Proper Hamiltonian Paths in Edge-Coloured Multigraphs

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Abstract

Given a c-edge-coloured multigraph, a proper Hamiltonian path is a path that contains all the vertices of the multigraph such that no two adjacent edges have the same colour. In this work we establish sufficient conditions for an edge-coloured multigraph to guarantee the existence of a proper Hamiltonian path, involving various parameters as the number of edges, the number of colours, the rainbow degree and the connectivity.

Keywords: Multigraph, Proper Hamiltonian Path, Edge-Coloured Graph

1 Introduction

The study of problems modelled by edge-coloured graphs have resulted in important developments recently. For instance, the research on long coloured cycles and paths for edge-coloured graphs has provided interesting results [3]. From a practical perspective, problems arising in molecular biology are often modeled using coloured graphs, i.e., graphs with coloured edges and/or vertices [9]. Given an edge-coloured graph, the original problems are equivalent to extract subgraphs coloured in a specified pattern. The most natural pattern in such a context is that of proper colourings, i.e., adjacent edges have different colours.

In this work we give sufficient conditions involving various parameters as the number of edges, rainbow degree, etc, in order to guarantee the existence of proper Hamiltonian paths in edge-coloured multigraphs where parallel edges with same colours are not allowed. Notice that the proper Hamiltonian path and proper Hamiltonian cycle problems are both NP-complete in the general case. However it is polynomial to find a proper Hamiltonian path in c-edge-coloured complete graphs, $c \geq 2$ [7]. It is also polynomial to find a proper Hamiltonian cycle in 2-edge-coloured complete graphs [4], but it is still open to determine the computational complexity for $c \geq 3$ [5]. Many other results for edge-coloured multigraphs can be found in the survey by Bang-Jensen and Gutin [2]. Results involving only degree conditions can be found in [1].

Formally, let $I_c = \{1, 2, \ldots, c\}$ be a set of $c \geq 2$ colours. Throughout this paper, G^c denotes a *c-edge-coloured multigraph* such that each edge is coloured with one colour in I_c and no two parallel edges joining the same pair of vertices have the same colour. Let n be the number of vertices and m be the number of edges of G^c. If H is a subgraph of G^c, then $N_H^i(x)$ denotes the set of vertices of H adjacent to x with an edge of colour i. Whenever H is isomorphic to G^c, we write $N^i(x)$ instead of $N_{G^c}^i(x)$. The *coloured i-degree* of a vertex x, denoted by $d^i(x)$, is the cardinality of $N^i(x)$. As usual $N(x)$ denotes the neighbourhood of x, $d(x)$ its degree and $\delta(G)$ the minimum degree among all vertices of G^c. The *rainbow degree* of a vertex x, denoted by $rd(x)$, is the number of different colours on the edges incident to x. The *rainbow degree* of a multigraph G^c, denoted by $rd(G^c)$, is the minimum rainbow degree among its vertices. An edge with endpoints x and y is denoted by xy, and its colour by $c(xy)$. A *rainbow complete multigraph* is the one having all possible coloured edges between any pair of vertices (its number of edges is therefore $c\binom{n}{2}$). The *complement* of a multigraph G^c denoted by $\overline{G^c}$, is a multigraph with the same vertices as G^c and an edge $vw \in E(\overline{G^c})$ on colour i if and only if $vw \notin E(G^c)$

on that colour. We say that an edge xy is a *missing edge* of G^c if $xy \in E(\overline{G^c})$. The graph G^i is the spanning subgraph of G^c with edges only in colour i. A subgraph of G^c is said to be *properly edge-coloured* if any two adjacent edges in this subgraph differ in colour. A *Hamiltonian path* (*cycle*) is a path (cycle) containing all vertices of the multigraph. A path is said to be *compatible* with a given matching M if the edges of the path are alternatively in M and not in M. We assume that the first and the last edge of the path are in M otherwise we just remove one (or both) of them in order to have this property. All multigraphs are assumed to be connected.

This paper is organized as follows: In Section 2 we present some preliminary results that will be useful for the rest of the paper. In Section 3 we study proper Hamiltonian paths in 2-edge-coloured multigraphs. In Section 4 we study proper Hamiltonian paths in c-edge-coloured multigraphs, for $c \geq 3$. Notice that in this work we focus only on edge-coloured multigraphs since it makes no sense to study such conditions for simple edge-coloured graphs.

