# Covering Arrays on Product Graphs

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#### Abstract

Two vectors x,y in  $\mathbb{Z}_g^n$  are qualitatively independent if for all pairs  $(a,b) \in \mathbb{Z}_q \times \mathbb{Z}_q$ , there exists  $i \in \{1, 2, \dots, n\}$  such that  $(x_i, y_i) = (a, b)$ . A covering array on a graph G, denoted by CA(n, G, g), is a  $|V(G)| \times n$ array on  $\mathbb{Z}_q$  with the property that any two rows which correspond to adjacent vertices in G are qualitatively independent. The number of columns in such array is called its *size*. Given a graph G, a covering array on G with minimum size is called *optimal*. Our primary concern in this paper is with constructions that make optimal covering arrays on large graphs those are obtained from product of smaller graphs. We consider four most extensively studied graph products in literature and give upper and lower bounds on the size of covering arrays on graph products. We find families of graphs for which the size of covering array on the Cartesian product achieves the lower bound. Finally, we present a polynomial time approximation algorithm with approximation ratio  $\log(\frac{V}{2^{k-1}})$  for constructing covering array on graph G = (V, E) with k > 1 prime factors with respect to the Cartesian product.

## 1 Introduction

A covering array CA(n, k, g) is a  $k \times n$  array on  $\mathbb{Z}_g$  with the property that any two rows are qualitatively independent. The number n of columns in such array is called its size. The smallest possible size of a covering array is denoted

$$CAN(k,g) = \min_{n \in \mathbb{N}} \{n : \text{ there exists a } CA(n,k,g)\}$$

Covering arrays are generalisations of both orthogonal arrays and Sperner systems. Bounds and constructions of covering arrays have been derived from algebra, design theory, graph theory, set systems and intersecting codes [1, 2, 3, 4]. Covering arrays have industrial applications in many disparate applications in which factors or components interact, for example, software and circuit testing, switching networks, drug screening and data compression [6, 7, 8]. In [17], the definition of a covering array has been extended to include a graph structure. **Definition 1.** (Covering arrays on graph). A covering array on a graph G with alphabet size g and k = |V(G)| is a  $k \times n$  array on  $\mathbb{Z}_g$ . Each row in the array corresponds to a vertex in the graph G. The array has the property that any two rows which correspond to adjacent vertices in G are qualitatively independent.

A covering array on a graph G will be denoted by CA(n, G, g). The smallest possible covering array on a graph G will be denoted

$$CAN(G,g) = \min_{n \in \mathbb{N}} \{n \ : \ \text{there exists a} \ CA(n,G,g) \}$$

Given a graph G and a positive integer g, a covering array on G with minimum size is called *optimal*. Seroussi and Bshouly proved that determining the existence of an optimal binary covering array on a graph is an NP-complete problem [7]. We start with a review of some definitions and results from product graphs in Section 2. In Section 3, we show that for all graphs  $G_1$  and  $G_2$ ,

$$\max_{i=1,2} \{ CAN(G_i, g) \} \le CAN(G_1 \square G_2, g) \le CAN(\max_{i=1,2} \{ \chi(G_i) \}, g)$$

We look for graphs  $G_1$  and  $G_2$  where the lower bound on  $CAN(G_1 \square G_2)$  is achieved. In Section 4, we give families of Cayley graphs that achieves this lower bound on covering array number on graph product. In Section 5, we present a polynomial time approximation algorithm with approximation ratio  $\log(\frac{V}{2^{k-1}})$  for constructing covering array on graph G = (V, E) having more than one prime factor with respect to the Cartesian product.

## 2 Preliminaries

In this section, we give several definitions from product graphs that we use in this article. A graph product is a binary operation on the set of all finite graphs. However among all possible associative graph products the most extensively studied in literature are the Cartesian product, the direct product, the strong product and the lexicographic product.

**Definition 2.** The Cartesian product of graphs G and H, denoted by  $G \Box H$ , is the graph with

$$V(G \Box H) = \{(g, h) | g \in V(G) \text{ and } h \in V(H)\},$$
  
$$E(G \Box H) = \{(g, h)(g', h') | g = g', hh' \in E(H), \text{ or } gg' \in E(G), h = h'\}.$$

The graphs G and H are called the *factors* of the product  $G \square H$ .

In general, given graphs  $G_1, G_2, ..., G_k$ , then  $G_1 \square G_2 \square \cdots \square G_k$ , is the graph with vertex set  $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ , and two vertices  $(x_1, x_2, ..., x_k)$ and  $(y_1, y_2, ..., y_k)$  are adjacent if and only if  $x_i y_i \in E(G_i)$  for exactly one index  $1 \le i \le k$  and  $x_j = y_j$  for each index  $j \ne i$ . **Definition 3.** The direct product of graphs  $G_1, G_2, ..., G_k$ , denoted by  $G_1 \times G_2 \times \cdots \times G_k$ , is the graph with vertex set  $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ , and for which vertices  $(x_1, x_2, ..., x_k)$  and  $(y_1, y_2, ..., y_k)$  are adjacent precisely if  $x_i y_i \in E(G_i)$  for each index *i*.

**Definition 4.** The strong product of graphs  $G_1, G_2, ..., G_k$ , denoted by  $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$ , is the graph with vertex set  $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ , and distinct vertices  $(x_1, x_2, ..., x_k)$  and  $(y_1, y_2, ..., y_k)$  are adjacent if and only if either  $x_i y_i \in E(G_i)$  or  $x_i = y_i$  for each  $1 \le i \le k$ . We note that in general  $E(\boxtimes_{i=1}^k G_i) \neq E(\square_{i=1}^k G_i) \cup E(\times_{i=1}^k G_i)$ , unless k = 2.

