

Collecting the above results, we can prove Theorem 1.1
proof of Theorem 1.1. The first part of the statement is proven in Theorem 3.2.
Now, let $G$ be a finite group. If $G$ has odd order or it is $\mathbb{Q}_{8}$, then by Theorems 3.1 and 3.3 we have that $G$ is the full automorphism group of a suitable HCS of odd order. If $G$ is a binary solvable group $\neq \mathbb{Q}_{8}$ or $\operatorname{AGL}(1, \mathrm{p})$ with $p$ prime, it is known [15, 6] that $G \in \mathcal{G}$.

Remark 3.1. Note that Theorem 3.3 would allow us to solve completely the problem under investigation in this paper, if one could show that any sufficiently large binary group has a 1-rotational action on an HCS of odd order. In this case, given a binary non-solvable group $H$ of order $4 m$, we could consider the group $G=H \times \mathbb{Z}_{d}$ with $d$ sufficiently large to ensure that $G$ has a 1-rotational action on an $\operatorname{HCS}(4 m d+1)$. By Theorem 3.3 we would have that $H$ is the full automorphism group of an HCS of odd order.

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[^0]:    *Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli 1-06123 Italy, email: buratti@dmi.unipg.it
    ${ }^{\dagger}$ Department of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, U.K., email: graham.lovegrove@virgin.net
    ${ }^{\ddagger}$ Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli 1-06123 Italy, email: traetta@dmi.unipg.it

