# 3-Rainbow index and forbidden subgraphs* 

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#### Abstract

A tree in an edge-colored connected graph $G$ is called $a$ rainbow tree if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an $S$-tree if it connects $S$ in $G$. A $k$-rainbow coloring of $G$ is an edgecoloring of $G$ having the property that for every set $S$ of $k$ vertices of $G$, there exists a rainbow $S$-tree in $G$. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is the $k$-rainbow index of $G$, denoted by $r x_{k}(G)$. The Steiner distance $d(S)$ of a set $S$ of vertices of $G$ is the minimum size of an $S$-tree $T$. The $k$-Steiner diameter $\operatorname{sdiam}_{k}(G)$ of $G$ is defined as the maximum Steiner distance of $S$ among all sets $S$ with $k$ vertices of $G$. In this paper, we focus on the 3 -rainbow index of graphs and find all finite families $\mathcal{F}$ of connected graphs, for which there is a constant $C_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph $G$, $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{\mathcal{F}}$.


Keywords: rainbow tree, $k$-rainbow index, 3 -rainbow index, forbidden subgraphs.

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## 1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here.

Let $G$ be a nontrivial connected graph with an edge-coloring c: $E(G) \rightarrow\{1,2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in $G$ is called a rainbow path if no two edges of the path are colored with the same color. The graph $G$ is called rainbow connected if for any two distinct vertices of $G$, there is a rainbow

[^0]path connecting them. For a connected graph $G$, the rainbow connection number of $G$, denoted by $r c(G)$, is defined as the minimum number of colors that are needed to make $G$ rainbow connected. These concepts were first introduced by Chartrand et al. in [4] and have been well-studied since then. For further details, we refer the reader to a survey paper [8] and a book [9].

In [5], Chartrand et al. generalized the concept of rainbow path to rainbow tree. A tree in an edge-colored graph $G$ is called a rainbow tree if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an $S$-tree if it connects $S$ in $G$. Let $G$ be a connected graph of order $n$. For a fixed integer $k$ with $2 \leq k \leq n$, a $k$-rainbow coloring of $G$ is an edge-coloring of $G$ having the property that for every $k$-subset $S$ of $G$, there exists a rainbow $S$-tree in $G$, and in this case, the graph $G$ is called $k$-rainbow connected. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is the $k$-rainbow index of $G$, denoted by $r x_{k}(G)$. Clearly, $r x_{2}(G)$ is just the rainbow connection number $r c(G)$ of $G$. In the sequel, we assume that $k \geq 3$. It is easy to see that $r x_{2}(G) \leq r x_{3}(G) \leq \cdots \leq r x_{n}(G)$. Recently, some results on the $k$-rainbow index have been published, especially on the 3-rainbow index. We refer to [3, 6] for more details.

The Steiner distance $d(S)$ of a set $S$ of vertices in $G$ is the minimum size of a tree in $G$ containing $S$. Such a tree is called a Steiner $S$-tree or simply a Steiner tree. The $k$-Steiner diameter $\operatorname{sdiam}_{k}(G)$ of $G$ is defined as the maximum Steiner distance of $S$ among all $k$-subsets $S$ of $G$. Then the following observation is immediate.

Observation 1. [5] For every connected graph $G$ of order $n \geq 3$ and each integer $k$ with $3 \leq k \leq n$,

$$
k-1 \leq \operatorname{sdiam}_{k}(G) \leq r x_{k}(G) \leq n-1
$$

The authors of [5] showed that the $k$-rainbow index of trees can achieve the upper bound.

Proposition 1. [5] Let $T$ be a tree of order $n \geq 3$. For each integer $k$ with $3 \leq k \leq n$,

$$
r x_{k}(T)=n-1
$$

From above, we notice that for a fixed integer $k$ with $k \geq 3$, the difference $r x_{k}(G)-$ $\operatorname{sdiam}_{k}(G)$ can be arbitrarily large. In fact, if $G$ is a star $K_{1, n}$, then we have $r x_{k}(G)-$ $\operatorname{sdiam}_{k}(G)=n-k$.

They also determined the precise values for the $k$-rainbow index of the cycle $C_{n}$ and the 3 -rainbow index of the complete graph $K_{n}$.