**Definition 5.** The lexicographic product of graphs  $G_1, G_2, ..., G_k$ , denoted by  $G_1 \circ G_2 \circ \cdots \circ G_k$ , is the graph with vertex set  $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ , and two vertices  $(x_1, x_2, ..., x_k)$  and  $(y_1, y_2, ..., y_k)$  are adjacent if and only if for some index  $j \in \{1, 2, ..., k\}$  we have  $x_j y_j \in E(G_j)$  and  $x_i = y_i$  for each index  $1 \le i < j$ .

Let G and H be graphs with vertex sets V(G) and V(H), respectively. A homomorphism from G to H is a map  $\varphi : V(G) \to V(H)$  that preserves adjacency: if uv is an edge in G, then  $\varphi(u)\varphi(v)$  is an edge in H. We say  $G \to H$  if there is a homomorphism from G to H, and  $G \equiv H$  if  $G \to H$  and  $H \to G$ . A weak homomorphism from G to H is a map  $\varphi : V(G) \to V(H)$ such that if uv is an edge in G, then either  $\varphi(u)\varphi(v)$  is an edge in H, or  $\varphi(u) = \varphi(v)$ . Clearly every homomorphism is automatically a weak homomorphism.

Let \* represent either the Cartesian, the direct or the strong product of graphs, and consider a product  $G_1 * G_2 * \ldots * G_k$ . For any index  $i, 1 \le i \le k$ , a projection map is defined as:

 $p_i : G_1 * G_2 * \ldots * G_k \to G_i$  where  $p_i(x_1, x_2, \ldots, x_k) = x_i$ .

By the definition of the Cartesian, the direct, and the strong product of graphs, each  $p_i$  is a weak homomorphism. In the case of direct product, as  $(x_1, x_2, \ldots, x_k)(y_1, y_2, \ldots, y_k)$  is an an edge of  $G_1 \times G_2 \times, \ldots, \times G_k$  if and only if  $x_i y_i \in E(G_i)$  for each  $1 \leq i \leq k$ , each projection  $p_i$  is actually a homomorphism. In the case of lexicographic product, the first projection map that is projection on first component is a weak homomorphism, where as in general the projections to the other components are not weak homomorphisms.

A graph is *prime* with respect to a given graph product if it is nontrivial and cannot be represented as the product of two nontrivial graphs. For the Cartesian product, it means that a nontrivial graph G is prime if  $G = G_1 \square G_2$ implies that either  $G_1$  or  $G_2$  is  $K_1$ . Similar observation is true for other three products. The uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product was first shown by Subidussi (1960), and independently by Vizing (1963). Prime factorization is not unique for the Cartesian product in the class of possibly disconnected simple graphs [9]. It is known that any connected graph factors uniquely into prime graphs with respect to the Cartesian product.

**Theorem 1.** (Sabidussi-Vizing) Every connected graph has a unique representation as a product of prime graphs, up to isomorphism and the order of the factors. The number of prime factors is at most  $\log_2 V$ .

For any connected graph G = (V, E), the prime factors of G with respect to the Cartesian product can be computed in  $O(E \log V)$  times and O(E) space. See Chapter 23, [9].

## **3** Graph products and covering arrays

Let \* represent either the Cartesian, the direct, the strong, or the lexicographic product operation. Given covering arrays  $CA(n_1, G_1, g)$  and  $CA(n_2, G_2, g)$ , one can construct covering array on  $G_1 * G_2$  as follows: the row corresponds to the vertex (a, b) is obtained by horizontally concatenating the row corresponds to the vertex a in  $CA(n_1, G_1, g)$  with the row corresponds to the vertex b in  $CA(n_2, G_2, g)$ . Hence an obvious upper bound for the covering array number is given by

$$CAN(G_1 * G_2, g) \le CAN(G_1, g) + CAN(G_2, g)$$

We now propose some improvements of this bound. A column of a covering array is *constant* if, for some symbol v, every entry in the column is v. In a *standardized* CA(n, G, g) the first column is constant. Because symbols within each row can be permuted independently, if a CA(n, G, g) exists, then a standardized CA(n, G, g) exists.

**Theorem 2.** Let  $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$ ,  $k \ge 2$  and g be a positive integer. Suppose for each  $1 \le i \le k$  there exists a  $CA(n_i, G_i, g)$ , then there exists a CA(n, G, g) where  $n = \sum_{i=1}^{k} n_i - k$ . Hence,  $CAN(G, g) \le \sum_{i=1}^{k} CAN(G_i, g) - k$ .

Proof. Without loss of generality, we assume that for each  $1 \leq i \leq g$ , the first column of  $CA(n_i, G_i, g)$  is a constant column on symbol *i* and for each  $g + 1 \leq i \leq k$ , the first column of  $CA(n_i, G_i, g)$  is a constant column on symbol 1. Let  $C_i$  be the array obtained from  $CA(n_i, G_i, g)$  by removing the first column. Form an array A with  $\prod_{i=1}^{k} |V(G_i)|$  rows and  $\sum_{i=1}^{k} n_i - k$  columns, indexing rows as  $(v_1, v_2, ..., v_k)$ , where  $v_i \in V(G_i)$ . Row  $(v_1, v_2, ..., v_k)$  is obtained by horizontally concatenating the rows correspond to the vertex  $v_i$  of  $C_i$ , for  $1 \leq i \leq k$ . Consider two distinct rows  $(u_1, u_2, ..., u_k)$  and

 $(v_1, v_2, \ldots, v_k)$  of A which correspond to adjacent vertices in G. Two distinct vertices  $(u_1, u_2, \ldots, u_k)$  and  $(v_1, v_2, \ldots, v_k)$  are adjacent if and only if either  $u_i v_i \in E(G_i)$  or  $u_i = v_i$  for each  $1 \leq i \leq k$ . Since the vertices are distinct,  $u_i v_i \in E(G_i)$  for at least one index i. When  $u_i = v_i$ , all pairs of the form (a, a) are covered. When  $u_i v_i \in E(G_i)$  all remaining pairs are covered because two different rows of  $C_i$  correspond to adjacent vertices in  $G_i$  are selected.

