

Solution to a problem on hamiltonicity of graphs under Ore- and Fan-type heavy subgraph conditions*

Bo Ning[†], Shenggui Zhang[‡] and Binlong Li[§]

Abstract

A graph G is called *claw- o -heavy* if every induced claw ($K_{1,3}$) of G has two end-vertices with degree sum at least $|V(G)|$. For a given graph S , G is called *S - f -heavy* if for every induced subgraph H of G isomorphic to S and every pair of vertices $u, v \in V(H)$ with $d_H(u, v) = 2$, there holds $\max\{d(u), d(v)\} \geq |V(G)|/2$. In this paper, we prove that every 2-connected claw- o -heavy and Z_3 - f -heavy graph is hamiltonian (with two exceptional graphs), where Z_3 is the graph obtained by identifying one end-vertex of P_4 (a path with 4 vertices) with one vertex of a triangle. This result gives a positive answer to a problem proposed in [B. Ning, S. Zhang, Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs, Discrete Math. 313 (2013) 1715–1725], and also implies two previous theorems of Faudree et al. and Chen et al., respectively.

Keywords: Induced subgraphs; Claw- o -heavy graphs; f -Heavy subgraphs; Hamiltonicity

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1 Introduction

Throughout this paper, the graphs considered are simple, finite and undirected. For terminology and notation not defined here, we refer the reader to Bondy and Murty [2].

Let G be a graph. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the set, and $d_G(v)$ the number, of neighbors of v in G . When there is no danger of ambiguity, we use $N(v)$

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and $d(v)$ instead of $N_G(v)$ and $d_G(v)$. If H and H' are two subgraphs of G , then we set $N_H(H') = \{v \in V(H) : N_G(v) \cap V(H') \neq \emptyset\}$. For two vertices $u, v \in V(H)$, the *distance* between u and v in H , denoted by $d_H(u, v)$, is the length of a shortest path connecting u and v in H . In particular, when we use the notation G to denote a graph, then for some subgraph H of G , we set $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$ (so, if G' is another graph defined on the same vertex set $V(G)$ and H is a subgraph of G' , we will not use $N_H(v)$ to denote $N_{G'}(v) \cap V(H)$).

We call H an *induced subgraph* of G , if for every $x, y \in V(H)$, $xy \in E(G)$ implies that $xy \in E(H)$. For a given graph S , G is called *S -free* if G contains no induced subgraph isomorphic to S . Following [8], G is called *S - o -heavy* if every induced subgraph of G isomorphic to S contains two nonadjacent vertices with degree sum at least $|V(G)|$ in G . Following [9], G is called *S - f -heavy* if for every induced subgraph H isomorphic to S and any two vertices $u, v \in V(H)$ such that $d_H(u, v) = 2$, there holds $\max\{d(u), d(v)\} \geq |V(G)|/2$. Note that an S -free graph is S - o -heavy (S - f -heavy).

The *claw* is the bipartite graph $K_{1,3}$. Note that a claw- f -heavy graph is also claw- o -heavy. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

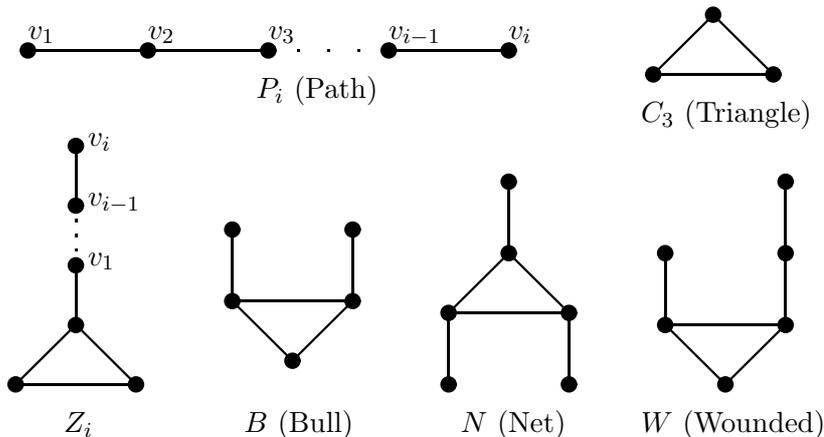


Fig. 1. Graphs P_i, C_3, Z_i, B, N and W .