2 Preliminary results

Lemma 2.1 *Let G be a connected non-coloured simple graph on n vertices, $n \geq 9$. If $m \geq \binom{n-2}{2} + 3$, then G has a matching M of size $|M| = \lfloor \frac{n}{2} \rfloor$.*

Proof. By a theorem in [6], a 2-connected graph on $n \geq 10$ vertices and $m \geq \binom{n-2}{2} + 5$ edges has a Hamiltonian cycle. So if we add a new vertex v to G and we join it to all the vertices of G we have that $G + \{v\}$ has $m \geq \binom{n-1}{2} + 5$ edges. Therefore $G + \{v\}$ has a Hamiltonian cycle, i.e., G has a Hamiltonian path and this implies that there exists a matching M in G of size $|M| = \lfloor \frac{n}{2} \rfloor$. □

Lemma 2.2 ([8]) *Let G be a simple non-coloured graph on $n \geq 14$ vertices. If $m \geq \binom{n-3}{2} + 4$ and $\delta(G) \geq 1$, then G has a matching M of size $|M| \geq \lceil \frac{n-2}{2} \rceil$.*

Lemma 2.3 *Let G^c be a 2-edge-coloured multigraph on $n \geq 14$ vertices coloured with $\{r, b\}$ (red and blue). If $rd(G^c) = 2$ and $m \geq \binom{n}{2} + \binom{n-3}{2} + 4$, then G^c has two matchings M^r and M^b of colours red and blue respectively, such that $|M^r| = \lfloor \frac{n}{2} \rfloor$ and $|M^b| \geq \lceil \frac{n-2}{2} \rceil$.*

Proof. Let E^r and E^b denote the set of edges coloured in red and blue respectively. Set $|E^r| = m^r$ and $|E^b| = m^b$. Observe that, as for every vertex x in G^c, $rd(x) = 2$, we have that $d^i(x) \geq 1$ for $i \in \{r, b\}$. Observe also that $m^i \geq \binom{n-3}{2} + 4$ for $i \in \{r, b\}$, since this threshold is tight when the multigraph is complete on one of the colours.

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Now, if n is odd, by Lemma 2.2 there exist two matchings M^r and M^b, each one of size $\frac{n-1}{2}$, so the result follows straightforward. Next, if n is even, suppose without loss of generality that $m^r \geq m^b$. Then $m^r \geq (\binom{n}{2} + \binom{n-3}{2} + 4)/2 > \binom{n-2}{2} + 3$. It is sufficient to show that G^r has a matching of size $\lfloor \frac{n}{2} \rfloor$ because G^b has one of size $\lceil \frac{n-2}{2} \rceil$ by Lemma 2.2. Since $\delta(G^r) \geq 1$, G^r is connected, thus, Lemma 2.1 implies that G^r has a matching of size $\lfloor \frac{n}{2} \rfloor$ as desired. □

Lemma 2.4 *Let G^c be a connected c-edge-coloured multigraph, $c \geq 2$. Suppose that G^c contains a proper path $P = x_1 y_1 x_2 y_2 \ldots x_p y_p$, $p \geq 3$, such that each edge $x_i y_i$ is red. If G^c does not contain a proper cycle C such that $V(C) = V(P)$ then there are at least $(c-1)(2p-2)$ missing edges in G^c.*

