3-Rainbow index and forbidden subgraphs^{*}

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Abstract

A tree in an edge-colored connected graph G is called a rainbow tree if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S*-tree if it connects S in G. A *k*-rainbow coloring of G is an edgecoloring of G having the property that for every set S of k vertices of G, there exists a rainbow *S*-tree in G. The minimum number of colors that are needed in a *k*-rainbow coloring of G is the *k*-rainbow index of G, denoted by $rx_k(G)$. The Steiner distance d(S) of a set S of vertices of G is the minimum size of an S-tree T. The *k*-Steiner diameter $sdiam_k(G)$ of G is defined as the maximum Steiner distance of S among all sets S with k vertices of G. In this paper, we focus on the 3-rainbow index of graphs and find all finite families \mathcal{F} of connected graphs, for which there is a constant $C_{\mathcal{F}}$ such that, for every connected \mathcal{F} -free graph G, $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$.

Keywords: rainbow tree, *k*-rainbow index, 3-rainbow index, forbidden subgraphs.

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1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, ..., t\}, t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow

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path connecting them. For a connected graph G, the rainbow connection number of G, denoted by rc(G), is defined as the minimum number of colors that are needed to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [4] and have been well-studied since then. For further details, we refer the reader to a survey paper [8] and a book [9].

In [5], Chartrand et al. generalized the concept of rainbow path to rainbow tree. A tree in an edge-colored graph G is called a rainbow tree if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an S-tree if it connects S in G. Let G be a connected graph of order n. For a fixed integer k with $2 \leq k \leq n$, a k-rainbow coloring of G is an edge-coloring of G having the property that for every k-subset S of G, there exists a rainbow S-tree in G, and in this case, the graph G is called k-rainbow connected. The minimum number of colors that are needed in a k-rainbow coloring of G is the k-rainbow index of G, denoted by $rx_k(G)$. Clearly, $rx_2(G)$ is just the rainbow connection number rc(G) of G. In the sequel, we assume that $k \geq 3$. It is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. Recently, some results on the k-rainbow index have been published, especially on the 3-rainbow index. We refer to [3, 6] for more details.

The Steiner distance d(S) of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a Steiner S-tree or simply a Steiner tree. The k-Steiner diameter $sdiam_k(G)$ of G is defined as the maximum Steiner distance of S among all k-subsets S of G. Then the following observation is immediate.

Observation 1. [5] For every connected graph G of order $n \ge 3$ and each integer k with $3 \le k \le n$,

$$k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1.$$

The authors of [5] showed that the k-rainbow index of trees can achieve the upper bound.

Proposition 1. [5] Let T be a tree of order $n \ge 3$. For each integer k with $3 \le k \le n$,

$$rx_k(T) = n - 1.$$

From above, we notice that for a fixed integer k with $k \geq 3$, the difference $rx_k(G) - sdiam_k(G)$ can be arbitrarily large. In fact, if G is a star $K_{1,n}$, then we have $rx_k(G) - sdiam_k(G) = n - k$.

They also determined the precise values for the k-rainbow index of the cycle C_n and the 3-rainbow index of the complete graph K_n .

Theorem 1. [5] For integers k and n with $3 \le k \le n$,

$$rx_k(C_n) = \begin{cases} n-2 & \text{if } k = 3 \text{ and } n \ge 4\\ n-1 & \text{if } k = n = 3 \text{ or } 4 \le k \le n. \end{cases}$$

Theorem 2. [5]

$$rx_3(K_n) = \begin{cases} 2 & \text{if } 3 \le n \le 5\\ 3 & \text{if } n \ge 6. \end{cases}$$

Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free if G does not contain any induced subgraph isomorphic to a graph from \mathcal{F} . Specifically, for $\mathcal{F} = \{X\}$ we say that G is X-free, for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)-free, and for $\mathcal{F} = \{X, Y, Z\}$ we say that G is (X, Y, Z)-free. The members of \mathcal{F} will be referred as forbidden induced subgraphs in this context. If $\mathcal{F} = \{X_1, X_2, \ldots, X_k\}$, we also refer to the graphs X_1, X_2, \ldots, X_k as a forbidden k-tuple, and for $|\mathcal{F}| = 2$ and 3 we also say forbidden pair and forbidden triple, respectively.