Bedrossian [1] characterized all connected forbidden pairs for a 2-connected graph to be hamiltonian.

Theorem 1. (Bedrossian [1]) *Let G be a 2-connected graph and let R and S be connected graphs other than P_3 . Then G being R -free and S -free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$ or W .*

Faudree and Gould [6] extended Bedrossian's result by giving a proof of the 'only if' part based on infinite families of non-hamiltonian graphs.

Theorem 2. (Faudree and Gould [6]) *Let G be a 2-connected graph of order at least 10 and let R and S be connected graphs other than P_3 . Then G being R -free and S -free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W .*

Li et al. [8] extended Bedrossian's result by restricting Ore's condition to pairs of induced subgraphs of a graph. Ning and Zhang [9] gave another extension of Bedrossian's theorem by restricting Ore's condition to induced claws and Fan's condition to other induced subgraphs of a graph.

Theorem 3. (Ning and Zhang [9]) *Let G be a 2-connected graph and S be a connected graph other than P_3 . Suppose that G is claw- o -heavy. Then G being S - f -heavy implies G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W .*

Motivated by Theorems 2 and 3, Ning and Zhang [9] proposed the following problem.

Problem 1. (Ning and Zhang [9]) *Is every claw- o -heavy and Z_3 - f -heavy graph of order at least 10 hamiltonian?*

The main goal of this paper is to give an affirmative solution to this problem. Our answer is the following theorem, where the graphs L_1 and L_2 are shown in Fig. 2.

Theorem 4. *Let G be a 2-connected graph. If G is claw- o -heavy and Z_3 - f -heavy, then G is either hamiltonian or isomorphic to L_1 or L_2 .*

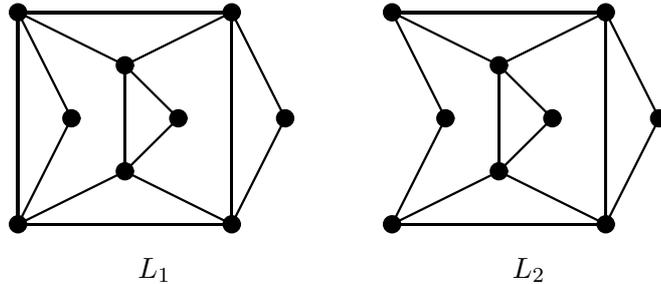


Fig. 2. Graphs L_1 and L_2 .

Theorem 4 extends the following two previous theorems.

Theorem 5. (Faudree et al. [7]) *If G is a 2-connected claw-free and Z_3 -free graph, then G is either hamiltonian or isomorphic to L_1 or L_2 .*

Theorem 6. (Chen et al. [5]) *If G is a 2-connected claw- f -heavy and Z_3 - f -heavy graph, then G is either hamiltonian or isomorphic to L_1 or L_2 .*

We remark that there are infinite 2-connected claw-*o*-heavy and Z_3 -*o*-heavy graphs which are non-hamiltonian, see [8].

Together with Theorem 3 and Theorem 4, we can obtain the following result which generalizes Theorem 2.

Theorem 7. *Let G be a 2-connected graph of order at least 10 and S be a connected graph other than P_3 . Suppose that G is claw-*o*-heavy. Then G being S -*f*-heavy implies G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W .*

2 Preliminaries

In this section, we will list some necessary preliminaries. First, we will introduce the closure theory of claw-*o*-heavy graphs proposed by Čada [4], which is an extension of the closure theory of claw-free graphs due to Ryjáček [10].

Let G be a graph of order n . A vertex $x \in V(G)$ is called *heavy* if $d(x) \geq n/2$; otherwise, it is called *light*. A pair of nonadjacent vertices $\{x, y\} \subset V(G)$ is called a *heavy pair* of G if $d(x) + d(y) \geq n$.