Proof. We show that there are at least $2p - 2$ missing edges in G^c per colour different from red. As there are $c - 1$ such colours the total number of missing edges will be $(c-1)(2p-2)$ as claimed. Let us consider some colour, say blue, different from red. The blue edge $x_1 y_p$ cannot be in G^c otherwise $x_1 y_1 \ldots x_p y_p x_1$ is a proper cycle. Suppose that the blue edge $x_1 x_i$ is present in G^c for some $i = 2, \ldots, p$. Then the blue edge $y_{i-1} y_p$ cannot be in G^c otherwise the proper cycle $x_1 x_i \ldots y_p y_{i-1} \ldots x_1$ contradicts our hypothesis. Therefore for each edge $y_{i-1} x_i$ either the blue edge $x_1 x_i$ or the blue edge $y_{i-1} y_p$ is missing. So there are $p - 1$ blue missing edges in G^c. Now suppose that the blue edge $x_1 y_i$ is present in G^c, for some $i = 2, \ldots, p - 2$. Then the blue edge $x_{i+1} y_p$ cannot be together with the blue edges $x_i y_{i+1}$, $y_{i-1} x_{i+2}$ or $y_{i-1} y_{i+1}$, $x_i x_{i+2}$ in G^c, otherwise the proper cycles $x_1 y_i x_i y_{i+1} x_{i+1} y_p \ldots x_{i+2} y_{i-1} \ldots x_1$ or $x_1 y_i x_i x_{i+2} \ldots y_p x_{i+1} y_{i+1} y_{i-1} \ldots x_1$ contradict again our hypothesis. Then for each edge $y_i x_{i+1}$, at least one of the edges $x_{i+1} y_p$, $x_1 y_i$ is missing in G^c for $i = 2, \ldots, p - 2$. Therefore there are $p - 3$ blue missing edges.

Up to now we have $2p - 3$ blue missing edges. To obtain the last missing edge observe that one of the blue edges $x_2 y_p$, $x_1 y_2$, $y_1 x_3$ ($x_2 y_p$, $x_1 x_3$, $y_1 y_2$) is missing in G^c, otherwise we obtain the proper cycle $x_1 y_2 x_2 y_p \ldots x_3 y_1 x_1$ ($x_1 x_3 \ldots y_p x_2 y_2 y_1 x_1$). We remark that the blue edges $x_2 y_p$ and $y_1 x_3$ ($y_1 y_2$) were not counted before. The edge $x_1 y_2$ ($x_1 x_3$) was supposed to exist, otherwise, to obtain the last missing edge we consider the symmetric case, i.e., using the blue edge $x_1 y_{p-1}$ (if it exists).

In conclusion there are $2p - 2$ blue missing edges in G^c as required. This completes the argument and the proof. □

Lemma 2.5 *Let G^c be a connected c-edge-coloured multigraph, $c \geq 2$. Let M be a matching of G^c in one colour, say red, of size $|M| \geq \lceil \frac{n-2}{2} \rceil$. Let $P = x_1 y_1 x_2 y_2 \ldots x_p y_p$, $p \geq 3$, be a longest proper path compatible with M.*

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Let $f(n,c)$ denote the minimum number of missing edges in G^c on colours different from red. Then the following holds:

$$f(n,c) = \begin{cases} (2n-4)(c-1) & \text{if } n \text{ is even, } |M| = \frac{n}{2} \text{ and } 2p < n \\ (2n-6)(c-1) & \text{if } n \text{ is odd, } |M| = \frac{n-1}{2} \text{ and } 2p < n-1 \\ (2n-8)(c-1) & \text{if } n \text{ is even, } |M| = \frac{n-2}{2} \text{ and } 2p < n-2 \end{cases}$$

Proof. Here we consider only the case n is even, $|M| = \frac{n}{2}$ and $2p < n$, as the two other cases are similar. Observe that, as the red matching M has $\frac{n}{2}$ edges and by hypothesis P uses p edges of M, there are precisely $\frac{n-2p}{2}$ edges of M in $G^c - P$. Let us denote these edges by $e_i = w_i z_i$, $w_i, z_i \in G^c - P$, $i = 1, \ldots, \frac{n-2p}{2}$.