Using the definition of strong product of graphs we have following result as a corollary.

**Corollary 1.** Let  $G = G_1 * G_2 * \cdots * G_k$ ,  $k \ge 2$  and g be a positive integer, where  $* \in \{\Box, \times\}$ . Then,  $CAN(G,g) \le \sum_{i=1}^{k} CAN(G_i,g) - k$ .

The lemma given below will be used in Theorem 3.

**Lemma 1.** (Meagher and Stevens [17]) Let G and H be graphs. If  $G \to H$  then  $CAN(G,g) \leq CAN(H,g)$ .

**Theorem 3.** Let  $G = G_1 \times G_2 \times \cdots \times G_k$ ,  $k \ge 2$  and g be a positive integer. Suppose for each  $1 \le i \le k$  there exists a  $CA(n_i, G_i, g)$ . Then there exists a CA(n, G, g) where  $n = \min_i n_i$ . Hence,  $CAN(G, g) \le \min_i CAN(G_i, g)$ .

*Proof.* Without loss of generality assume that  $n_1 = \min_i n_i$ . It is known that  $G_1 \times G_2 \times \cdots \times G_k \to G_1$ . Using Lemma 1, we have  $CAN(G,g) \leq CAN(G_1,g)$ .

**Theorem 4.** Let  $G = G_1 \circ G_2 \circ \cdots \circ G_k$ ,  $k \ge 2$  and g be a positive integer. Suppose for each  $1 \le i \le k$  there exists a  $CA(n_i, G_i, g)$ . Then there exists a CA(n, G, g) where  $n = \sum_{i=1}^{k} n_i - k + 1$ . Hence,  $CAN(G, g) \le \sum_{i=1}^{k} CAN(G_i, g) - k + 1$ .

Proof. We assume that for each  $1 \leq i \leq k$ , the first column of  $CA(n_i, G_i, g)$ is a constant column on symbol 1. Let  $C_1 = CA(n_1, G_1, g)$ . For each  $2 \leq i \leq k$  remove the first column of  $CA(n_i, G_i, g)$  to form  $C_i$  with  $n_i - 1$  columns. Without loss of generality assume first column of each  $CA(n_i, G_i, g)$  is constant vector on symbol 1 while for each  $2 \leq i \leq k$ ,  $C_i$  is the array obtained from  $CA(n_i, G_i, g)$  by removing the first column. Form an array A with  $\prod_{i=1}^{k} |V(G_i)|$  rows and  $\sum_{i=1}^{k} n_i - k + 1$  columns, indexing rows as  $(v_1, v_2, ..., v_k)$ ,  $v_i \in V(G_i)$ . Row  $(v_1, v_2, ..., v_k)$  is obtained by horizontally concatenating the rows correspond to the vertex  $v_i$  of  $C_i$ , for  $1 \leq i \leq k$ . If two vertices  $(v_1, v_2, ..., v_k)$  and  $(u_1, u_2, ..., u_k)$  are adjacent in G then either  $v_1u_1 \in E(G_1)$ or  $v_ju_j \in E(G_j)$  for some  $j \ge 2$  and  $v_i = u_i$  for each i < j. In first case rows from  $C_1$  covers each ordered pair of symbols while in second case rows from  $C_j$  covers each ordered pair of symbol probably except (1, 1). But this pair appears in each  $C_i$  for i < j. Hence A is a covering array on G.

**Definition 6.** A proper colouring on a graph is an assignment of colours to each vertex such that adjacent vertices receive a different colour. The chromatic number of a graph G,  $\chi(G)$ , is defined to be the size of the smallest set of colours such that a proper colouring exists with that set.

**Definition 7.** A maximum clique in a graph G is a maximum set of pairwise adjacent vertices. The maximum clique number of a graph G,  $\omega(G)$ , is defined to be the size of a maximum clique.

Since there are homomorphisms  $K_{\omega(G)} \to G \to K_{\chi(G)}$ , we can find bound on the size of a covering array on a graph from the graph's chromatic number and clique number. For all graphs G,

$$CAN(K_{\omega(G)}, g) \le CAN(G, g) \le CAN(K_{\chi(G)}, g).$$

We have the following results on proper colouring of product graphs [15]

$$\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}.$$

For other graph products there are no explicit formulae for chromatic number but following bounds are mentioned in [9].

$$\chi(G_1 \times G_2) \le \min\{\chi(G_1), \chi(G_2)\}$$
$$\chi(G_1 \boxtimes G_2) \le \chi(G_1 \circ G_2) \le \chi(G_1)\chi(G_2).$$

A proper colouring of  $G_1 * G_2$  with  $\chi(G_1 * G_2)$  colours is equivalent to a homomorphism from  $G_1 * G_2$  to  $K_{\chi(G_1 * G_2)}$  for any  $* \in \{\Box, \times, \boxtimes, \circ\}$ . Hence

 $CAN(G_1 \square G_2, g) \le CAN(K_{\max\{\chi(G_1), \chi(G_2)\}}, g)$  $CAN(G_1 \times G_2, g) \le CAN(K_{\min\{\chi(G_1), \chi(G_2)\}}, g)$  $CAN(G_1 \boxtimes G_2, g) \le CAN(K_{\chi(G_1)\chi(G_2)}, g)$  $CAN(G_1 \circ G_2, g) \le CAN(K_{\chi(G_1)\chi(G_2)}, g).$ 

Note that  $G_1 \to G_1 * G_2$  and  $G_2 \to G_1 * G_2$  for  $* \in \{\Box, \boxtimes, \circ\}$  which gives

$$max\{CAN(G_1,g), CAN(G_2,g)\} \le CAN(G_1 * G_2,g).$$

We now describe colouring construction of covering array on graph G. If G is a k-colourable graph then build a covering array CA(n, k, g) and without loss of generality associate row i of CA(n, k, g) with colour i for  $1 \le i \le k$ . In order to construct CA(n, G, g), we assign row i of CA(n, k, g) to all the vertices having colour i in G.