Let G be a graph and $x \in V(G)$. Define $B_x^o(G) = \{uv : \{u, v\} \subset N(x), d(u) + d(v) \geq |V(G)|\}$. Let G_x^o be a graph with vertex set $V(G_x^o) = V(G)$ and edge set $E(G_x^o) = E(G) \cup B_x^o(G)$. Suppose that $G_x^o[N(x)]$ consists of two disjoint cliques C_1 and C_2 . For a vertex $y \in V(G) \setminus (N(x) \cup \{x\})$, if $\{x, y\}$ is a heavy pair in G and there are two vertices $x_1 \in C_1$ and $x_2 \in C_2$ such that $x_1y, x_2y \in E(G)$, then y is called a *join vertex* of x in G . If $N(x)$ is not a clique and $G_x^o[N(x)]$ is connected, or $G_x^o[N(x)]$ consists of two disjoint cliques and there is some join vertex of x , then the vertex x is called an *o -eligible vertex* of G . The *locally completion of G at x* , denoted by G'_x , is the graph with vertex set $V(G'_x) = V(G)$ and edge set $E(G'_x) = E(G) \cup \{uv : u, v \in N(x)\}$.

Let G be a claw-*o*-heavy graph. The *closure* of G , denoted by $cl_o(G)$, is the graph such that:

- (1) there is a sequence of graphs G_1, G_2, \dots, G_t such that $G = G_1$, $G_t = cl_o(G)$, and for any $i \in \{1, 2, \dots, t-1\}$, there is an *o -eligible vertex* x_i of G_i , such that $G_{i+1} = (G_i)'_{x_i}$; and
- (2) there is no *o -eligible vertex* in G_t .

Theorem 8. (Čada [4]) *Let G be a claw-*o*-heavy graph. Then*

- (1) *the closure $cl_o(G)$ is uniquely determined;*
- (2) *there is a C_3 -free graph H such that $cl_o(G)$ is the line graph of H ; and*
- (3) *the circumferences of $cl_o(G)$ and G are equal.*

Now we introduce some new terminology and notations. Let G be a claw- o -heavy graph and C be a maximal clique of $cl_o(G)$. We call $G[C]$ a *region* of G . For a vertex v of G , we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices $u, v \in V(G)$, we say u and v are *associated* if u, v are contained in a common region of G ; otherwise u and v are *dissociated*. For a region R of G , we denote by I_R the set of interior vertices of R , and by F_R the set of frontier vertices of R .

From the definition of the closure, it is not difficult to get the following lemma.

Lemma 1. *Let G be a claw- o -heavy graph. Then*

- (1) *every vertex is either an interior vertex of a region or a frontier vertex of two regions;*
- (2) *every two regions are either disjoint or have only one common vertex; and*
- (3) *every pair of dissociated vertices have degree sum less than $|V(G)|$ in $cl_o(G)$ (and in G).*

Proof. In the proof of the lemma, we let $G' = cl_o(G)$.

(1) Let v be an arbitrary vertex of G . Since G' is closed, $N_{G'}(v)$ is either a clique or a disjoint union of two cliques in G' . Thus v is contained in one or two regions of G , and the assertion is true.

(2) Let R and R' be two regions of G , and C and C' be the two maximal cliques of G' corresponding to R and R' , respectively. If C and C' have two common vertices, say u and v , then u and v will be o -eligible vertices of G' , contradicting the definition of the closure of G . This implies that C and C' (and then, R and R') have at most one common vertex.

(3) Let u, v be two nonadjacent vertices with $d_{G'}(u) + d_{G'}(v) \geq n = |V(G)|$. Then u, v have at least two common neighbors in G' . Suppose that u and v are not in a common clique of G' . Let x be a common neighbor of u and v in G' . Since $N_{G'}(x)$ is not a clique in G' , it is the disjoint union of two cliques, one containing u and the other containing v . Since $uv \in B_x^o(G')$, x is an o -eligible vertex of G' , a contradiction. Thus we conclude that u, v are in a common clique of G' , i.e., u and v are associated. □ □

The next lemma provides some structural information on regions.

Lemma 2. *Let G be a claw- o -heavy graph and R be a region of G . Then*

- (1) *R is nonseparable;*
- (2) *if v is a frontier vertex of R , then v has an interior neighbor in R or R is complete and has no interior vertices;*
- (3) *for any two vertices $u, v \in R$, there is an induced path of G from u to v such that every*

internal vertex of the path is an interior vertex of R ; and

(4) for two vertices u, v in R , if $\{u, v\}$ is a heavy pair of G , then u, v have two common neighbors in I_R .

Proof. Let G_1, G_2, \dots, G_t be the sequence of graphs, and x_1, x_2, \dots, x_{t-1} the sequence of vertices in the definition of $cl_o(G)$.