Suppose first that there is no proper C cycle such that $V(C) = V(P)$. Let blue be some colour different from red. By Lemma 2.4 there are $2p-2$ blue missing edges in the subgraph induced by $V(P)$. Furthermore there are no blue edges between the vertices x_1, y_p and the endpoints of every edge e_i. Otherwise if such an edge exists for some i, say $x_1 w_i$, then the path $z_i w_i x_1 y_1 \ldots x_p y_p$ contradicts the maximality property of P. Thus, there are $2(n-2p)$ blue missing edges. In adittion, for each edge $y_j x_{j+1}$, $j = 1, \ldots, p-1$, at least two of the blue edges $y_j w_i$, $y_j z_i$, $x_{j+1} w_i$ and $x_{j+1} z_i$ are missing in G^c, otherwise if at least three among them exist, we can easily find a path longer than P, a contradiction. So, in this case there are $(n-2p)(p-1)$ blue missing edges. Summing up we obtain $(n - 2 + pn - 2p^2)$ blue missing edges in G^c. As there are $c-1$ colours different from red, we finally have a total of $(n - 2 + pn - 2p^2)(c-1)$ missing edges in G^c. For n and c fixed, the minimum value of this function is obtained for $p = \frac{n-2}{2}$. Thus $f(n,c) = [n - 2 + \frac{n-2}{2}n - 2(\frac{n-2}{2})^2](c-1) = (2n-4)(c-1)$ as required.

Suppose next that there is a proper cycle C such that $V(C) = V(P)$. Then every edge (if any) between a vertex of C and the endpoints of the edges $e_i = w_i z_i$ should be red. Otherwise if such a non red edge exists, say $x_j w_i$ for some i and j, $x_i \in C$, then appropriately using the segment $x_j w_i z_i$ along with C, we may find a path longer than P, a contradiction. Therefore there are at least $(2pn - 4p^2)(c-1)$ missing edges in G^c. Again, by minimizing the function we obtain $f(n,c) = (2n-4)(c-1)$ for $p = \frac{n-2}{2}$. $\qquad\square$

3   2-edge-coloured multigraphs

In this section we study the existence of proper Hamiltonian paths in 2-edge-coloured multigraphs. We present two main results. The first one involves the

number of edges. The second one involves both the number of edges and the rainbow degree. All results are tight.

Theorem 3.1 *Let G^c be a 2-edge-coloured multigraph on $n \geq 8$ vertices coloured with $\{r, b\}$. If $m \geq \binom{n}{2} + \binom{n-2}{2} + 1$, then G^c has a proper Hamiltonian path.*

For the extremal example, $n \geq 8$, consider a rainbow complete 2-edge-coloured multigraph on $n - 2$ vertices, n odd. Add two new vertices x_1 and x_2. Then add a red edge $x_1 x_2$ and all red edges between $\{x_1, x_2\}$ and the complete graph. Although the resulting graph has $\binom{n}{2} + \binom{n-2}{2}$ edges, it has no proper Hamiltonian path, since there is no blue matching of size $(n-1)/2$.

Proof. By induction on n. For $n = 8, 9$ by a rather tedious but easy analysis the result can be shown. Suppose now that $n \geq 10$. As G^c has at least $\binom{n}{2} + \binom{n-2}{2}$ edges then $|E(\overline{G^c})| \leq 2n - 4$. By a theorem in [1], if every vertex $x \in G^c$ has $d^r(x) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(x) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper Hamiltonian path. Thus, we can assume that there exists a vertex $x \in G^c$ such that $d^r(x) \leq \lceil \frac{n+1}{2} \rceil - 1$, otherwise there is nothing to prove.

Suppose first that there exist two distinct neighbours y, z of x such that $c(xy) = b$ and $c(xz) = r$. We then construct a new multigraph G'^c by replacing the vertices x, y, z to a new vertex s such that $N^r(s) = N^r_{G^c - \{x,z\}}(y)$ and $N^b(s) = N^b_{G^c - \{x,y\}}(z)$. We remark that $N^r_{G^c - \{x,z\}}(y)$ and $N^b_{G^c - \{x,y\}}(z)$ cannot both be empty, otherwise $|E(\overline{G^c})| \geq 3n - 5 - \lceil \frac{n+1}{2} \rceil > 2n - 4$, a contradiction. By doing, in the worst case we remove at most $n - 1$ blue and $\lceil \frac{n+1}{2} \rceil - 1$ red edges from x, $n - 3$ blue edges from y, $n - 3$ red edges from z and one red and one blue between y and z. Therefore G'^c has at least $\binom{n}{2} + \binom{n-2}{2} + 1 - (n - 1) - (\lceil \frac{n+1}{2} \rceil - 1) - 2(n - 3) - 2 \geq \binom{n-2}{2} + \binom{n-4}{2} + 1$ edges. Thus by induction, G'^c has a proper Hamiltonian path P. From this path P we can easily obtain a proper Hamiltonian path in G^c.