In [7], Holub et al. considered the question: For which families \mathcal{F} of connected graphs, a connected \mathcal{F} -free graph G satisfies $rc(G) \leq diam(G) + C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is a constant (depending on \mathcal{F}), and they gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where N denotes the *net*, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 3. [7] Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies $rc(G) \leq diam(G) + C_X$, if and only if $X = P_3$.

Theorem 4. [7] Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq$ $diam(G) + C_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ ($r \geq 4$) and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.

Let $k \geq 3$ be a positive integer. From Observation 1, we know that the k-rainbow index is lower bounded by the k-Steiner diameter. So we wonder an analogous question concerning the k-rainbow index of graphs. In this paper, we will consider the following question.

For which families \mathcal{F} of connected graphs, there is a constant $C_{\mathcal{F}}$ such that $rx_k(G) \leq sdiam_k(G) + C_{\mathcal{F}}$ if a connected graph G is \mathcal{F} -free ?

In general, it is very difficult to give answers to the above question, even if one considers the case k = 4. So, in this paper we pay our attention only on the case k = 3. In Sections 3, 4 and 5, we give complete answers for the 3-rainbow index when $|\mathcal{F}| = 1, 2$ and 3, respectively. Finally, we give a complete characterization for an arbitrary finite family \mathcal{F} .

2 Preliminaries

In this section, we introduce some further terminology and notation that will be used in the sequel. Throughout the paper, \mathbb{N} denotes the set of all positive integers.

Let G be a graph. We use V(G), E(G), and |G| to denote the vertex set, edge set, and the order of G, respectively. For $A \subseteq V(G)$, |A| denotes the number of vertices in A, and G[A] denotes the subgraph of G induced by the vertex set A. For two disjoint subsets X and Y of V(G), we use E[X, Y] to denote the set of edges of G between X and Y. For graphs X and G, we write $X \subseteq G$ if X is a subgraph of G, $X \subseteq G$ if X is an induced subgraph of G, and $X \cong G$ if X is isomorphic to G. In an edge-colored graph G, we use c(uv) to denote the color assigned to an edge $uv \in E(G)$.

Let G be a connect graph. For $u, v \in V(G)$, a path in G from u to v will be referred as a (u, v)-path, and, whenever necessary, it will be considered with orientation from u to v. The distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The eccentricity of a vertex v is $ecc(v) := max_{x \in V(G)}d_G(v, x)$. The diameter of G is $diam(G) := max_{x \in V(G)}ecc(x)$, and the radius of G is rad(G) := $min_{x \in V(G)ecc(x)}$. One can easily check that $rad(G) \leq diam(G) \leq 2rad(G)$. A vertex x is central in G if ecc(x) = rad(G). Let $D \subseteq V(G)$ and $x \in V(G) \setminus D$. Then we call a path $P = v_0v_1 \dots v_k$ is a v-D path if $v_0 = v$ and $V(P) \cap D = v_k$, and $d_G(v, D) := min_{w \in D}d_G(v, w)$.

For a set $S \subseteq V(G)$ and $k \in \mathbb{N}$, we use $N_G^k(S)$ to denote the *neighborhood at* distance k of S, i.e., the set of all vertices of G at distance k from S. In the special case when k = 1, we simply write $N_G(S)$ for $N_G^1(S)$ and if |S| = 1 with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subseteq V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$. Finally, we will also use the closed neighborhood of a vertex $x \in V(G)$ defined by $N_G^k[x] = (\bigcup_{i=1}^k N_G^i(x)) \cup \{x\}$.