(1) Suppose that R has a cut-vertex y . We prove by induction that y would be a cut-vertex of $G_i[V(R)]$ for all $i \in [1, t]$. Since y is a cut-vertex of $G_1[V(R)] = R$, we assume that $2 \leq i \leq t$. By the induction hypothesis, y is a cut-vertex of $G_{i-1}[V(R)]$. Let R' and R'' be two components of $G_{i-1}[V(R)] - y$, u be a vertex of R' and v be a vertex of R'' . Then u and v have at most one common neighbor y in R . Note that each two maximal cliques of $cl_o(G)$ is either disjoint or have only one common vertex (see Lemma 1 (1)). This implies that u and v have no common neighbors in $G_{i-1} - V(R)$. Hence $\{u, v\}$ is not a heavy pair of G . Note that an o -eligible vertex of G_{i-1} will be an interior vertex of $cl_o(G)$. This implies that y is not an o -eligible vertex of G_{i-1} . Thus $x_{i-1} \neq y$. Note that x_{i-1} has no neighbors in R' or has no neighbors in R'' . This implies that there are no new edges in G_i between R' and R'' . Thus y is also a cut-vertex of $G_i[V(R)]$. By induction, we can see that y is a cut-vertex of $cl_o(G)[V(R)]$, contradicting the fact that $V(R)$ is a clique in $cl_o(G)$.

(2) Note that $cl_o(G)[V(R)]$ is complete. If R has no interior vertex, then R contains no o -eligible vertex of G . Since the locally completion of G at every o -eligible vertex does not add an edge in R , $R = cl_o(G)[V(R)]$ is complete.

Now we assume that R has at least one interior vertex. Suppose that v has no interior neighbors in R , i.e., $N(v) \cap I_R = \emptyset$. Using induction, we will prove that $N_{G_i}(v) \cap I_R = \emptyset$. Since $N_{G_1}(v) \cap I_R = \emptyset$, we assume that $2 \leq i \leq t$. By the induction hypothesis, $N_{G_{i-1}}(v) \cap I_R = \emptyset$. Note that x_{i-1} is either nonadjacent to v or nonadjacent to every vertex in $N_{G_{i-1}}(v) \cap V(R)$. This implies that there are no new edges of G_i between v and $G_i[V(R)] - v$. Hence $N_{G_i}(v) \cap I_R = \emptyset$. Thus by the induction hypothesis, we can see that $N_{cl_o(G)}(v) \cap I_R = \emptyset$, a contradiction.

(3) We use induction on $t - i$ (t is the subscript of $G_t = cl_o(G)$) to prove that there is an induced path of $G_i[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R . Note that uv is an edge in $G_t[V(R)]$. We are done if $i = t$. Now suppose that there is an induced path P of $G_i[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R . We will prove that there is an induced path of $G_{i-1}[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R . If P is also a path of $G_{i-1}[V(R)]$, then we are done. So we assume that there is an

edge $u'v' \in E(P)$ such that $u'v' \notin E(G_{i-1})$. This implies that $u', v' \in N(x_{i-1})$. Since P is an induced path of G_i , x_{i-1} has the only two neighbors u', v' on P . We also note that $x_{i-1} \in V(R)$ is an interior vertex. Thus $P' = (P - u'v') \cup u'xv'$ (with the obvious meaning) is an induced path of $G_{i-1}[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R . Thus by the induction hypothesis, the proof is complete.

(4) Since every vertex in F_R has at least one neighbor in $G - R$ and every vertex in $G - R$ has at most one neighbor in F_R , we have $|N_{G-R}(F_R \setminus \{u, v\})| \geq |F_R \setminus \{u, v\}|$. Furthermore, we have $n = |I_R \setminus \{u, v\}| + |F_R \setminus \{u, v\}| + |V(G - R)| + 2$. Thus, we get

$$\begin{aligned}
n &\leq d(u) + d(v) \\
&= d_{I_R}(u) + d_{I_R}(v) + d_{F_R}(u) + d_{F_R}(v) + d_{G-R}(u) + d_{G-R}(v) \\
&\leq d_{I_R}(u) + d_{I_R}(v) + 2|F_R \setminus \{u, v\}| + d_{G-R}(u) + d_{G-R}(v) \\
&\leq d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |N_{G-R}(F_R \setminus \{u, v\})| + |N_{G-R}(u)| + |N_{G-R}(v)| \\
&= d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |N_{G-R}(F_R)| \\
&\leq d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |V(G - R)|,
\end{aligned}$$

and

$$d_{I_R}(u) + d_{I_R}(v) \geq n - |F_R \setminus \{u, v\}| - |V(G - R)| = |I_R \setminus \{u, v\}| + 2.$$