Suppose now that there does not exist two distinct neighbours y, z of x such that $c(xy) = b$ and $c(xz) = r$. Suppose first that both y and z exist but they are not distinct, i.e., $y = z$. In this case, it is easy to observe that $G^c - \{x\}$ has $(n - 1)(n - 2)$ edges, i.e., it is a rainbow complete multigraph. Therefore, it contais a proper Hamiltonian path starting at y. This path can be easily extended to a proper Hamiltonian path of G^c by adding one of the edges xy in the appropriate colour. Suppose next that all edges incident to x are on the same colour, say b. Observe that for every vertex $w \neq x$, there exists at least one red edge wu, $u \in G^c - \{x, w\}$, otherwise $|E(\overline{G^c})| \geq 2n - 3 > 2n - 4$, which is a contradiction. In the following we distinguish between to cases depending

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on the neighbourhood of x. Assume first that $N^b(x) \leq n - 2$. Consider a neighbour y of x and remove all its blue incident edges. Then remove x from G^c and call this multigraph G''^c. In G''^c, y is monochromatic in red and G''^c has at least $\binom{n-1}{2} + \binom{n-3}{2} + 1$ edges. Thus by the inductive hypothesis, G''^c has a proper Hamiltonian path. This path starts at y since it was monochromatic. So we have a proper Hamiltonian path in G^c. Assume next that $N^b(x) = n-1$. If for some neighbour y of x, $N^b(y) \leq n - 3$, we complete the argument as before. Otherwise for every vertex y, $N^b(y) = n - 2$. It follows that the underlying blue subgraph G''^b of $G''^c = G^c - \{x\}$ is complete. Furthermore, G''^c has at least $n^2 - 4n + 5$ edges. Now remove all the blue edges from G''^c. This new (red) graph has $n - 1$ vertices and at least $\binom{n-2}{2} + 1$ edges. Therefore by a theorem in [6], it has a Hamiltonian path P. Now since G''^b is complete, we can appropriately use some blue edges of G''^b along with the edges of P to define a proper Hamiltonian path P' in G''^c that always starts with an edge on colour red. Finally, we can join x to the first vertex of P' in order to obtain a proper Hamiltonian path in G^c. $\qquad\square$

Theorem 3.2 *Let G^c be a 2-edge-coloured multigraph on $n \geq 14$ vertices coloured with $\{r, b\}$. If $rd(G^c) = 2$ and $m \geq \binom{n}{2} + \binom{n-3}{2} + 4$, then G^c has a proper Hamiltonian path.*

For the extremal example, $n \geq 14$ odd, consider a complete blue graph, say A, on $n - 3$ vertices. Add three new vertices v_1, v_2, v_3 and join them to a same vertex v in A with blue edges. Finally, superpose the obtained graph with a complete red graph on n vertices. Although the resulting 2-edge-coloured multigraph has $\binom{n}{2} + \binom{n-3}{2} + 3$ edges, it has no proper Hamiltonian path since one of the vertices v_1, v_2, v_3 cannot belong to such a path.

Proof. Let us suppose that G^c does not have a proper Hamiltonian path. We will show that $\overline{G^c}$ has more than $3n - 10$ edges, i.e., G^c has less than $\binom{n}{2} + \binom{n-3}{2} + 4$ edges, contradicting the hypothesis of the theorem. We distinguish between two cases depending on the parity of n.
Case A: *n is even.* By Lemma 2.3 G^c has two matchings M^r, M^b such that $|M^r| = \frac{n}{2}$ and $|M^b| \geq \frac{n-2}{2}$. Take two longest proper paths, say $P = x_1 y_1 x_2 y_2 \ldots x_p y_p$ and $P' = x_1' y_1' x_2' y_2' \ldots x_{p'}' y_{p'}'$, compatibles with M^r and M^b, respectively.

Notice now that if $2p = n$ or $2p' = n$ then we are finished. In addition, if $2p' < n - 2$, then by Lemma 2.5 there are at least $2n - 4$ blue missing edges and $2n - 8$ red ones. This gives a total of $4n - 12 > 3n - 10$ missing edges, which is a contradiction. Consequently, in what follows we may suppose that

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$2p = 2p' = n - 2$.

