

The Union-Closed Sets Conjecture for Small Families

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Abstract

We prove that the union-closed sets conjecture is true for separating union-closed families \mathcal{A} with $|\mathcal{A}| \leq 2 \left(m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right)$ where m denotes the number of elements in \mathcal{A} .

1 Introduction

A family \mathcal{A} of sets is said to be *union-closed* if for any two member sets $A, B \in \mathcal{A}$ their union $A \cup B$ is also a member of \mathcal{A} .

A well-known conjecture is the *Union-Closed Sets Conjecture* which is also called *Frankl's conjecture*:

Conjecture 1.1. *Any finite non-empty union-closed family of sets has an element that is contained in at least half of its member sets.*

There are many papers considering this conjecture. So it is known to be true if \mathcal{A} has at most 12 elements [8] or at most 50 member sets [4, 7] or if the number of member sets is large compared to the number m of elements, that is $|\mathcal{A}| \geq \frac{2}{3}2^m$ [1]. Nevertheless, the conjecture is still far from being proved or disproved. A good survey on the current state of this conjecture is given by Bruhn and Schaudt [2].

In this paper we consider the case that the number of member-sets is small compared to the number of elements. But first we recall some basic definitions and results. Let \mathcal{A} be a union-closed set. We call $U(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ the *universe* of \mathcal{A} . For an element $x \in U(\mathcal{A})$ the cardinality of $|\{A \in \mathcal{A} : x \in A\}|$ is called the *frequency* of x . Thus the union-closed sets conjecture states that there exists an element $x \in U(\mathcal{A})$ of frequency at least $\frac{1}{2}|\mathcal{A}|$.

A family \mathcal{A} is called *separating* if for any two distinct elements $x, y \in U(\mathcal{A})$ there exists a set $A \in \mathcal{A}$ that contains exactly one of the elements x and y . We

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can restrict ourselves to separating union-closed families: If there exist elements x and y such that each member set $A \in \mathcal{A}$ that contains x also contains y , then we can delete x from each such set and obtain a new family of the same cardinality that is still union-closed. Falgas-Ravry showed that there are some sets in \mathcal{A} satisfying certain conditions which help us to analyze small separating union-closed families:

Theorem 1.2 (Falgas-Ravry [3]). *Let \mathcal{A} be a separating union-closed family and let x_1, \dots, x_m be the elements of $U(\mathcal{A})$ labeled in order of increasing frequency. Then there exist sets $X_0, \dots, X_m \in \mathcal{A}$ such that*

$$x_i \notin X_i \quad \forall i \in \{1, \dots, m\} \quad (1)$$

and

$$\{x_{i+1}, \dots, x_m\} \subset X_i \quad \forall i \in \{0, \dots, m\} \quad (2)$$

Proof. As \mathcal{A} is separating, for any $1 \leq i < j \leq m$ there exists a set $X_{ij} \in \mathcal{A}$ such that $x_i \notin X_{ij}$ and $x_j \in X_{ij}$. For all $1 \leq i \leq m-1$ let $X_i = \bigcup_{j=i+1}^m X_{ij}$ and set $X_0 = U(\mathcal{A})$. \square

The previous theorem directly implies that the conjecture is satisfied for small families:

Lemma 1.3. *Any separating family on m elements with at most $2m$ member sets satisfies the Union-Closed Sets Conjecture.*

Proof. Consider the sets X_0, \dots, X_{m-1} constructed in Theorem 1.2 and observe that the most frequent element x_m is contained in all these sets. As these sets are pairwise different, x_m is contained in at least m of all member sets of \mathcal{A} . \square

In this paper we show that the Union-Closed Sets Conjecture is also satisfied for families that contain (slightly) more than $2m$ member sets. Considering such families is motivated by a result of Hu (see also [2]):

Theorem 1.4 (Hu [5]). *Suppose there is a $c > 2$ so that any separating union-closed family \mathcal{A}' with $|\mathcal{A}'| \leq c|U(\mathcal{A}')|$ satisfies the Union-Closed Sets Conjecture. Then, for every union-closed family \mathcal{A} , there is an element $x \in U(\mathcal{A})$ of frequency*

$$|\{A \in \mathcal{A} : x \in A\}| \geq \frac{c-2}{2(c-1)}|\mathcal{A}|. \quad (3)$$

Therefore, if the Union-Closed Sets Conjecture is satisfied for 'small' families, then for any union-closed family there exists an element that appears with a frequency at least a constant fraction of the number of member sets. In this paper we push the bound over $2m$, but for increasing m it still converges slowly towards $2m$.

2 Frankl's Conjecture for Small Families

Combining and extending the idea of the proof of Theorem 1.2 and an argument of Knill [6] we get the main result of this paper.

Theorem 2.1. *The Union-Closed Sets Conjecture is true for separating union-closed families \mathcal{A} with a universe containing m elements satisfying*

$$|\mathcal{A}| \leq 2 \left(m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right).$$

Proof. Let \mathcal{A} be a separating union-closed family, let the elements x_1, \dots, x_m of $U(\mathcal{A})$ be labeled in order of increasing frequency and set $n = |\mathcal{A}|$. Assume that each element appears in at most $m + c$ member sets. We compute an upper bound on the size of n .