Theorem 1. [5] For integers $k$ and $n$ with $3 \leq k \leq n$,

$$
r x_{k}\left(C_{n}\right)= \begin{cases}n-2 & \text { if } k=3 \text { and } n \geq 4 \\ n-1 & \text { if } k=n=3 \text { or } 4 \leq k \leq n .\end{cases}
$$

Theorem 2. [5]

$$
r x_{3}\left(K_{n}\right)= \begin{cases}2 & \text { if } 3 \leq n \leq 5 \\ 3 & \text { if } n \geq 6 .\end{cases}
$$

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph $G$ is $\mathcal{F}$-free if $G$ does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F}=\{X\}$ we say that $G$ is $X$-free, for $\mathcal{F}=\{X, Y\}$ we say that $G$ is $(X, Y)$-free, and for $\mathcal{F}=\{X, Y, Z\}$ we say that $G$ is $(X, Y, Z)$-free. The members of $\mathcal{F}$ will be referred as forbidden induced subgraphs in this context. If $\mathcal{F}=\left\{X_{1}, X_{2}, \ldots, X_{k}\right\}$, we also refer to the graphs $X_{1}, X_{2}, \ldots, X_{k}$ as a forbidden $k$-tuple, and for $|\mathcal{F}|=2$ and 3 we also say forbidden pair and forbidden triple, respectively.

In [7], Holub et al. considered the question: For which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph $G$ satisfies $r c(G) \leq \operatorname{diam}(G)+C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$ ), and they gave a complete answer for $|\mathcal{F}| \in\{1,2\}$ in the following two results (where $N$ denotes the net, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 3. [7] Let $X$ be a connected graph. Then there is a constant $C_{X}$ such that every connected $X$-free graph $G$ satisfies $r c(G) \leq \operatorname{diam}(G)+C_{X}$, if and only if $X=P_{3}$.

Theorem 4. 77 Let $X, Y$ be connected graphs such that $X, Y \neq P_{3}$. Then there is a constant $C_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $r c(G) \leq$ $\operatorname{diam}(G)+C_{X Y}$, if and only if (up to symmetry) either $X=K_{1, r}(r \geq 4)$ and $Y=P_{4}$, or $X=K_{1,3}$ and $Y$ is an induced subgraph of $N$.

Let $k \geq 3$ be a positive integer. From Observation 1, we know that the $k$-rainbow index is lower bounded by the $k$-Steiner diameter. So we wonder an analogous question concerning the $k$-rainbow index of graphs. In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $C_{\mathcal{F}}$ such that $r x_{k}(G) \leq$ sdiam $_{k}(G)+C_{\mathcal{F}}$ if a connected graph $G$ is $\mathcal{F}$-free ?

In general, it is very difficult to give answers to the above question, even if one considers the case $k=4$. So, in this paper we pay our attention only on the case $k=3$. In Sections 3,4 and 5 , we give complete answers for the 3 -rainbow index when $|\mathcal{F}|=1,2$ and 3 , respectively. Finally, we give a complete characterization for an arbitrary finite family $\mathcal{F}$.

## 2 Preliminaries

In this section, we introduce some further terminology and notation that will be used in the sequel. Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

Let $G$ be a graph. We use $V(G), E(G)$, and $|G|$ to denote the vertex set, edge set, and the order of $G$, respectively. For $A \subseteq V(G),|A|$ denotes the number of vertices in $A$, and $G[A]$ denotes the subgraph of $G$ induced by the vertex set $A$. For two disjoint subsets $X$ and $Y$ of $V(G)$, we use $E[X, Y]$ to denote the set of edges of $G$ between $X$ and $Y$. For graphs $X$ and $G$, we write $X \subseteq G$ if $X$ is a subgraph of $G, X \subseteq G$ if $X$
is an induced subgraph of $G$, and $X \cong G$ if $X$ is isomorphic to $G$. In an edge-colored graph $G$, we use $c(u v)$ to denote the color assigned to an edge $u v \in E(G)$.

Let $G$ be a connect graph. For $u, v \in V(G)$, a path in $G$ from $u$ to $v$ will be referred as a $(u, v)$-path, and, whenever necessary, it will be considered with orientation from $u$ to $v$. The distance between $u$ and $v$ in $G$, denoted by $d_{G}(u, v)$, is the length of a shortest $(u, v)$-path in $G$. The eccentricity of a vertex $v$ is $\operatorname{ecc}(v):=\max _{x \in V(G)} d_{G}(v, x)$. The diameter of $G$ is $\operatorname{diam}(G):=\max _{x \in V(G)} \operatorname{ecc}(x)$, and the radius of $G$ is $\operatorname{rad}(G):=$ $\min _{x \in V(G) e c c(x)}$. One can easily check that $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2 \operatorname{rad}(G)$. A vertex $x$ is central in $G$ if $\operatorname{ecc}(x)=\operatorname{rad}(G)$. Let $D \subseteq V(G)$ and $x \in V(G) \backslash D$. Then we call a path $P=v_{0} v_{1} \ldots v_{k}$ is a $v-D$ path if $v_{0}=v$ and $V(P) \cap D=v_{k}$, and $d_{G}(v, D):=\min _{w \in D} d_{G}(v, w)$.