**Definition 8.** An orthogonal array OA(k,g) is a  $k \times g^2$  array with entries from  $\mathbb{Z}_g$  having the properties that in every two rows, each ordered pair of symbols from  $\mathbb{Z}_q$  occurs exactly once.

**Theorem 5.** [5] If g is prime or power of prime, then one can construct OA(g+1,g).

The set of rows in an orthogonal array OA(k,g) is a set of k pairwise qualitatively independent vectors from  $\mathbb{Z}_g^{g^2}$ . For g = 2, by Theorem 5, there are three qualitatively independent vectors from  $\mathbb{Z}_2^4$ . Here we give some examples where the lower bound on  $CAN(G_1 \square G_2, g)$  is achieved, that is,  $CAN(G_1 \square G_2, g) = max\{CAN(G_1, g), CAN(G_2, g)\}.$ 

**Example 1.** If  $G_1$  and  $G_2$  are bicolorable graphs, then  $\chi(G_1 \Box G_2) = 2$ . Let  $x_1$  and  $x_2$  be two qualitatively independent vectors in  $\mathbb{Z}_g^{g^2}$ . Assign vector  $x_i$  to all the vertices of  $G_1 \Box G_2$  having colour *i* for i = 1, 2 to get a covering array with  $CAN(G_1 \Box G_2, g) = g^2$ .

**Example 2.** If  $G_1$  and  $G_2$  are complete graphs, then  $CAN(G_1 \square G_2, g) = max\{CAN(G_1, g), CAN(G_2, g)\}.$ 

**Example 3.** If  $G_1$  is bicolorable and  $G_2$  is a complete graph on  $k \ge 3$  vertices, then  $CAN(G_1 \square G_2, g) = CAN(G_2, g)$ . In general, if  $\chi(G_1) \le \chi(G_2)$  and  $G_2$  is a complete graph, then  $CAN(G_1 \square G_2, g) = CAN(G_2, g)$ .

**Example 4.** If  $P_m$  is a path of length m and  $C_n$  is an odd cycle of length n, then  $\chi(P_m \square C_n) = 3$ . Using Theorem 5, we get a set of three qualitatively independent vectors in  $\mathbb{Z}_g^{g^2}$  for  $g \ge 2$ . Then the colouring construction of covering arrays gives us a covering array on  $P_m \square C_n$  with  $CAN(P_m \square C_n, g) = g^2$ .

**Lemma 2.** [9] Let  $G_1$  and  $G_2$  be graphs and Q be a clique of  $G_1 \boxtimes G_2$ . Then  $Q = p_1(Q) \boxtimes p_2(Q)$ , where  $p_1(Q)$  and  $p_2(Q)$  are cliques of  $G_1$  and  $G_2$ , respectively.

Hence a maximum size clique of  $G_1 \boxtimes G_2$  is product of maximum size cliques from  $G_1$  and  $G_2$ . That is,  $\omega(G_1 \boxtimes G_2) = \omega(G_1)\omega(G_2)$ . Using the graph homomorphism, this results into another lower bound on  $CAN(G_1 \boxtimes G_2, g)$  as  $CAN(K_{\omega(G_1)\omega(G_2)}, g) \leq CAN(G_1 \boxtimes G_2, g)$ . Following are some examples where this lower bound can be achieved.

**Example 5.** If  $G_1$  and  $G_2$  are nontrivial bipartite graphs then  $\omega(G_1 \boxtimes G_2) = \chi(G_1 \boxtimes G_2)$  which is 4. Hence  $CAN(G_1 \boxtimes G_2, g) = CAN(K_4, g)$ , which is of optimal size.

**Example 6.** If  $G_1$  and  $G_2$  are complete graphs, then  $G_1 \boxtimes G_2$  is again a complete graph. Hence  $CAN(G_1 \boxtimes G_2, g) = CAN(K_{\omega(G_1 \boxtimes G_2)}, g)$ .

**Example 7.** If  $G_1$  is a bipartite graph and  $G_2$  is a complete graph on  $k \ge 2$  vertices, then  $\omega(G_1 \boxtimes G_2) = \chi(G_1 \boxtimes G_2) = 2k$ . Hence  $CAN(G_1 \boxtimes G_2, g) = CAN(K_{2k}, g)$ .

**Example 8.** If  $P_m$  is a path of length m and  $C_n$  is an odd cycle of length n, then  $\omega(P_m \boxtimes C_n) = 4$  and  $\chi(P_m \boxtimes C_n) = 5$ . Here we have  $CAN(K_4, g) \leq CAN(G, g) \leq CAN(K_5, g)$ . For  $g \geq 4$ , using Theorem 5, we get a set of five qualitatively independent vectors in  $\mathbb{Z}_g^{g^2}$ . Then the colouring construction of covering arrays gives us a covering array on  $P_m \boxtimes C_n$  with  $CAN(P_m \boxtimes C_n, g) = g^2$ .

## 4 Optimal size covering arrays over the Cartesian product of graphs

**Definition 9.** Two graphs  $G_1 = (V, E)$  and  $G_2 = (V', E')$  are said to be isomorphic if there is a bijection mapping  $\varphi$  from the vertex set V to the vertex set V' such that  $(u, v) \in E$  if and only if  $(\varphi(u), \varphi(v)) \in E'$ . The mapping  $\varphi$  is called an isomorphism. An automorphism of a graph is an isomorphism from the graph to itself.