This implies that u, v have two common neighbors in I_R . □ □

Let G be a graph and Z be an induced copy of Z_3 in G . We denote the vertices of Z as in Fig. 3, and say that Z is *center-heavy* in G if a_1 is a heavy vertex of G . If every induced copy of Z_3 in G is center-heavy, then we say that G is *Z_3 -center-heavy*.

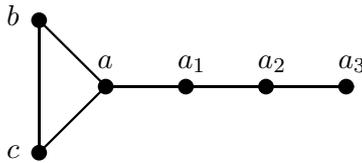


Fig. 3. The Graph Z_3 .

Lemma 3. *Let G be a claw-o-heavy and Z_3 -f-heavy graph. Then $cl_o(G)$ is Z_3 -center-heavy.*

Proof. Let Z be an arbitrary induced copy of Z_3 in $G' = cl_o(G)$. We denote the vertices of Z as in Fig. 3, and will prove that a_1 is heavy in G' .

Let R be the region of G containing $\{a, b, c\}$. Recall that I_R is the set of interior vertices of R , and F_R is the set of frontier vertices of R .

Claim 1. $|N_R(a_2) \cup N_R(a_3)| \leq 1$.

Proof. Note that every vertex in $G - R$ has at most one neighbor in R . If $N_R(a_2) = \emptyset$, then the assertion is obviously true. Now we assume that $N_R(a_2) \neq \emptyset$. Let x be the vertex in $N_R(a_2)$. Clearly $x \neq a$ and $a_1x \notin E(G')$. If $a_3x \notin E(G')$, then $\{a_2, a_1, a_3, x\}$ induces a claw in G' , a contradiction. This implies that $a_3x \in E(G')$, and x is the unique vertex in $N_{G'}(a_3) \cap V(R)$. Thus $N_R(a_2) \cup N_R(a_3) = \{x\}$. \square \square

Claim 2. Let x, y be two vertices in $I_R \cup \{a\}$. If $xy \in E(G)$ and $d(x) + d(y) \geq n$, then x, y have a common neighbor in I_R .

Proof. Note that every vertex in F_R has at least one neighbor in $G - R$ and every vertex in $G - R$ has at most one neighbor in R . By Claim 1, $|V(G - R)| \geq |F_R| + 1$. Moreover, since a is not the neighbor of a_2 and a_3 in R , $|V(G - R)| \geq |F_R \setminus \{a\}| + |N_{G-R}(a)| + 1$.

If $x, y \in I_R$, then

$$\begin{aligned} n &\leq d(x) + d(y) \\ &= d_{I_R}(x) + d_{I_R}(y) + d_{F_R}(x) + d_{F_R}(y) \\ &\leq d_{I_R}(x) + d_{I_R}(y) + 2|F_R| \\ &\leq d_{I_R}(x) + d_{I_R}(y) + |F_R| + |V(G - R)| - 1, \end{aligned}$$

and

$$d_{I_R}(x) + d_{I_R}(y) \geq n - |F_R| - |V(G - R)| + 1 = |I_R| + 1.$$

This implies that x, y have a common neighbor in I_R .

If one of x, y , say y is a , then

$$\begin{aligned} n &\leq d(x) + d(a) \\ &= d_{I_R}(x) + d_{I_R}(a) + d_{F_R}(x) + d_{F_R}(a) + d_{G-R}(a) \\ &\leq d_{I_R}(x) + d_{I_R}(a) + |F_R| + |F_R \setminus \{a\}| + d_{G-R}(a) \\ &\leq d_{I_R}(x) + d_{I_R}(a) + |F_R| + |V(G - R)| - 1, \end{aligned}$$

and

$$d_{I_R}(x) + d_{I_R}(a) \geq n - |F_R| - |V(G - R)| + 1 = |I_R| + 1.$$

This implies that x, a have a common neighbor in I_R . \square \square

By Lemma 2 (3), G has an induced path P from a to a_3 such that every vertex of P is either in $\{a, a_1, a_2, a_3\}$ or an interior vertex outside R . Let a, a'_1, a'_2, a'_3 be the first four vertices of P .