Suppose first that there exists a proper cycle C in G^c such that $V(C) = V(P)$. Let $e = wz$ be the red edge of $M^r - E(C)$. If there exists a blue edge e' between w or z and some vertex of C, we can easily obtain a proper Hamiltonian path considering e, e' and the rest of C in the appropriate direction. Otherwise as the multigraph is connected, all edges e' between the endpoints of e and C are red. Now as $rd(G^c) = 2$, there must exist a blue edge e'' between w and z and therefore we can obtain a proper Hamiltonian path just as before but starting with e'' instead of e.

Next suppose that there exists no proper cycle C in G^c such that $V(C) = V(P)$. By Lemma 2.5 there are at least $2n - 4$ blue missing edges. Consider now the path P' and let v_1, w_1 be the two vertices of $G^c - P'$. It is clear that if there exists a blue edge joining v_1 and w_1, then $|M^b| = \frac{n}{2}$. Thus, by symmetry on the colours there are at least $2n - 4$ red missing edges. This gives a total of $4n - 8 > 3n - 10$ blue and red missing edges, a contradiction. Otherwise, assume that there is no blue edge between v_1 and w_1. In this case we will count the red missing edges assuming that we cannot extend P' to a proper Hamiltonian path. If there exists no cycle C' in G^c such that $V(C') = V(P')$, then by Lemma 2.4 there are $2p' - 2 = n - 4$ red missing edges. By summing up we obtain $3n - 8 > 3n - 10$ missing edges, which is a contradiction. Finally, assume that there exists a proper cycle C' in G^c such that $V(C') = V(P')$. Set $C = c_1 c_2 \ldots c_{2p'} c_1$ where $c(c_i c_{i+1}) = r$ for $i = 1, 3, \ldots, 2p' - 1$. If there are three or more red edges between $\{v_1, w_1\}$ and $\{c_i, c_{i+1}\}$, for some $i = 1, 3, \ldots, 2p' - 1$, then either the edges $v_1 c_i$ and $w_1 c_{i+1}$, or $v_1 c_{i+1}$ and $w_1 c_i$ are red. Suppose $v_1 c_i$ and $w_1 c_{i+1}$ are red. In this case, the path $v_1 c_i c_{i-1} \ldots c_1 c_{2p'} \ldots c_{i+1} w_1$ is a proper Hamiltonian one. Otherwise, there are at most two red edges between $\{v_1, w_1\}$ and $\{c_i, c_{i+1}\}$, for all $i = 1, 3, \ldots, 2p' - 1$, then there are $2p' - 2 = n - 4$ red missing edges. If we sum up we obtain a total of $3n - 8 > 3n - 10$ missing edges, which is a contradiction.

Case B: *n is odd.* By Lemma 2.3 G^c has two matchings M^r, M^b such that $|M^r| = |M^b| = \frac{n-1}{2}$. As in Case A, we consider two longest proper paths P and P' compatibles with the matchings M^r and M^b, respectively. Suppose first that $2p < n - 1$ and $2p' < n - 1$. By Lemma 2.5 there are at least $2n - 6$ blue and $2n - 6$ red missing edges. We obtain a total of $4n - 12 > 3n - 10$ missing edges, which is a contradiction.

Suppose next $2p = 2p' = n - 1$ (the cases where $2p < n - 1$ and $2p' = n - 1$, or $2p = n - 1$ and $2p' < n - 1$ are similar). In the rest of the proof, we will consider only the path P since, by symmetry, the same arguments may be applied for P'. In this case we will count the blue missing edges assuming