The set of all automorphisms of a graph G forms a group, denoted Aut(G), the automorphism group of G.

**Theorem 6.** Let  $G_1$  be a graph having the property that  $Aut(G_1)$  contains a fixed point free automorphism which maps every vertex to its neighbour. Then for any bicolourable graph  $G_2$ ,

$$CAN(G_1 \square G_2, g) = CAN(G_1, g).$$

Proof. Consider the set  $\Gamma = \{\phi \in Aut(G_1) \mid \phi(u) \in N(u) - \{u\} \text{ for all } u \in V(G_1)\}$  where N(u) denotes the set of neighbours of u. From the assumption,  $\Gamma$  is not empty. Consider a 2-colouring of  $G_2$  with colours 0 and 1. Let  $W_0 = \{(u, v) \in V(G_1 \square G_2) \mid \text{colour}(v) = 0\}$  and  $W_1 = \{(u, v) \in V(G_1 \square G_2) \mid \text{colour}(v) = 0\}$  and  $W_1 = \{(u, v) \in V(G_1 \square G_2) \mid \text{colour}(v) = 1\}$ . Note that  $W_0$  and  $W_1$  partition  $V(G_1 \square G_2)$  in two two parts. Let the rows of covering array  $CA(G_1, g)$  be indexed by  $u_1, u_2, \ldots, u_k$ . Form an array C with  $|V(G_1 \square G_2)|$  rows and  $CAN(G_1, g)$  columns, indexing rows as (u, v) for  $1 \leq u \leq |V(G_1)|, 1 \leq v \leq |V(G_2)|$ . If  $(u, v) \in W_0$ , row (u, v) is row u of  $CA(G_1, g)$ ; otherwise if  $(u, v) \in W_1$ , row (u, v) is row  $\phi(u)$  of  $CA(G_1, g)$ . We verify that C is a  $CA(G_1 \square G_2, g)$ . Consider two adjacent vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  of C.

(i) Let  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_i$ , then  $(u_1, v_1) \sim (u_2, v_2)$  if and only if  $u_1 \sim u_2$  and  $v_1 = v_2$ . When  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_0$ , rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are rows  $u_1$  and  $u_2$  of  $CA(G_1, g)$  respectively. As  $u_1 \sim u_2$ , rows  $u_1$  and  $u_2$  are qualitatively independent in  $CA(G_1, g)$ . When  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_1$ , rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are rows  $\phi(u_1)$  and  $\phi(u_2)$  of  $CA(G_1, g)$  respectively. As  $\phi(u_1) \sim \phi(u_2)$ , rows  $\phi(u_1)$  and  $\phi(u_2)$  are qualitatively independent in  $CA(G_1, g)$ . Therefore, rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are qualitatively independent in C.

(ii) Let  $(u_1, v_1) \in W_0$  and  $(u_2, v_2) \in W_1$ . In this case,  $(u_1, v_1) \sim (u_2, v_2)$  if and only if  $u_1 = u_2$  and  $v_1 \sim v_2$ . Let  $u_1 = u_2 = u$ . Rows  $(u, v_1)$  and  $(u, v_2)$ are rows u and  $\phi(u)$  of  $CA(G_1, g)$ . As  $\phi$  is a fixed point free automorphism that maps every vertex to its neighbour, u and  $\phi(u)$  are adjacent in  $G_1$ . Therefore, the rows indexed by u and  $\phi(u)$  are qualitatively independent in  $CA(G_1, g)$ ; therefore, rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are qualitatively independent in C.

**Definition 10.** Let H be a finite group and S be a subset of  $H \setminus \{id\}$  such that S = -S (i.e., S is closed under inverse). The Cayley graph of H generated by S, denoted Cay(H, S), is the undirected graph G = (V, E) where V = H and  $E = \{(x, sx) \mid x \in H, s \in S\}$ . The Cayley graph is connected if and only if S generates H.

Through out this article by S = -S we mean, S is closed under inverse for a given group operation

**Definition 11.** A circulant graph G(n, S) is a Cayley graph on  $\mathbb{Z}_n$ . That is, it is a graph whose vertices are labelled  $\{0, 1, \ldots, n-1\}$ , with two vertices labelled i and j adjacent iff  $i - j \pmod{n} \in S$ , where  $S \subset \mathbb{Z}_n$  with S = -S and  $0 \notin S$ .

**Corollary 2.** Let  $G_1(n, S)$  be a circulant graph and  $G_2$  be a bicolorable graph, then  $CAN(G_1(n, S) \square G_2, g) = CAN(G_1(n, S), g)$ .

*Proof.* Let *i* and *j* be any two adjacent vertices in  $G_1(n, S)$ . We define a mapping  $\phi$  from  $\mathbb{Z}_n$  as follows:

$$\phi(k) = k + j - i \pmod{n}$$

It is easy to verify that  $\phi$  is an automorphism and it sends every vertex to its neighbour. Hence  $\phi \in \Gamma$  and the result follows.