A set $D \subseteq V(G)$ is called *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. In addition, if G[D] is connected, then we call D a *connected dominating set*. A *clique* of a graph G is a subset $Q \subseteq V(G)$ such that G[Q] is complete. A clique is maximum if G has no clique Q' with |Q'| > |Q|. For a graph G, a subset $I \subseteq V(G)$ is called an *independent set* of G if no two vertices of I are adjacent in G. An independent set is maximum if G has no independent set I' with |I'| > |I|.

For two positive integers a and b, the Ramsey number R(a, b) is the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_a (i.e., a complete subgraph on a vertices all of whose edges are colored red) or there is a blue K_b . Ramsey [10] showed that R(a, b) is finite for any a and b.

Finally, we will use P_n to denote the path on n vertices. An edge is called a *pendant* edge if one of its end vertices has degree one.

3 Families with one forbidden subgraph

In this section, we characterize all possible connected graphs X such that every connected X-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, where C_X is a constant.

Theorem 5. Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_X$, if and only if $X = P_3$.

Proof. We have that the graph G is a complete graph since G is P_3 -free. Then from Theorem 2, it follows that $rx_3(G) \leq 3 = sdiam_3(G) + 1$.

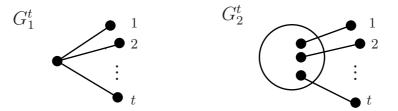


Figure 1: The graphs G_1^t and G_2^t .

Let t be an arbitrarily large integer, set $G_1^t = K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of the complete graph K_t (see Figure 1). We also use K_t^h to denote G_2^t . Since $rx_3(G_1^t) = t$ but $sdiam_3(G_1^t) = 3$, X is an induced subgraph of G_1^t . Clearly, $rx_3(G_2^t) \ge t + 2$ but $sdiam_3(G_2^t) = 5$, and G_2^t is $K_{1,3}$ -free. Hence, $X = K_{1,2} = P_3$. The proof is thus complete.

4 Forbidden pairs

The following statement, which is the main result of this section, characterizes all possible forbidden pairs X, Y for which there is a constant C_{XY} such that $rx_3(G) \leq sdiam_3(G) + C_{XY}$ if G is (X, Y)-free. Since any P_3 -free graph is a complete graph, we exclude the case that one of X, Y is P_3 .

Theorem 6. Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$, if and only if (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.

The proof of Theorem 6 will be divided into two parts. We prove the necessity in Proposition 2, and then we establish the sufficiency in Theorem 7.

Proposition 2. Let $X, Y \neq P_3$ be a pair of connected graphs for which there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $rx_3(G) \leq$ $sdiam_3(G) + C_{XY}$. Then, (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.

Proof. Let t be an arbitrarily large integer, and set $G_3^t = C_t$. We will also use the graphs G_1^t and G_2^t shown in Figure 1.

Consider the graph G_1^t . Since $sdiam_3(G_1^t) = 3$ but $rx_3(G_1^t) = t$, we have, up to symmetry, $X = K_{1,r}, r \ge 3$. Then we consider the graphs G_2^t and G_3^t . It is easy to verify that $sdiam_3(G_2^t) = 5$ but $rx_3(G_2^t) \ge t + 2$, and $sdiam_3(G_3^t) = \lceil \frac{2}{3}t \rceil$ while $rx_3(G_3^t) \ge t - 2 \ge \frac{3}{2}(sdiam_3(G_3^t) - 1) - 2$, respectively. Clearly, G_2^t and G_3^t are both $K_{1,3}$ -free, so neither of them contains X, implying that both G_2^t and G_3^t contain Y. Since the maximum common induced subgraph of them is P_4 , we get that $Y = P_4$. This completes the proof.

Next, we can prove that the converse of Proposition 2 is true.

Theorem 7. Let G be a connected $(P_4, K_{1,r})$ -free graph for some $r \ge 3$. Then $rx_3(G) \le sdiam_3(G) + r + 3$.

Proof. Let G be a connected $(P_4, K_{1,r})$ -free graph $(r \ge 3)$. Then, $sdiam_3(G) \ge 2$. For simplicity, we set V = V(G). Let $S \subseteq V$ be the maximum clique of G.

Claim 1: S is a dominating set.