For a set $S \subseteq V(G)$ and $k \in \mathbb{N}$, we use $N_{G}^{k}(S)$ to denote the neighborhood at distance $k$ of $S$, i.e., the set of all vertices of $G$ at distance $k$ from $S$. In the special case when $k=1$, we simply write $N_{G}(S)$ for $N_{G}^{1}(S)$ and if $|S|=1$ with $x \in S$, we write $N_{G}(x)$ for $N_{G}(\{x\})$. For a set $M \subseteq V(G)$, we set $N_{M}(S)=N_{G}(S) \cap M$ and $N_{M}(x)=N_{G}(x) \cap M$. Finally, we will also use the closed neighborhood of a vertex $x \in V(G)$ defined by $N_{G}^{k}[x]=\left(\cup_{i=1}^{k} N_{G}^{i}(x)\right) \cup\{x\}$.

A set $D \subseteq V(G)$ is called dominating if every vertex in $V(G) \backslash D$ has a neighbor in $D$. In addition, if $G[D]$ is connected, then we call $D$ a connected dominating set. A clique of a graph $G$ is a subset $Q \subseteq V(G)$ such that $G[Q]$ is complete. A clique is maximum if $G$ has no clique $Q^{\prime}$ with $\left|Q^{\prime}\right|>|Q|$. For a graph $G$, a subset $I \subseteq V(G)$ is called an independent set of $G$ if no two vertices of $I$ are adjacent in $G$. An independent set is maximum if $G$ has no independent set $I^{\prime}$ with $\left|I^{\prime}\right|>|I|$.

For two positive integers $a$ and $b$, the Ramsey number $R(a, b)$ is the smallest integer $n$ such that in any two-coloring of the edges of a complete graph on $n$ vertices $K_{n}$ by red and blue, either there is a red $K_{a}$ (i.e., a complete subgraph on $a$ vertices all of whose edges are colored red) or there is a blue $K_{b}$. Ramsey [10] showed that $R(a, b)$ is finite for any $a$ and $b$.

Finally, we will use $P_{n}$ to denote the path on $n$ vertices. An edge is called a pendant edge if one of its end vertices has degree one.

## 3 Families with one forbidden subgraph

In this section, we characterize all possible connected graphs $X$ such that every connected $X$-free graph $G$ satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{X}$, where $C_{X}$ is a constant.

Theorem 5. Let $X$ be a connected graph. Then there is a constant $C_{X}$ such that every connected $X$-free graph $G$ satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{X}$, if and only if $X=P_{3}$.

Proof. We have that the graph $G$ is a complete graph since $G$ is $P_{3}$-free. Then from Theorem 2, it follows that $r x_{3}(G) \leq 3=\operatorname{siam}_{3}(G)+1$.


Figure 1: The graphs $G_{1}^{t}$ and $G_{2}^{t}$.

Let $t$ be an arbitrarily large integer, set $G_{1}^{t}=K_{1, t}$, and let $G_{2}^{t}$ denote the graph obtained by attaching a pendant edge to each vertex of the complete graph $K_{t}$ (see Figure 1). We also use $K_{t}^{h}$ to denote $G_{2}^{t}$. Since $r x_{3}\left(G_{1}^{t}\right)=t$ but $\operatorname{sdiam}_{3}\left(G_{1}^{t}\right)=3, X$ is an induced subgraph of $G_{1}^{t}$. Clearly, $r x_{3}\left(G_{2}^{t}\right) \geq t+2$ but $\operatorname{sdiam}_{3}\left(G_{2}^{t}\right)=5$, and $G_{2}^{t}$ is $K_{1,3}$-free. Hence, $X=K_{1,2}=P_{3}$. The proof is thus complete.

## 4 Forbidden pairs

The following statement, which is the main result of this section, characterizes all possible forbidden pairs $X, Y$ for which there is a constant $C_{X Y}$ such that $r x_{3}(G) \leq$ $\operatorname{sdiam}_{3}(G)+C_{X Y}$ if $G$ is $(X, Y)$-free. Since any $P_{3}$-free graph is a complete graph, we exclude the case that one of $X, Y$ is $P_{3}$.