Note that a'_1 is either a_1 or an interior vertex in the region containing $\{a, a_1\}$. This implies that $d_{G'}(a_1) \geq d_{G'}(a'_1) \geq d(a'_1)$. If a'_1 is heavy in G , then a_1 is heavy in G' and we are done. So we assume that a'_1 is not heavy in G .

If $abca$ is also a triangle in G , then the subgraph induced by $\{a, b, c, a'_1, a'_2, a'_3\}$ is a Z_3 . Since G is Z_3 - f -heavy and a'_1 is not heavy in G , b and a'_3 are heavy in G . By Lemma 1 (3), b and a'_3 are associated, a contradiction. Thus we conclude that one edge of $\{ab, ac, bc\}$ is not in $E(G)$.

Note that R is not complete. By Lemma 2 (2), a has a neighbor in I_R .

Claim 3. $d_{I_R}(a) = 1$.

Proof. Suppose that $d_{I_R}(a) \geq 2$. Let x, y be two arbitrary vertices in $N_{I_R}(a)$. If $xy \in E(G)$, then $\{a, x, y, a'_1, a'_2, a'_3\}$ induces a Z_3 in G . Note that a'_1 is not heavy in G . Thus x and a'_3 are heavy in G . Note that x and a'_3 are dissociated, a contradiction. This implies that $N_{I_R}(a)$ is an independent set.

Since $\{a, x, y, a'_1\}$ induces a claw in G , and $\{a'_1, x\}, \{a'_1, y\}$ are not heavy pairs of G by Lemma 1 (3), we have $\{x, y\}$ is a heavy pair of G . We assume without loss of generality that x is heavy in G .

If a is also heavy in G , then by Claim 2, a, x have a common neighbor in I_R , contradicting the fact that $N_{I_R}(a)$ is an independent set. So we conclude that a is not heavy in G .

Since $\{x, y\}$ is a heavy pair of G , by Lemma 2 (4), x, y have two common neighbors in I_R . Let x', y' be two vertices in $N_{I_R}(x) \cap N_{I_R}(y)$. Clearly $ax', ay' \notin E(G)$. If $x'y' \in E(G)$, then $\{x, x', y', a, a'_1, a'_2\}$ induces a Z_3 in G . Since a is light, x', a'_2 are heavy. Note that x' and a'_2 are dissociated, a contradiction. Thus we obtain that $x'y' \notin E(G)$.

Note that $\{x, x', y', a\}$ induces a claw in G , and a is light in G . So one vertex of $\{x', y'\}$, say x' , is heavy in G . By Claim 2, x, x' have a common neighbor x'' in I_R . Clearly $ax'' \notin E(G)$. Thus $\{x, x', x'', a, a'_1, a'_2\}$ induces a Z_3 . Since a is not heavy in G , x', a'_2 are heavy in G , a contradiction. \square \square

Now let $N_{I_R}(a) = \{x\}$.

Claim 4. $N_R(a) = V(R) \setminus \{a\}$.

Proof. Suppose that $V(R) \setminus (\{a\} \cup N_R(a)) \neq \emptyset$. By Lemma 2 (1), $R - x$ is connected. Let y be a vertex in $V(R) \setminus (\{a\} \cup N_R(a))$ such that a, y have a common neighbor z in $R - x$. Note that z is a frontier vertex of R . Let z' be a vertex in $N_{G-R}(z)$. Then $\{z, y, a, z'\}$ induces a claw in G . Since $\{a, z'\}, \{y, z'\}$ are not heavy pairs of G , $\{a, y\}$ is a heavy

pair of G . By Lemma 2 (4), a, y have two common neighbors in I_R , contradicting Claim 3. □ □