Proof. Assume that there is a vertex y at distance 2 from S. Let yxu be a shortest path from y to S, where $u \in S$. Because S is the maximum clique, there is some $v \in S$ such that $vx \notin E(G)$. Thus the path $vuxy \cong P_4$, a contradiction. So S is a dominating set.

Let X be the maximum independent set of $G[V \setminus S]$ and $Y = V \setminus (S \cup X)$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$. Furthermore, for any independent set W of graph G[Y], $|N_X(W)| \ge |W|$ since X is maximum.

Claim 2: There is a vertex $v \in S$ such that v is adjacent to all the vertices in X.

Proof. Suppose that the claim fails. Let u be the vertex of S with the largest number of neighbors in X. Set $X_1 = N_X(u)$, $X_2 = X \setminus X_1$. Then, $X_2 \neq \emptyset$ according to our assumption. Pick a vertex w in X_2 . Then, $uw \notin E(G)$. Let v be a neighbor of w in S. For any vertex z in X_1 , G[w, v, u, z] can not be an induced P_4 , so vz must be an edge of G. Thus, $N_X(v) \supseteq N_X(u) \cup \{w\}$, contradicting the maximum of u.

Let z be the vertex in S which is adjacent to all the vertices of X. Set $X = \{x_1, x_2, \ldots, x_\ell\}$. Then, $0 \leq \ell \leq r-1$ since G is $K_{1,r}$ -free. Now we demonstrate a 3-rainbow coloring of G using at most $\ell + 6$ colors. Assign color *i* to the edge zx_i , and i + 1 to the edge $x_i y$ where $1 \leq i \leq \ell$ and $y \in Y$. Color E[S, Y] with color $\ell + 2$ and E(G[Y]) with color $\ell + 3$. Give a 3-rainbow coloring of G[S] using colors from $\{\ell + 4, \ell + 5, \ell + 6\}$. And color the remaining edges arbitrarily (e.g., all of them with color 1). Next, we prove that this coloring is a 3-rainbow coloring of G.

Let $W = \{u, v, w\}$ be a 3-subset of V.

(i) $\{u, v, w\} \subseteq S \cup X$. There is a rainbow tree containing W.

(ii) $\{u, v\} \subseteq S \cup X, w \in Y$. We can find a rainbow tree containing an edge in E[S, Y] that connects W.

 $(iii) \ u \in S \cup X, \{v, w\} \subseteq Y.$

a) If $vw \in E(G)$, then there is a rainbow tree containing the edge vw that connects W.

b) If $vw \notin E(G)$, then we have $|N_X(\{v, w\})| \ge |\{v, w\}| = 2$. So there are two vertices x_i and $x_j (i \ne j)$ in X adjacent to v and w, respectively. As $i + 1 \ne j + 1$, so either $i + 1 \ne c(zu)$ or $j + 1 \ne c(zu)$. Without loss of generality, we assume that $i + 1 \ne c(zu)$ and s is a neighbor of w in S. Then there is a rainbow tree containing the edges zu, uv, sw, sz if $u = x_i$ or the edges zu, zx_i, x_iv, sw, sz if $u \ne x_i$. $(iv) \{u, v, w\} \subseteq Y.$

a) If $\{uv, vw, uw\} \cap E(G) \neq \emptyset$, for example, $uv \in E(G)$, then we have a rainbow tree containing the edges zx_i, x_iu, uv, sw, sz where x_i is a neighbor of u in X and s is a neighbor of w in S.

b) If $\{uv, vw, uw\} \cap E(G) = \emptyset$, then we have $|N_X\{u, v, w\}| \ge |\{u, v, w\}| = 3$, so we can find three distinct vertices x_i, x_j, x_k in X such that $\{x_iu, x_jv, x_kw\} \subseteq E(G)$. We may assume that i < j < k, so $k + 1 \notin \{i, j, k, i + 1, j + 1\}$ and $k \neq i + 1$. Then there is a rainbow tree containing the edges $zx_i, x_iu, zx_k, x_kw, sv, sz$ where s is a neighbor of v in S.