Theorem 6. Let $X, Y \neq P_{3}$ be a pair of connected graphs. Then there is a constant $C_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+$ $C_{X Y}$, if and only if (up to symmetry) $X=K_{1, r}, r \geq 3$ and $Y=P_{4}$.

The proof of Theorem 6 will be divided into two parts. We prove the necessity in Proposition 2, and then we establish the sufficiency in Theorem 7 ,

Proposition 2. Let $X, Y \neq P_{3}$ be a pair of connected graphs for which there is a constant $C_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $r x_{3}(G) \leq$ $\operatorname{sdiam}_{3}(G)+C_{X Y}$. Then, (up to symmetry) $X=K_{1, r}, r \geq 3$ and $Y=P_{4}$.

Proof. Let $t$ be an arbitrarily large integer, and set $G_{3}^{t}=C_{t}$. We will also use the graphs $G_{1}^{t}$ and $G_{2}^{t}$ shown in Figure 1.

Consider the graph $G_{1}^{t}$. Since $\operatorname{sdiam}_{3}\left(G_{1}^{t}\right)=3$ but $r x_{3}\left(G_{1}^{t}\right)=t$, we have, up to symmetry, $X=K_{1, r}, r \geq 3$. Then we consider the graphs $G_{2}^{t}$ and $G_{3}^{t}$. It is easy to verify that $\operatorname{sdiam}_{3}\left(G_{2}^{t}\right)=5$ but $r x_{3}\left(G_{2}^{t}\right) \geq t+2$, and $\operatorname{sdiam}_{3}\left(G_{3}^{t}\right)=\left\lceil\frac{2}{3} t\right\rceil$ while $r x_{3}\left(G_{3}^{t}\right) \geq t-2 \geq \frac{3}{2}\left(\operatorname{sdiam}_{3}\left(G_{3}^{t}\right)-1\right)-2$, respectively. Clearly, $G_{2}^{t}$ and $G_{3}^{t}$ are both $K_{1,3}$-free, so neither of them contains $X$, implying that both $G_{2}^{t}$ and $G_{3}^{t}$ contain $Y$. Since the maximum common induced subgraph of them is $P_{4}$, we get that $Y=P_{4}$. This completes the proof.

Next, we can prove that the converse of Proposition 2 is true.

Theorem 7. Let $G$ be a connected $\left(P_{4}, K_{1, r}\right)$-free graph for some $r \geq 3$. Then $r x_{3}(G) \leq$ $\operatorname{sdiam}_{3}(G)+r+3$.

Proof. Let $G$ be a connected $\left(P_{4}, K_{1, r}\right)$-free graph $(r \geq 3)$. Then, $\operatorname{sdiam}_{3}(G) \geq 2$. For simplicity, we set $V=V(G)$. Let $S \subseteq V$ be the maximum clique of $G$.

Claim 1: $S$ is a dominating set.
Proof. Assume that there is a vertex $y$ at distance 2 from $S$. Let $y x u$ be a shortest path from $y$ to $S$, where $u \in S$. Because $S$ is the maximum clique, there is some $v \in S$ such that $v x \notin E(G)$. Thus the path vuxy $\cong P_{4}$, a contradiction. So $S$ is a dominating set.

Let $X$ be the maximum independent set of $G[V \backslash S]$ and $Y=V \backslash(S \cup X)$. Then for any vertex $y \in Y, y$ is adjacent to some $x \in X$. Furthermore, for any independent set $W$ of graph $G[Y],\left|N_{X}(W)\right| \geq|W|$ since $X$ is maximum.

Claim 2: There is a vertex $v \in S$ such that $v$ is adjacent to all the vertices in $X$.
Proof. Suppose that the claim fails. Let $u$ be the vertex of $S$ with the largest number of neighbors in $X$. Set $X_{1}=N_{X}(u), X_{2}=X \backslash X_{1}$. Then, $X_{2} \neq \emptyset$ according to our assumption. Pick a vertex $w$ in $X_{2}$. Then, $u w \notin E(G)$. Let $v$ be a neighbor of $w$ in $S$. For any vertex $z$ in $X_{1}, G[w, v, u, z]$ can not be an induced $P_{4}$, so $v z$ must be an edge of $G$. Thus, $N_{X}(v) \supseteq N_{X}(u) \cup\{w\}$, contradicting the maximum of $u$.