By Claims 3 and 4, we can see that $|I_R| = 1$. Recall that one edge of $\{ab, bc, ac\}$ is not in $E(G)$. By Claim 4, $ab, ac \in E(G)$. This implies that $bc \notin E(G)$, and $\{a, b, c, a'_1\}$ induces a claw in G . Since $\{b, a'_1\}, \{c, a'_1\}$ are not heavy pairs of G , $\{b, c\}$ is a heavy pair of G . By Lemma 2 (4), b and c have two common neighbors in I_R , contradicting the fact that $|I_R| = 1$. □ □

Following [3], we define \mathcal{P} to be the class of graphs obtained by taking two vertex-disjoint triangles $a_1a_2a_3a_1, b_1b_2b_3b_1$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} = a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$, for $k_i \geq 3$ or by a triangle $a_i b_i c_i a_i$. We denote the graphs in \mathcal{P} by P_{l_1, l_2, l_3} , where $l_i = k_i$ if a_i, b_i are joined by a path P_{k_i} , and $l_i = T$ if a_i, b_i are joined by a triangle. Note that $L_1 = P_{T, T, T}$ and $L_2 = P_{3, T, T}$.

Theorem 9. (Brousek [3]) *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph $H \in \mathcal{P}$.*

3 Proof of Theorem 4

Let $G' = cl_o(G)$. If G' is hamiltonian, then so is G by Theorem 8, and we are done. Now we assume that G' is not hamiltonian. By Theorem 9, G' contains an induced subgraph $H = P_{l_1, l_2, l_3} \in \mathcal{P}$. We denote the vertices of H by a_i, b_i, c_i and c_i^j as in Section 2. By Lemma 3, G' is Z_3 -center-heavy.

Claim 1. For $i \in \{1, 2, 3\}$, $l_i = 3$ or T ; and at most one of $\{l_1, l_2, l_3\}$ is 3.

Proof. If one of $\{l_1, l_2, l_3\}$ is at least 4, say $l_1 \geq 4$, then the subgraph of G' induced by $\{a_1, a_2, a_3, c_1^1, c_1^2, c_1^3\}$ is a Z_3 (we set $c_1^3 = b_1$ if $l_1 = 4$). Thus c_1^1 is heavy in G' . If $l_2 = T$, then the subgraph of G' induced by $\{a_2, a_1, a_3, b_2, b_1, c_1^{l_1-2}\}$ is a Z_3 , implying b_2 is heavy in G' . But c_1^1 and b_2 are dissociated, a contradiction. If $l_2 \neq T$, then the subgraph of G' induced by $\{a_2, a_1, a_3, c_2^1, \dots, c_2^{l_2-2}, b_2, b_1\}$ is a Z_r with $r \geq 3$, implying c_2^1 is heavy in G' . But c_1^1 and c_2^1 are dissociated, a contradiction again. Thus we conclude that $l_i = 3$ or T for all $i = 1, 2, 3$.

If two of $\{l_1, l_2, l_3\}$ equal 3, say $l_1 = l_2 = 3$, then the subgraphs of G' induced by $\{a_1, a_2, a_3, c_1^1, b_1, b_2\}$ and by $\{a_2, a_1, a_3, c_2^1, b_2, b_1\}$ are Z_3 's. This implies that c_1^1 and c_2^1 are heavy in G' . But c_1^1 and c_2^1 are dissociated, a contradiction. Thus we conclude that at most one of $\{l_1, l_2, l_3\}$ is 3. □ □

By Claim 1, we assume without loss of generality that $l_2 = l_3 = T$ and $l_1 = 3$ or T . If G' has only the nine vertices in H , then $G' = L_1$ or L_2 , and G has no o -eligible vertices. This implies that $G = L_1$ or L_2 . Now we assume that G' has a tenth vertex.

Let A be the region containing $\{a_1, a_2, a_3\}$ and B be the region containing $\{b_1, b_2, b_3\}$. For $l_i = T$, let C_i be the region containing $\{a_i, b_i, c_i\}$; and if $l_1 = 3$, then let C_1^1 and C_1^2 be the regions containing $\{a_1, c_1^1\}$ and $\{b_1, c_1^1\}$, respectively.

Claim 2. $|V(A)| = |V(B)| = |V(C_i)| = 3$; and if $l_1 = 3$, then $|V(C_1^1)| = |V(C_1^2)| = 2$.