Thus the coloring is a 3-rainbow coloring of G using at most $\ell + 6 \leq r + 5 \leq sdiam_3(G) + r + 3$ colors. The proof is complete.

Combining Proposition 2 and Theorem 7, we can easily get Theorem 6.

Remark When the maximum independent set of $G[V \setminus S]$, X, satisfies $|X| = \ell \ge 4$, we just need $\ell + 5$ colors in the proof of Theorem 7: for the edges $x_{\ell}y$, we can color them with color 1 instead of color $\ell + 1$. It only matters when the case $\{u, v, w\} \subseteq Y$ and $\{uv, vw, uw\} \cap E(G) = \emptyset$ happens. Suppose $\{x_iu, x_jv, x_kw\} \subseteq E(G)$ and i < j < k. If $i \neq 1$ or $k \neq \ell$, it is the case in the proof above. So we turn to the case when i = 1 and k = l. If j = 2, then $j + 1 < 4 \le \ell$ (that is why we need the condition $\ell \ge 4$). Thus, there is a rainbow tree containing the edges $zx_j, x_jv, zx_k, x_kw, su, sz$ where s is a neighbor of u in S. If $j \neq 2$, then there is a rainbow tree containing the edges $zx_i, x_iu, zx_j, x_jv, sw, sz$.

5 Forbidden triples

Now, we continue to consider more and obtain an analogous result which characterizes all forbidden triples \mathcal{F} for which there is a constant $C_{\mathcal{F}}$ such that G being \mathcal{F} -free implies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$. We exclude the cases which are covered by Theorems 5 and 6. We set:

$$\begin{split} \mathfrak{F}_{1} &= \{\{P_{3}\}\},\\ \mathfrak{F}_{2} &= \{\{K_{1,r}, P_{4}\} \mid r \geq 3\},\\ \mathfrak{F}_{3} &= \{\{K_{1,r}, Y, P_{\ell}\} \mid r \geq 3, Y \stackrel{\text{IND}}{\subseteq} K_{s}^{h}, s \geq 3, \ell > 4\}. \end{split}$$

Theorem 8. Let \mathcal{F} be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. Then there is a constant $C_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_3$.

First of all, we prove the necessity of the triples given by Theorem 8.

Proposition 3. Let $X, Y, Z \neq P_3$ be connected graphs, $\{X, Y, Z\} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_2$, for which there is a constant C_{XYZ} such that every connected (X, Y)-free graph G

satisfies $rx_3(G) \leq sdiam_3(G) + C_{XYZ}$. Then, (up to symmetry) $X = K_{1,r}(r \geq 3), Y \subseteq K_s^h(s \geq 3)$, and $Z = P_\ell(\ell > 4)$.

Proof. Let t be an arbitrarily large integer, and let G_1^t, G_2^t, G_3^t be the graphs defined in the proof of Proposition 2.

Firstly, we consider the graph G_1^t . Up to symmetry, we have $X = K_{1,r}, r \ge 3$ (for the case r = 2 is excluded by the assumptions). Secondly, we consider the graph G_2^t . The graph G_2^t does not contain X, since it is $K_{1,3}$ -free. Thus, up to symmetry, we have G_2^t contains Y, implying $Y \stackrel{\text{IND}}{\subseteq} K_s^h$ for some $s \ge 3$ (for the case $s \le 2$ is excluded by the assumptions). Finally, we consider the graphs G_3^t and G_3^{t+1} . Clearly, they are $(K_{1,3}, K_3^h)$ -free, so both of them contain neither X nor Y. Hence, we get that $Z = P_\ell$ for some $\ell > 4$ (for the case $\ell \le 4$ is excluded by the assumptions).

This completes the proof.

It is easy to observe that if $X \subseteq X'$, then every (X, Y, Z)-free graph is also (X', Y, Z)-free. Thus, when proving the sufficiency of Theorem 8, we will be always interested in *maximal triples* of forbidden subgraphs, i.e., triples X, Y, Z such that, if replacing one of X, Y, Z, say X, with a graph $X' \neq X$ such that $X \subseteq X'$, then the statement under consideration is not true for (X', Y, Z)-free graphs.