For a group H and  $S \subseteq H$ , we denote conjugation of S by elements of itself as

$$S^{S} = \{ss's^{-1} | s, s' \in S\}$$

**Corollary 3.** Let H be a finite group and  $S \subseteq H \setminus \{id\}$  is a generating set for H such that S = -S and  $S^S = S$ . Then for  $G_1 = Cay(H, S)$  and any bicolorable graph  $G_2$ ,

$$CAN(G_1 \square G_2, g) = CAN(G_1, g)$$

Proof. We will show that there exists a  $\phi \in Aut(G_1)$  such that  $\phi$  is stabilizer free. Define  $\phi : H \to H$  as  $\phi(h) = sh$  for some  $s \in S$ . It it easy to check that  $\phi$  is bijective and being  $s \neq id$  it is stabilizer free. Now to prove it is a graph homomorphism we need to show it is an adjacency preserving map. It is sufficient to prove that  $(h, s'h) \in E(G_1)$  implies  $(sh, ss'h) \in E(G_1)$ . As  $ss'h = ss's^{-1}sh$  and  $ss's^{-1} \in S$ , we have  $(sh, ss'h) \in E(G_1)$ . Hence  $\phi \in \Gamma$ and Theorem 6 implies the result.

**Example 9.** For any abelian group H and S be a generating set such that S = -S and  $id \notin S$ , we always get  $S^S = S$ .

**Example 10.** For  $H = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  and  $S = \{\pm i, \pm j\}$ , we have  $S^S = S$  and S = -S.

**Example 11.** For  $H = D_8 = \langle a, b | a^2 = 1 = b^4$ ,  $aba = b^3 \rangle$  and  $S = \{ab, ba\}$ , we have  $S^S = S$  and S = -S.

**Example 12.** For  $H = S_n$  and S = set of all even cycles, we have  $S^S = S$  and S = -S

**Theorem 7.** Let H be a finite group and S be a generating set for H such that

- 1. S = -S and  $id \notin S$
- 2.  $S^S = S$
- 3. there exist  $s_1$  and  $s_2$  in S such that  $s_1 \neq s_2$  and  $s_1s_2 \in S$

then for  $G_1 = Cay(H, S)$  and any three colourable graph  $G_2$ 

$$CAN(G_1 \square G_2, g) = CAN(G_1, g)$$

Proof. Define three distinct automorphisms of  $G_1$ ,  $\sigma_i : H \to H$ , for i = 0, 1, 2, as  $\sigma_0(u) = u$ ,  $\sigma_1(u) = s_1 u$ ,  $\sigma_2(u) = s_2^{-1} u$ . Consider a three colouring of  $G_2$  using the colours 0, 1 and 2. Let  $W_i = \{(u, v) \in V(G_1 \square G_2) \mid \text{colour}(v) = i\}$  for i = 0, 1, 2. Note that  $W_0$ ,  $W_1$ , and  $W_2$  partition  $V(G_1 \square G_2)$  into three parts. Let the rows of covering array  $CA(G_1, g)$  be indexed by  $u_1, u_2, \ldots, u_k$ . Using  $CA(G_1, g)$ , form an array C with  $|V(G_1 \square G_2)|$  rows and  $CAN(G_1, g)$  columns, indexing rows as (u, v) for  $1 \le u \le |V(G_1)|$ ,  $1 \le v \le |V(G_2)|$ . If  $(u, v) \in W_i$ , row (u, v) is row  $\sigma_i(u)$  of  $CA(G_1, g)$ . Consider two adjacent vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  of C.

(i) Let  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_i$ . In this case,  $(u_1, v_1) \sim (u_2, v_2)$ if and only if  $u_1 \sim u_2$  and  $v_1 = v_2$ . When  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_0$ , rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are rows  $u_1$  and  $u_2$  of  $CA(G_1, g)$ . As  $u_1 \sim u_2$ in  $G_1$ , the rows  $u_1$  and  $u_2$  are qualitatively independent in  $CA(G_1, g)$ . Let  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_1$  (res.  $W_2$ ). Similarly, as  $s_1u_1 \sim s_1u_2$  (res.  $s_2^{-1}u_1 \sim s_2^{-1}u_1$ ) the rows indexed by  $s_1u_1$  and  $s_1u_2$  (res.  $s_2^{-1}u_1$  and  $s_2^{-1}u_2$ ) are qualitatively independent in  $CA(G_1, g)$ . Hence the rows  $(u_1, v_1)$  and  $(u_2, v_2)$  are qualitatively independent in C.

(ii) Let  $(u_1, v_1) \in W_i$  and  $(u_2, v_2) \in W_j$  for  $0 \le i \ne j \le 2$ . In this case,  $(u_1, v_1) \sim (u_2, v_2)$  if and only if  $u_1 = u_2$  and  $v_1 \sim v_2$ . Let  $u_1 = u_2 = u$ .

Let  $(u, v_1) \in W_0$  and  $(u, v_2) \in W_1$ , then rows  $(u, v_1)$  and  $(u, v_2)$  are rows uand  $s_1u$  of  $CA(G_1, g)$  respectively. Then as  $u \sim s_1u$  the rows indexed by  $(u, v_1) \in W_0$  and  $(u, v_2) \in W_1$  are qualitatively independent in C.

Let  $(u, v_1) \in W_0$  and  $(u, v_2) \in W_2$ . Then, as  $u \sim s_2^{-1}u$ , the rows indexed by  $(u, v_1) \in W_0$  and  $(u, v_2) \in W_2$  are qualitatively independent in C.

Let  $(u, v_1) \in W_1$  and  $(u, v_2) \in W_2$ . Then, as  $s_1 u \sim s_2^{-1} u$ , the rows indexed by  $(u, v_1) \in W_1$  and  $(u, v_2) \in W_2$  are qualitatively independent in C.