Proof. Suppose that $|V(A)| \geq 4$. Let x be a vertex in $V(A) \setminus \{a_1, a_2, a_3\}$. Then the subgraphs of G' induced by $\{a_2, a_1, x, b_2, b_3, c_3\}$ and by $\{a_3, a_1, x, b_3, b_2, c_2\}$ are Z_3 's. This implies that b_2 and b_3 are heavy in G' . Since there are two vertices a_1, x nonadjacent to b_2 and b_3 , b_2 and b_3 have at least two common neighbors in G' . Let y be a common neighbor of b_2 and b_3 in G' other than b_1 . Then $y \in V(B)$, and the subgraphs of G' induced by $\{b_2, b_1, y, a_2, a_3, c_3\}$ is a Z_3 . Thus a_2 is heavy in G' . By Lemma 1 (3), a_2 and b_3 are associated, a contradiction. Thus we conclude that $|V(A)| = 3$, and similarly, $|V(B)| = 3$.

Suppose that $|V(C_i)| \geq 4$ for $l_i = T$. We assume up to symmetry that $|V(C_2)| \geq 4$. Let x be a vertex in $V(C_2) \setminus \{a_2, b_2, c_2\}$. Then the subgraph of G' induced by $\{a_2, c_2, x, a_3, b_3, b_1\}$ is a Z_3 , implying that a_3 is heavy in G . If $l_1 = T$, then the subgraph of G' induced by $\{b_2, c_2, x, b_1, a_1, a_3\}$ is a Z_3 ; if $l_1 = 3$, then the subgraph of G' induced by $\{b_2, c_2, x, b_1, c_1, a_1\}$ is a Z_3 . In any case, we have b_1 is heavy in G' . But a_3 and b_1 are dissociated in G , a contradiction.

Suppose that $l_1 = 3$ and $|V(C_1^1)| \geq 3$. Let x be a vertex in $V(C_1^1) \setminus \{a_1, c_1^1\}$. Then the subgraphs of G' induced by $\{a_1, c_1^1, x, a_2, b_2, b_3\}$ and by $\{c_1^1, a_1, x, b_1, b_2, c_2\}$ are Z_3 's. This implies that a_2 and b_1 are heavy in G' . But a_2 and b_1 are dissociated, a contradiction. Thus we conclude that $|V(C_1^1)| = 2$, and similarly, $|V(C_1^2)| = 2$. \square \square

In the following, we set $S = \{v \in V(G') : N_{G'}(v) \cap V(H) \neq \emptyset\}$.

Claim 3. $l_1 = 3$, and for $x \in S$, $xc_2, xc_3 \in E(G')$.

Proof. By Claim 2, all the neighbors of $a_1, a_2, a_3, b_1, b_2, b_3$ and c_1^1 (if $l_1 = 3$) are in H . Note that G' has at least 10 vertices. The vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and c_1^1 (if $l_1 = 3$) are not heavy in G' .

Let x be a vertex in S . Suppose that $l_1 = T$. Note that x cannot be adjacent to all the three vertices c_1, c_2, c_3 . We assume up to symmetry that $xc_1 \in E(G')$ and $xc_2 \notin E(G')$. Then the subgraph of G' induced by $\{a_2, b_2, c_2, a_1, c_1, x\}$ is a Z_3 , implying a_1 is heavy in G' , a contradiction. Thus we conclude that $l_1 = 3$.

Suppose that one edge of xc_2, xc_3 is not in $E(G')$, say $xc_2 \notin E(G')$. Then the subgraph of G' induced by $\{a_2, b_2, c_2, a_3, c_3, x\}$ is a Z_3 , implying a_3 is heavy in G' , a contradiction. Thus we conclude that $xc_2, xc_3 \in E(G')$. \square \square

Let x be a vertex in S . By Claim 3, $xc_2, xc_3 \in E(G')$. If G' has only ten vertices, then $C = a_1a_2a_3c_3xc_2b_2b_3b_1c_1a_1$ is a Hamilton cycle of G' , a contradiction. Suppose now that G' has an eleventh vertex. Since G' is 2-connected, let x' be a vertex in $S \setminus \{x\}$. By Claim 3, $x'c_2, x'c_3 \in E(G')$. Thus $xx' \in E(G')$. Note that $N_{G'}(x)$ is neither a clique nor a disjoint union of two cliques of G' . This implies that x is an o -eligible vertex of G' , a contradiction.

The proof is complete. \square

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