Let $z$ be the vertex in $S$ which is adjacent to all the vertices of $X$. Set $X=$ $\left\{x_{1}, x_{2}, \ldots, x_{\ell}\right\}$. Then, $0 \leq \ell \leq r-1$ since $G$ is $K_{1, r}$-free. Now we demonstrate a 3 -rainbow coloring of $G$ using at most $\ell+6$ colors. Assign color $i$ to the edge $z x_{i}$, and $i+1$ to the edge $x_{i} y$ where $1 \leq i \leq \ell$ and $y \in Y$. Color $E[S, Y]$ with color $\ell+2$ and $E(G[Y])$ with color $\ell+3$. Give a 3-rainbow coloring of $G[S]$ using colors from $\{\ell+4, \ell+5, \ell+6\}$. And color the remaining edges arbitrarily (e.g., all of them with color 1 ). Next, we prove that this coloring is a 3 -rainbow coloring of $G$.

Let $W=\{u, v, w\}$ be a 3 -subset of $V$.
(i) $\{u, v, w\} \subseteq S \cup X$. There is a rainbow tree containing $W$.
(ii) $\{u, v\} \subseteq S \cup X, w \in Y$. We can find a rainbow tree containing an edge in $E[S, Y]$ that connects $W$.
(iii) $u \in S \cup X,\{v, w\} \subseteq Y$.
a) If $v w \in E(G)$, then there is a rainbow tree containing the edge $v w$ that connects $W$.
b) If $v w \notin E(G)$, then we have $\left|N_{X}(\{v, w\})\right| \geq|\{v, w\}|=2$. So there are two vertices $x_{i}$ and $x_{j}(i \neq j)$ in $X$ adjacent to $v$ and $w$, respectively. As $i+1 \neq j+1$, so either $i+1 \neq c(z u)$ or $j+1 \neq c(z u)$. Without loss of generality, we assume that $i+1 \neq c(z u)$ and $s$ is a neighbor of $w$ in $S$. Then there is a rainbow tree containing the edges $z u, u v, s w, s z$ if $u=x_{i}$ or the edges $z u, z x_{i}, x_{i} v, s w, s z$ if $u \neq x_{i}$.
(iv) $\{u, v, w\} \subseteq Y$.
a) If $\{u v, v w, u w\} \cap E(G) \neq \emptyset$, for example, $u v \in E(G)$, then we have a rainbow tree containing the edges $z x_{i}, x_{i} u, u v, s w, s z$ where $x_{i}$ is a neighbor of $u$ in $X$ and $s$ is a neighbor of $w$ in $S$.
b) If $\{u v, v w, u w\} \cap E(G)=\emptyset$, then we have $\left|N_{X}\{u, v, w\}\right| \geq|\{u, v, w\}|=3$, so we can find three distinct vertices $x_{i}, x_{j}, x_{k}$ in $X$ such that $\left\{x_{i} u, x_{j} v, x_{k} w\right\} \subseteq E(G)$. We may assume that $i<j<k$, so $k+1 \notin\{i, j, k, i+1, j+1\}$ and $k \neq i+1$. Then there is a rainbow tree containing the edges $z x_{i}, x_{i} u, z x_{k}, x_{k} w, s v, s z$ where $s$ is a neighbor of $v$ in $S$.

Thus the coloring is a 3 -rainbow coloring of $G$ using at most $\ell+6 \leq r+5 \leq$ $\operatorname{sdiam}_{3}(G)+r+3$ colors. The proof is complete.

Combining Proposition 2 and Theorem [7, we can easily get Theorem 6,
Remark When the maximum independent set of $G[V \backslash S], X$, satisfies $|X|=\ell \geq 4$, we just need $\ell+5$ colors in the proof of Theorem [7: for the edges $x_{\ell} y$, we can color them with color 1 instead of color $\ell+1$. It only matters when the case $\{u, v, w\} \subseteq Y$ and $\{u v, v w, u w\} \cap E(G)=\emptyset$ happens. Suppose $\left\{x_{i} u, x_{j} v, x_{k} w\right\} \subseteq E(G)$ and $i<j<k$. If $i \neq 1$ or $k \neq \ell$, it is the case in the proof above. So we turn to the case when $i=1$ and $k=l$. If $j=2$, then $j+1<4 \leq \ell$ (that is why we need the condition $\ell \geq 4$ ). Thus, there is a rainbow tree containing the edges $z x_{j}, x_{j} v, z x_{k}, x_{k} w, s u, s z$ where $s$ is a neighbor of $u$ in $S$. If $j \neq 2$, then there is a rainbow tree containing the edges $z x_{i}, x_{i} u, z x_{j}, x_{j} v, s w, s z$.