For every vertex $c \in V(G)$ and $i \in \mathbb{N}$, we set $\alpha_i(G, c) = \max\{|M| | M \subseteq N_G^i[c], M \text{ is independent}\}$ and $\alpha_i^0(G, c) = \max\{|M^0| | M^0 \subseteq N_G^i(c), M^0 \text{ is independent}\}.$

Lemma 1. [2] Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha(r, s, i)$ such that, for every connected $(K_{1,r}, K_s^h)$ -free graph G and for every $c \in V(G)$, $\alpha_i(G, c) < \alpha(r, s, i)$.

We use the proof of Lemma 1 to get the following corollary concerning $\alpha_i^0(G, c)$ for each integer $i \ge 1$.

Corollary 1. Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha^0(r, s, i)$ such that, for every connected $(K_{1,r}, K_s^h)$ -free graph G and for every $c \in V(G)$, $\alpha_i^0(G, c) < \alpha^0(r, s, i)$.

Proof. For the sake of completeness, here we give a brief proof concentrating on the upper bound of $\alpha_i^0(G, c)$. We prove the corollary by induction on *i*.

For i = 1, we have $\alpha^0(r, s, 1) = r$, for otherwise G contains a $K_{1,r}$ as an induced subgraph.

Let, to the contrary, *i* be the smallest integer for which $\alpha^0(r, s, i)$ does not exist(i.e., $\alpha_i^0(G, c)$ can be arbitrarily large), choose a graph *G* and a vertex $c \in V(G)$ such that $\alpha_i^0(G, c) \geq (r-2)R(s(2r-3), \alpha^0(r, s, i-1))$, and let $M^0 = \{x_1^0, \ldots, x_k^0\} \subseteq N_G^i(c)$ be an independent set in *G* of size $\alpha_i^0(G, c)$. Obviously, $k \geq (r-2)R(s(2r-3), \alpha^0(r, s, i-1))$. Let Q_j be a shortest (x_j^0, c) -path in *G*, $j = 1, \ldots, k$. We denote $M^1 \subseteq N_G^{i-1}(c)$ the set of all successors of the vertices from M^0 on Q_j , $j = 1, \ldots, k$, and x_j^1 the successor in M^1). Every vertex in M^1 has at most r-2 neighbors in M^0 since *G* is $K_{1,r}$ -free.

Thus, $|M^1| \ge \frac{k}{r-2} \ge R(s(2r-3), \alpha^0(r, s, i-1))$. By the induction assumption and the definition of Ramsey number, $G[M^1]$ contains a complete subgraph $K_{s(2r-3)}$. Choose the notation such that $V(K_{s(2r-3)}) = \{x_1^1, \ldots, x_{s(2r-3)}^1\}$, and set $\widetilde{M^0} = N_{M^0}(K_{s(2r-3)})$. Using a matching between $K_{s(2r-3)}$ and $\widetilde{M^0}$, we can find in G an induced K_s^h with vertices of degree 1 in $\widetilde{M^0}$, a contradiction. For more details about finding the K_s^h , we refer the reader to [2].

Armed with Corollary 1, we can get the following important theorem.

Theorem 9. Let $r \geq 3, s \geq 3$, and $\ell > 4$ be fixed integers. Then there is a constant $C(r, s, \ell)$ such that every connected $(K_{1,r}, K_s^h, P_\ell)$ -free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C(r, s, \ell)$.

Proof. We have $diam(G) \leq \ell - 2$ since G is P_{ℓ} -free. Let c be a central vertex of G, i.e., $ecc(c) = rad(G) \leq diam(G) \leq \ell - 2$. And we set $S_i = \bigcup_{j=1}^i N_G^j[c]$ for an integer $i \geq 1$.