**Theorem 8.** Let H be a finite group and S is a generating set for H such that

- 1. S = -S and  $id \notin S$
- 2.  $S^S = S$
- 3.  $\exists s_1 \text{ and } s_2 \text{ in } S \text{ such that } s_1 \neq s_2 \text{ and } s_1 s_2, s_1 s_2^{-1} \in S$

then for  $G_1 = Cay(H, S)$  and any four colourable graph  $G_2$ 

$$CAN(G_1 \square G_2, g) = CAN(G_1, g)$$

Proof. Define four distinct automorphisms of  $G_1$ ,  $\sigma_i : H \to H$ , i = 0, 1, 2, 3as  $\sigma_0(u) = u$ ,  $\sigma_1(u) = s_1u$ ,  $\sigma_2(u) = s_2u$  and  $\sigma_3(u) = s_1s_2u$ . Consider a four colouring of  $G_2$  using the colours 0, 1, 2 and 3. Let  $W_i = \{(u, v) \in V(G_1 \square G_2) \mid \text{colour}(v) = i\}$  for i = 0, 1, 2, 3. Let the rows of covering array  $CA(G_1, g)$  be indexed by  $u_1, u_2, \ldots, u_k$ . Form an array C with  $|V(G_1 \square G_2)|$ rows and  $CAN(G_1, g)$  columns, indexing rows as (u, v) for  $1 \le u \le |V(G_1)|$ ,  $1 \le v \le |V(G_2)|$ . If  $(u, v) \in W_i$ , row (u, v) is row  $\sigma_i(u)$  of  $CA(G_1, g)$ . Consider two adjacent vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  of C.

(i) Let  $(u_1, v_1)$  and  $(u_2, v_2)$  belong to  $W_i$ . It is easy to verify that  $(u_1, v_1)$  and  $(u_2, v_2)$  are qualitatively independent.

(ii) Let  $(u_1, v_1) \in W_i$  and  $(u_2, v_2) \in W_j$  for  $0 \le i \ne j \le 3$ . In this case,  $(u_1, v_1) \sim (u_2, v_2)$  if and only if  $u_1 = u_2$  and  $v_1 \sim v_2$ . Let  $u_1 = u_2 = u$ .

Let  $(u, v_1) \in W_0$  and  $(u, v_2) \in W_i$  for i = 1, 2, 3, then row  $(u, v_1)$  and  $(u, v_2)$  are rows u and  $\sigma_i(u)$  of  $CA(G_1, g)$  respectively. Then as  $u \sim \sigma_i(u)$  the rows  $(u, v_1)$  and  $(u, v_2)$  are qualitatively independent.

Let  $(u, v_1) \in W_1$  and  $(u, v_2) \in W_2$ . Then rows  $(u, v_1)$  and  $(u, v_2)$  are rows  $s_1u$ and  $s_2u$  of  $CA(G_1, g)$ . As  $s_1u = s_1s_2^{-1}s_2u$  and  $s_1s_2^{-1} \in S$ , we get  $s_1u \sim s_2u$ . Hence the rows  $(u, v_1) \in W_1$  and  $(u, v_2) \in W_2$  are qualitatively independent. Similarly, as  $s_1u = s_1s_2^{-1}s_1^{-1}s_1s_2u$  and  $s_1s_2^{-1}s_1^{-1} \in S$  being  $S^S = S$ , we have  $s_1u \sim s_1s_2u$ . Hence the rows  $(u, v_1) \in W_1$  and  $(u, v_2) \in W_3$  are qualitatively



Figure 1:  $Cay(Q_8, \{-1, \pm i, \pm j\}) \Box K_3$ 

independent.

Let  $(u, v_1) \in W_2$  and  $(u, v_2) \in W_3$ . As  $s_2u = s_1^{-1}s_1s_2u$  and  $s_1^{-1} \in S$ , we get  $s_2u \sim s_1s_2u$ . Hence the rows  $(u, v_1) \in W_2$  and  $(u, v_2) \in W_3$  are qualitatively independent.

**Example 13.**  $G = Q_8$  and  $S = \{\pm i, \pm j, \pm k\}$ . Here  $s_1 = i$  and  $s_2 = j$ . **Example 14.**  $G = Q_8$  and  $S = \{-1, \pm i, \pm j\}$ . Here  $s_1 = -1$  and  $s_2 = i$ .

# 5 Approximation algorithm for covering array on graph

In this section, we present an approximation algorithm for construction of covering array on a given graph G = (V, E) with k > 1 prime factors with respect to the Cartesian product. In 1988, G. Seroussi and N H. Bshouty proved that the decision problem whether there exists a binary covering array of strength  $t \ge 2$  and size  $2^t$  on a given t-uniform hypergraph is NP-complete [13]. Also, construction of an optimal size covering array on a graph is at least as hard as finding its optimal size.

We give an approximation algorithm for the Cartesian product with approximation ratio  $O(\log_s |V|)$ , where s can be obtained from the number of

symbols corresponding to each vertex. The following result by Bush is used in our approximation algorithm.

**Theorem 9.** [10] Let g be a positive integer. If g is written in standard form:

$$g = p_1^{n_1} p_2^{n_2} \dots p_l^{n_l}$$

where  $p_1, p_2, \ldots, p_l$  are distinct primes, and if

$$r = \min(p_1^{n_1}, p_2^{n_2}, \dots, p_l^{n_l}),$$

then one can construct OA(s, g) where  $s = 1 + \max(2, r)$ .

We are given a wighted connected graph G = (V, E) with each vertex having the same weight q. In our approximation algorithm, we use a technique from [9] for prime factorization of G with respect to the Cartesian product. This can be done in  $O(E \log V)$  time. For details see [9]. After obtaining prime factors of G, we construct strength two covering array  $C_1$  on maximum size prime factor. Then using rows of  $C_1$ , we produce a covering array on G.

#### **APPROX** CA(G,g):

**Input:** A weighted connected graph G = (V, E) with k > 1 prime factors with respect to the Cartesian product. Each vertex has weight g;  $g = p_1^{n_1} p_2^{n_2} \dots p_l^{n_l}$  where  $p_1, p_2, \dots, p_l$  are primes. Output:  $CA(ug^2, G, g)$ .