## 5 Forbidden triples

Now, we continue to consider more and obtain an analogous result which characterizes all forbidden triples $\mathcal{F}$ for which there is a constant $C_{\mathcal{F}}$ such that $G$ being $\mathcal{F}$-free implies $r x_{3}(G) \leq \operatorname{siam}_{3}(G)+C_{\mathcal{F}}$. We exclude the cases which are covered by Theorems 5 and 6 . We set:

$$
\begin{aligned}
& \mathfrak{F}_{1}=\left\{\left\{P_{3}\right\}\right\}, \\
& \mathfrak{F}_{2}=\left\{\left\{K_{1, r}, P_{4}\right\} \mid r \geq 3\right\}, \\
& \mathfrak{F}_{3}=\left\{\left\{K_{1, r}, Y, P_{\ell}\right\} \mid r \geq 3, Y \stackrel{\text { IND }}{\subseteq} K_{s}^{h}, s \geq 3, \ell>4\right\} .
\end{aligned}
$$

Theorem 8. Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}|=3$ such that $\mathcal{F} \nsupseteq \mathcal{F}^{\prime}$ for any $\mathcal{F}^{\prime} \in \mathfrak{F}_{1} \cup \mathfrak{F}_{2}$. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph $G$ satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_{3}$.

First of all, we prove the necessity of the triples given by Theorem 8 .
Proposition 3. Let $X, Y, Z \neq P_{3}$ be connected graphs, $\{X, Y, Z\} \nsupseteq \mathcal{F}^{\prime}$ for any $\mathcal{F}^{\prime} \in \mathfrak{F}_{2}$, for which there is a constant $C_{X Y Z}$ such that every connected $(X, Y)$-free graph $G$
satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{X Y Z}$. Then, (up to symmetry) $X=K_{1, r}(r \geq 3), Y \stackrel{\text { IND }}{\subseteq}$ $K_{s}^{h}(s \geq 3)$, and $Z=P_{\ell}(\ell>4)$.

Proof. Let $t$ be an arbitrarily large integer, and let $G_{1}^{t}, G_{2}^{t}, G_{3}^{t}$ be the graphs defined in the proof of Proposition 2.

Firstly, we consider the graph $G_{1}^{t}$. Up to symmetry, we have $X=K_{1, r}, r \geq 3$ (for the case $r=2$ is excluded by the assumptions). Secondly, we consider the graph $G_{2}^{t}$. The graph $G_{2}^{t}$ does not contain $X$, since it is $K_{1,3}-$ free. Thus, up to symmetry, we have $G_{2}^{t}$ contains $Y$, implying $Y \subseteq K_{s}^{h}$ for some $s \geq 3$ (for the case $s \leq 2$ is excluded by the assumptions). Finally, we consider the graphs $G_{3}^{t}$ and $G_{3}^{t+1}$. Clearly, they are $\left(K_{1,3}, K_{3}^{h}\right)$-free, so both of them contain neither $X$ nor $Y$. Hence, we get that $Z=P_{\ell}$ for some $\ell>4$ (for the case $\ell \leq 4$ is excluded by the assumptions).

This completes the proof.
It is easy to observe that if $X \stackrel{\text { ind }}{\subseteq} X^{\prime}$, then every $(X, Y, Z)$-free graph is also $\left(X^{\prime}, Y, Z\right)$-free. Thus, when proving the sufficiency of Theorem 8, we will be always interested in maximal triples of forbidden subgraphs, i.e., triples $X, Y, Z$ such that, if replacing one of $X, Y, Z$, say $X$, with a graph $X^{\prime} \neq X$ such that $X \stackrel{\text { IND }}{\subseteq} X^{\prime}$, then the statement under consideration is not true for $\left(X^{\prime}, Y, Z\right)$-free graphs.

For every vertex $c \in V(G)$ and $i \in \mathbb{N}$, we set $\alpha_{i}(G, c)=\max \left\{|M| \mid M \subseteq N_{G}^{i}[c], M\right.$ is independent $\}$ and $\alpha_{i}^{0}(G, c)=\max \left\{\left|M^{0}\right| \mid M^{0} \subseteq N_{G}^{i}(c), M^{0}\right.$ is independent $\}$.

Lemma 1. [2] Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha(r, s, i)$ such that, for every connected $\left(K_{1, r}, K_{s}^{h}\right)$-free graph $G$ and for every $c \in V(G), \alpha_{i}(G, c)<\alpha(r, s, i)$.

We use the proof of Lemma 1 to get the following corollary concerning $\alpha_{i}^{0}(G, c)$ for each integer $i \geq 1$.