For $i \in \{1, \dots, m\}$ we set

$$M_i = \bigcup_{A \in \mathcal{A}: x_i \notin A} A \tag{4}$$

to be the union of all sets containing x_i and we set $M_0 = U$. If the sets X_i , $i \in \{0, \dots, m\}$, are chosen as in Theorem 1.2, then we have $X_i \subset M_i$ for all $i \in \{0, \dots, m-1\}$ and thus

$$\{x_{i+1}, \dots, x_m\} \subseteq M_i. \tag{5}$$

Let $\tilde{U} = \{x_i : \exists A \in \mathcal{A} \text{ with } \max_{x_j \in A} j\}$ be the set of all x_i which are the elements with the highest index in some set A .

For $x_i \in \tilde{U}$ we set

$$A_i = \bigcup_{A \in \mathcal{A}: i = \max\{j: x_j \in A\}} A. \tag{6}$$

By definition $x_i \in A_i$. Now consider $j > i$. As $x_j \notin A_i$ we have $A_i \subset M_j$. Together with (5) we have

$$x_i \in M_j \quad \forall x_i \in \tilde{U}, j \in \{0, \dots, m-1\}, i \neq j. \tag{7}$$

Observe that every non-empty member set of \mathcal{A} touches \tilde{U} . Following an argument of Knill [6] let $\hat{U} \subseteq \tilde{U}$ be minimal such that every non-empty set of \mathcal{A} touches \hat{U} . Then for all $x_i \in \hat{U}$ there exists a set $A \in \mathcal{A}$ with $\hat{U} \cap A = \{x_i\}$; if not, $\hat{U} \setminus \{x_i\}$ still touches every member set of \mathcal{A} contradicting the minimality of \hat{U} . Therefore as \mathcal{A} is union-closed, for each $B \subseteq \hat{U}$ there exists a set $P_B \in \mathcal{A}$ with $P_B \cap \hat{U} = B$. Let $\mathcal{P} = \{P_B : B \subseteq \hat{U}\}$. The sets in \mathcal{P} are pairwise disjoint and each element $x_i \in \hat{U}$ is contained in exactly half of the sets. Setting $k = |\hat{U}|$, we conclude that there are 2^k sets in \mathcal{P} containing in total $k2^{k-1}$ elements from \hat{U} .

Note, that \mathcal{P} might contain the sets M_i for $x_i \in \hat{U}$ and one additional set M_j with $\hat{U} \subset M_j$. But then $\{M_0, \dots, M_{m-1}\}$ contains $m - k$ sets that are not in \mathcal{P} and each of these sets contains all elements of \hat{U} .

Before we compute an upper bound for the number of elements in \mathcal{A} we summarize the previous observations:

- Each of the k elements in \hat{U} appears in at most $m + c$ member sets,
- the 2^k sets in \mathcal{P} contain in total $k2^{k-1}$ copies of elements of \hat{U} ,
- there are $m - k$ additional member sets, each containing all elements of \hat{U} and
- all remaining member sets contain at least one element of \hat{U} .

We conclude:

$$n \leq k(m + c) + (2^k - k)2^{k-1} + (m - k)(1 - k) \quad (8)$$

$$= m + kc + (2 - k)2^{k-1} + k^2 - k. \quad (9)$$

Suppose the Union-Closed Sets Conjecture is wrong, that is, $n > 2(m + c)$ or $\frac{n}{2} - m > c$. Then

$$n \leq m + k\left(\frac{n}{2} - m\right) + (2 - k)2^{k-1} + k^2 - k \quad (10)$$

or

$$n \geq 2\frac{(k-1)m + (k-2)2^{k-1} + k - k^2}{k-2} \quad (11)$$

$$\geq 2\left(m + 2^{k-1} + \frac{m}{k-2} - k - 3\right). \quad (12)$$

We conclude that the conjecture is true for all n satisfying

$$n \leq 2\left(m + \min_{k \in \mathbb{N}} \left(2^{k-1} + \frac{m}{k-2} - k - 3\right)\right). \quad (13)$$

The function $f_m(k) := 2^{k-1} + \frac{m}{k-2} - k - 3$ is convex. Živković et al. [8] showed that the Union-Closed Sets Conjecture is satisfied for $m \leq 12$ so we can assume that $m \geq 13$. In this case the minimum of $f_m(k)$ is obtained in the interval $[5, \log_2(m)]$ and we get

$$f_m(k) = \max\left\{2^{k-1}, \frac{m}{k-2}\right\} + \left(\min\left\{2^{k-1}, \frac{m}{k-2}\right\} - 3 - k\right) \quad (14)$$

$$\geq \max\left\{2^{k-1}, \frac{m}{k-2}\right\} \quad (15)$$

$$\geq \min_{k'} \left(\max\left\{2^{k'-1}, \frac{m}{k'-2}\right\}\right) \quad (16)$$

$$\geq \max_{k'} \left(\min\left\{2^{k'-1}, \frac{m}{k'-2}\right\}\right). \quad (17)$$

The last inequality is due to the fact that 2^{k-1} is increasing in k while $\frac{m}{k-2}$ is decreasing in k .

Setting $k' = \log_2(m) - \log_2 \log_2(m) + 2$ we get

$$\begin{aligned}
\log_2 \left(\frac{m}{k' - 2} \right) &= \log_2(m) - \log_2(\log_2(m) - \log_2 \log_2(m)) \\
&= \log_2(m) - \log_2 \log_2(m) - \log_2 \left(1 - \frac{\log_2 \log_2(m)}{\log_2(m)} \right) \\
&\leq \log_2(m) - \log_2 \log_2(m) + 1 \\
&= \log_2(2^{k'}).
\end{aligned}$$

Inserting this result in (17) and (13) we finally obtain that the Union-Closed Sets Conjecture is true for all n satisfying

$$n \leq 2 \left(m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right). \quad (18)$$

□

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References

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