**Step 1:** Compute  $s = 1 + \max\{2, r\}$  where  $r = \min(p_1^{n_1}, p_2^{n_2}, \dots, p_l^{n_l})$ .

Step 2: Factorize G into prime factors with respect to the Cartesian product; say  $G = \Box_{i=1}^k G_i$  where  $G_i = (V_i, E_i)$  is a prime factor. **Step 3:** Suppose  $V_1 \ge V_2 \ge \ldots \ge V_k$ . For prime factor  $G_1 = (V_1, E_1)$  do

- 1. Find the smallest positive integer u such that  $s^u \ge V_1$ . That is, u = $\lceil \log_s V_1 \rceil$ .
- 2. Let OA(s, g) be an orthogonal array and denote its *i*th row by  $R_i$  for  $i = 1, 2, \ldots, s$ . Total  $s^u$  many row vectors  $(R_{i_1}, R_{i_2}, \ldots, R_{i_u})$ , each of length  $ug^2$ , are formed by horizontally concatenating u rows  $R_{i_1}$ ,  $R_{i_2}$ ,  $\ldots, R_{i_u}$  where  $1 \leq i_1, \ldots, i_u \leq s$ .
- 3. Form an  $V_1 \times ug^2$  array  $C_1$  by choosing any  $V_1$  rows out of  $s^u$  concatenated row vectors. Each row in the array corresponds to a vertex in the graph  $G_1$ .

**Step 4:** From  $C_1$  we can construct an  $V \times ug^2$  array C. Index the rows of C by  $(u_1, u_2, \ldots, u_k)$ ,  $u_i \in V(G_i)$ . Set the row  $(u_1, u_2, \ldots, u_k)$  to be identical to the row corresponding to  $u_1 + u_2 + \ldots + u_k \mod V_1$  in  $C_1$ . Return C.

**Theorem 10.** Algorithm APPROX CA(G,g) is a polynomial-time  $\rho(V)$  approximation algorithm for covering array on graph problem, where

$$\rho(V) \le \lceil \log_s \frac{V}{2^{k-1}} \rceil.$$

Proof. Correctness: The verification that C is a  $CA(ug^2, G, g)$  is straightforward. First, we show that  $C_1$  is a covering array of strength two with  $|V_1|$ parameters. Pick any two distinct rows of  $C_1$  and consider the sub matrix induced by these two rows. In the sub matrix, there must be a column  $(R_i, R_j)^T$  where  $i \neq j$ . Hence each ordered pair of values appears at least once. Now to show that C is a covering array on G, it is sufficient to show that the rows in C for any pair of adjacent vertices  $u = (u_1, u_2, \ldots, u_k)$  and  $v = (v_1, v_2, \ldots, v_k)$  in G will be qualitatively independent. We know u and v are adjacent if and only if  $(a_i, b_i) \in E(G_i)$  for exactly one index  $1 \leq i \leq k$ and  $a_j = b_j$  for  $j \neq i$ . Hence  $u_1 + u_2 + \ldots + u_k \neq v_1 + v_2 + \ldots + v_k \mod V_1$ and in Step 6, two distinct rows from  $C_1$  are assigned to the vertices u and v.

**Complexity :** The average order of l in Step 1 is  $\ln \ln g$  [14]. Thus, the time to find s in Step 1 is  $O(\ln \ln g)$ . The time to factorize graph G = (V, E) in Step 2 is  $O(E \log V)$ . In Step 3(1), the smallest positive integer u can be found in  $O(\log_s V_1)$  time. In Step 3(2), forming one row vector requires  $\log_s V_1$  assignments; hence, forming  $V_1$  row vectors require  $O(V_1 \log V_1)$  time. Thus the total running time of APPROX CA(G,g) is  $O(E \log V + \ln \ln g)$ . Observing that, in practice,  $\ln \ln g \leq E \log V$ , we can restate the running time of APPROX CA(G,g) as  $O(E \log V)$ .

**Approximation ratio:** We show that APPROX CA(G,g) returns a covering array that is at most  $\rho(V)$  times the size of an optimal covering array on G. We know the smallest n for which a CA(n, G, g) exists is  $g^2$ , that is,  $CAN(G,g) \ge g^2$ . The algorithm returns a covering array on G of size  $ug^2$  where

$$u = \lceil \log_s V_1 \rceil.$$

As G has k prime factors, the maximum number of vertices in a factor can be  $\frac{V}{2^{k-1}}$ , that is,  $V_1 \leq \frac{V}{2^{k-1}}$ . Hence

$$u = \lceil \log_s V_1 \rceil \le \lceil \log_s \frac{V}{2^{k-1}} \rceil.$$

By relating to the size of the covering array returned to the optimal size, we obtain our approximation ratio

$$\rho(V) \le \lceil \log_s \frac{V}{2^{k-1}} \rceil.$$

## 6 Conclusions

One motivation for introducing a graph structure was to optimise covering arrays for their use in testing software and networks based on internal structure. Our primary concern in this paper is with constructions that make optimal covering arrays on large graphs from smaller ones. Large graphs are obtained by considering either the Cartesian, the direct, the strong, or the Lexicographic product of small graphs. Using graph homomorphisms, we have

$$\max_{i=1,2} \{ CAN(G_i, g) \} \le CAN(G_1 \square G_2, g) \le CAN(\max_{i=1,2} \{ \chi(G_i) \}, g).$$

We gave several classes of Cayley graphs where the lower bound on covering array number  $CAN(G_1 \square G_2)$  is achieved. It is an interesting problem to find out other classes of graphs for which lower bound on covering array number of product graph can be achieved. We gave an approximation algorithm for construction of covering array on a graph G having more than one factor with respect to the Cartesian product. Clearly, another area to explore is to consider in details the other graph products, that is, the direct, the strong, and the Lexicographic product.

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