Corollary 1. Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha^{0}(r, s, i)$ such that, for every connected ( $K_{1, r}, K_{s}^{h}$ )-free graph $G$ and for every $c \in V(G), \alpha_{i}^{0}(G, c)<\alpha^{0}(r, s, i)$.

Proof. For the sake of completeness, here we give a brief proof concentrating on the upper bound of $\alpha_{i}^{0}(G, c)$. We prove the corollary by induction on $i$.

For $i=1$, we have $\alpha^{0}(r, s, 1)=r$, for otherwise $G$ contains a $K_{1, r}$ as an induced subgraph.

Let, to the contrary, $i$ be the smallest integer for which $\alpha^{0}(r, s, i)$ does not exist(i.e., $\alpha_{i}^{0}(G, c)$ can be arbitrarily large), choose a graph $G$ and a vertex $c \in V(G)$ such that $\alpha_{i}^{0}(G, c) \geq(r-2) R\left(s(2 r-3), \alpha^{0}(r, s, i-1)\right)$, and let $M^{0}=\left\{x_{1}^{0}, \ldots, x_{k}^{0}\right\} \subseteq N_{G}^{i}(c)$ be an independent set in $G$ of size $\alpha_{i}^{0}(G, c)$. Obviously, $k \geq(r-2) R\left(s(2 r-3), \alpha^{0}(r, s, i-1)\right)$. Let $Q_{j}$ be a shortest $\left(x_{j}^{0}, c\right)$-path in $G, j=1, \ldots, k$. We denote $M^{1} \subseteq N_{G}^{i-1}(c)$ the set of all successors of the vertices from $M^{0}$ on $Q_{j}, j=1, \ldots, k$, and $x_{j}^{1}$ the successor of $x_{j}^{0}$ on $Q_{j}$ (note that some distinct vertices in $M^{0}$ can have a common successor in $M^{1}$ ). Every vertex in $M^{1}$ has at most $r-2$ neighbors in $M^{0}$ since $G$ is $K_{1, r}$-free.

Thus, $\left|M^{1}\right| \geq \frac{k}{r-2} \geq R\left(s(2 r-3), \alpha^{0}(r, s, i-1)\right)$. By the induction assumption and the definition of Ramsey number, $G\left[M^{1}\right]$ contains a complete subgraph $K_{s(2 r-3)}$. Choose the notation such that $V\left(K_{s(2 r-3)}\right)=\left\{x_{1}^{1}, \ldots, x_{s(2 r-3)}^{1}\right\}$, and set $\widetilde{M^{0}}=N_{M^{0}}\left(K_{s(2 r-3)}\right)$. Using a matching between $K_{s(2 r-3)}$ and $\widetilde{M^{0}}$, we can find in $G$ an induced $K_{s}^{h}$ with vertices of degree 1 in $\widetilde{M^{0}}$, a contradiction. For more details about finding the $K_{s}^{h}$, we refer the reader to [2].

Armed with Corollary 1, we can get the following important theorem.
Theorem 9. Let $r \geq 3, s \geq 3$, and $\ell>4$ be fixed integers. Then there is a constant $C(r, s, \ell)$ such that every connected $\left(K_{1, r}, K_{s}^{h}, P_{\ell}\right)$-free graph $G$ satisfies $r x_{3}(G) \leq$ $\operatorname{sdiam}_{3}(G)+C(r, s, \ell)$.

Proof. We have $\operatorname{diam}(G) \leq \ell-2$ since $G$ is $P_{\ell}$-free. Let $c$ be a central vertex of $G$, i.e., $\operatorname{ecc}(c)=\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq \ell-2$. And we set $S_{i}=\cup_{j=1}^{i} N_{G}^{j}[c]$ for an integer $i \geq 1$.

Claim: $r x_{3}\left(G\left[S_{i} \cup N_{G}^{i+1}(c)\right]\right) \leq r x_{3}\left(G\left[S_{i}\right]\right)+\alpha_{i+1}^{0}(G, c)+3$
Proof. Let $X=\left\{x_{1}, x_{2}, \ldots, x_{\alpha_{i+1}^{0}(G, c)}\right\}$ be the maximum independent set of $N_{G}^{i+1}(c)$ and $Y=N_{G}^{i+1}(c) \backslash X$. Then for any vertex $y \in Y, y$ is adjacent to some $x \in X$ and $s \in S$. Further more, for any independent set $W$ of graph $G[Y]$, we have $\left|N_{X}(W)\right| \geq|W|$ since $X$ is maximum.