Claim: $rx_3(G[S_i \cup N_G^{i+1}(c)]) \le rx_3(G[S_i]) + \alpha_{i+1}^0(G, c) + 3$

Proof. Let $X = \{x_1, x_2, \ldots, x_{\alpha_{i+1}^0(G,c)}\}$ be the maximum independent set of $N_G^{i+1}(c)$ and $Y = N_G^{i+1}(c) \setminus X$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$ and $s \in S$. Further more, for any independent set W of graph G[Y], we have $|N_X(W)| \ge |W|$ since X is maximum.

Now we demonstrate a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$ using at most $k + \alpha_{i+1}^0(G, c) + 3$ colors, where $k = rx_3(G[S_i])$. We color the edges of $G[S_i]$ using colors $1, 2, \ldots, k$. Color $E[S_i, Y]$ with color k + 1 and E(G[Y]) with color k + 2. And assign color j + k + 2 to the edges $E[\{x_j\}, S_i]$, and j + k + 3 to the edges $E[\{x_j\}, Y]$ where $1 \leq j \leq \alpha_{i+1}^0(G, c)$. With the same argument as the proof of Theorem 7, we can prove that this coloring is a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$.

From the proof of Corollary 1, it follows that $\alpha_1^0(G,c) \leq r-1$ and $\alpha_i^0(G,c) \leq (r-2)R(s(2r-3),\alpha^0(r,s,i-1))-1$ for each integer $i \geq 2$. Let $\mathcal{R}(r,s) = \sum_{i=2}^{ecc(c)} R(s(2r-3),\alpha^0(r,s,i-1))$. Recall that $ecc(c) \leq \ell-2$. Repeated application of Claim gives the following:

$$rx_{3}(G) \leq rx_{3}(G[N_{G}^{ecc(c)-1}[c]]) + \alpha_{ecc(c)}^{0}(G,c) + 3$$

$$\leq \dots$$

$$\leq rx_{3}(c) + \alpha_{1}^{0}(G,c) + \dots + \alpha_{ecc(c)}^{0}(G,c) + 3ecc(c)$$

$$\leq 0 + r + (r-2)\mathcal{R}(r,s) + 2(\ell-2)$$

$$\leq sdiam_{3}(G) + (r-2)(\mathcal{R}(r,s) + 1) + 2(\ell-1).$$

Thus, we complete our proof.

Remark The same as the remark in Section 4: for $i \ge 1$, every time $\alpha_{i+1}^0(G, c) \ge 4$ happens, we can save one color in the Claim of Theorem 9.

6 Forbidden k-tuples for any $k \in \mathbb{N}$

Let $\mathcal{F} = \{X_1, X_2, X_3, \ldots, X_k\}$ be a finite family of connected graphs with $k \geq 4$ for which there is a constant $k_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph satisfies $rx_3(G) \leq$ $sdiam_3(G) + C_{\mathcal{F}}$. Let t be an arbitrarily large integer, and let G_1^t, G_2^t and G_3^t be defined in Proposition 2. For the graph G_1^t , Up to symmetry, we suppose that $X_1 = K_r, r \geq 3$ (for the case r = 2 has been discussed in Section 3). Then, we consider the graphs G_2^t and G_3^t . Notice that G_2^t and G_3^t are both $K_{1,3}$ -free, so neither of them contains X_1 , implying that G_2^t or G_3^t contains X_i , where $i \neq 1$. We may assume that X_2 is an induced subgraph of G_2^t . If G_3^t contains X_2 , then $X_2 = P_4$, which is just the case in Section 4. So we turn to the case that G_3^t contains X_i for some i > 2. Now consider the graphs $G_3^t, G_3^{t+1}, G_3^{t+2}, \ldots, G_3^{t+k}$, each of which contains at least one of X_3, X_4, \ldots, X_k as an induced subgraph due to the analysis above. So it is forced that at least one of these $X_i(i \geq 3)$ is isomorphic to P_l for some $l \geq 5$, which goes back to the case in Section 5. Thus, the conclusion comes out.

Theorem 10. Let \mathcal{F} be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph satisfies $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$, if and only if \mathcal{F} contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.

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