Now we demonstrate a 3-rainbow coloring of $G\left[S_{i} \cup N_{G}^{i+1}(c)\right]$ using at most $k+$ $\alpha_{i+1}^{0}(G, c)+3$ colors, where $k=r x_{3}\left(G\left[S_{i}\right]\right)$. We color the edges of $G\left[S_{i}\right]$ using colors $1,2, \ldots, k$. Color $E\left[S_{i}, Y\right]$ with color $k+1$ and $E(G[Y])$ with color $k+2$. And assign color $j+k+2$ to the edges $E\left[\left\{x_{j}\right\}, S_{i}\right]$, and $j+k+3$ to the edges $E\left[\left\{x_{j}\right\}, Y\right]$ where $1 \leq j \leq \alpha_{i+1}^{0}(G, c)$. With the same argument as the proof of Theorem 7 , we can prove that this coloring is a 3-rainbow coloring of $G\left[S_{i} \cup N_{G}^{i+1}(c)\right]$.

From the proof of Corollary 1, it follows that $\alpha_{1}^{0}(G, c) \leq r-1$ and $\alpha_{i}^{0}(G, c) \leq$ $(r-2) R\left(s(2 r-3), \alpha^{0}(r, s, i-1)\right)-1$ for each integer $i \geq 2$. Let $\mathcal{R}(r, s)=\sum_{i=2}^{\text {ecc(c) }} R(s(2 r-$ 3), $\alpha^{0}(r, s, i-1)$. Recall that $\operatorname{ecc}(c) \leq \ell-2$. Repeated application of Claim gives the following:

$$
\begin{aligned}
r x_{3}(G) & \leq r x_{3}\left(G\left[N_{G}^{e c c(c)-1}[c]\right]\right)+\alpha_{e c c(c)}^{0}(G, c)+3 \\
& \leq \cdots \\
& \leq r x_{3}(c)+\alpha_{1}^{0}(G, c)+\cdots+\alpha_{e c c(c)}^{0}(G, c)+3 \operatorname{ecc}(c) \\
& \leq 0+r+(r-2) \mathcal{R}(r, s)+2(\ell-2) \\
& \leq \operatorname{sdiam}_{3}(G)+(r-2)(\mathcal{R}(r, s)+1)+2(\ell-1) .
\end{aligned}
$$

Thus, we complete our proof.
Remark The same as the remark in Section 4: for $i \geq 1$, every time $\alpha_{i+1}^{0}(G, c) \geq 4$ happens, we can save one color in the Claim of Theorem 9 .

## 6 Forbidden $k$-tuples for any $k \in \mathbb{N}$

Let $\mathcal{F}=\left\{X_{1}, X_{2}, X_{3}, \ldots, X_{k}\right\}$ be a finite family of connected graphs with $k \geq 4$ for which there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $r x_{3}(G) \leq$ $\operatorname{sdiam}_{3}(G)+C_{\mathcal{F}}$. Let $t$ be an arbitrarily large integer, and let $G_{1}^{t}, G_{2}^{t}$ and $G_{3}^{t}$ be defined in Proposition 2. For the graph $G_{1}^{t}$, Up to symmetry, we suppose that $X_{1}=K_{r}, r \geq 3$ (for the case $r=2$ has been discussed in Section 3). Then, we consider the graphs $G_{2}^{t}$ and $G_{3}^{t}$. Notice that $G_{2}^{t}$ and $G_{3}^{t}$ are both $K_{1,3}$-free, so neither of them contains $X_{1}$, implying that $G_{2}^{t}$ or $G_{3}^{t}$ contains $X_{i}$, where $i \neq 1$. We may assume that $X_{2}$ is an induced subgraph of $G_{2}^{t}$. If $G_{3}^{t}$ contains $X_{2}$, then $X_{2}=P_{4}$, which is just the case in Section 4. So we turn to the case that $G_{3}^{t}$ contains $X_{i}$ for some $i>2$. Now consider the graphs $G_{3}^{t}, G_{3}^{t+1}, G_{3}^{t+2}, \ldots, G_{3}^{t+k}$, each of which contains at least one of $X_{3}, X_{4}, \ldots, X_{k}$ as an induced subgraph due to the analysis above. So it is forced that at least one of these $X_{i}(i \geq 3)$ is isomorphic to $P_{l}$ for some $l \geq 5$, which goes back to the case in Section 5. Thus, the conclusion comes out.

Theorem 10. Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $r x_{3}(G) \leq \operatorname{sdiam}_{3}(G)+C_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}^{\prime} \in \mathfrak{F}_{1} \cup \mathfrak{F}_{2} \cup \mathfrak{F}_{3}$.

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