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that we cannot extend P to a proper Hamiltonian path. Now let v be the unique vertex in $G^c - P$. It is clear that if there is a proper cycle C in G^c such that $V(C) = V(P)$, we can trivially obtain a proper Hamiltonian path since the multigraph is connected. Then, as there is no proper cycle C in G^c such that $V(C) = V(P)$, by Lemma 2.4 there are $2p - 2 = n - 3$ blue missing edges. If there exists a blue edge between x_1 and x_i, for some $i = 2, \ldots, p$, then the blue edge vy_{i-1} cannot exist in G^c, otherwise we would obtain the proper Hamiltonian path $vy_{i-1} \ldots x_1 x_i \ldots y_p$. We can complete the argument in a similar way if both edges $y_p y_i$ and vx_{i+1}, $i = 1, \ldots, p - 1$ exist in G^c and are on colour blue. Note that since there is no proper cycle C in G^c such that $V(C) = V(P)$, then the blue edges $x_1 x_i$ and $y_p y_{i-1}$, $i = 2, \ldots, p$ cannot exist simultaneously in G^c. Therefore there are $p - 1 = \frac{n-3}{2}$ blue missing edges. If we make the sum and multiply it by two (since the same number of red missing edges is obtained with P'), we conclude that there are $3n - 9 > 3n - 10$ missing edges, which is a contradiction. This completes the argument and the proof of the theorem. $\qquad\square$

4   c-edge-coloured multigraphs, $c \geq 3$

In this section we study the existence of proper Hamiltonian paths in c-edge-coloured multigraphs, for $c \geq 3$. We present three main results that involve: (1) the number of edges, (2) the number of edges and the connectivity of the multigraph, (3) the number of edges and the rainbow degree. All results are tight.

   In the next lemma we present a key result that reduces the case $c \geq 4$ to $c = 3$.

Lemma 4.1 *Let ℓ be a positive integer. Let G^c be a c-edge-coloured connected multigraph on n vertices and $m \geq c\,\ell + 1$ edges, $c \geq 4$. There exists one colour c_j such that if we colour the edges of G^{c_j} with another colour and we delete parallel edges with the same colour, then the resulting $(c-1)$-edge-coloured multigraph G^{c-1} is connected and has $m' \geq (c-1)\ell + 1$ edges. Furthermore, if G^{c-1} has a proper Hamiltonian path then G^c has one too. Also, if $rd(G^c) = c$, then $rd(G^{c-1}) = c - 1$.*

Proof. Let c_i denote the colour i, for $i = 1, \ldots, c$, and denote by $|c_i|$ the number of edges of G^c with colour i. Let c_j be the colour with the least number of edges. Colour the edges on colour c_j with another colour, say c_l, and delete (if necessary) parallel edges with that colour. Call this multigraph G^{c-1}. By this, we delete at most $|c_j|$ edges. It is clear that this multigraph

is connected since we deleted just parallel edges. Also if G^{c-1} has a proper Hamiltonian path, then this path is also proper Hamiltonian in G^c but perhaps with some edges on colour c_j (in the case that they have been recoloured with c_l). Observe also that, if $rd(G^c) = c$ then $rd(G^{c-1}) = c-1$ since only the colour c_j was removed. We will show now that $m' \geq (c-1)\ell + 1$. Now, if $|c_j| > \ell$, then clearly $m' \geq (c-1)\ell + 1$ edges since for all i, $|c_i| > \ell$. Otherwise $|c_j| \leq \ell$. Now, $m = \sum_{i=1}^{c} |c_i| \geq c\,\ell + 1$ and therefore $\sum_{i=1, i\neq j}^{c} |c_i| \geq c\,\ell - |c_j| + 1 = (c-1)\ell + \ell - |c_j| + 1$. This last expression is greater than or equal to $(c-1)\ell + 1$ since $\ell - |c_j| \geq 0$. Finally, we have that G^{c-1} has $m' \geq (c-1)\ell + 1$ edges as desired. $\qquad\square$

In view of Theorems 4.4,4.6 and 4.8 we need the following definition.

Definition 4.2 Let G^c be a 3-edge-coloured multigraph coloured with $\{r, b, g\}$. Suppose that there exist two distinct vertices $x, y \in G^c$ such that y is a neighbour of x and either $|N(x)| = 1$ or $N^r(x) = N^g(x) = \emptyset$. First remove the vertex x. Then, remove all the edges (if any) in colours either b, r or b, g, incident to y. Finally rename the vertex y to s. We call this process the *contraction* of x, y to s.

Definition 4.3 Let G^c be a 3-edge-coloured multigraph coloured with $\{r, b, g\}$. Suppose that there exist three different vertices $x, y, z \in G^c$ such that $c(xy) = b$ and $c(xz) = r$. Now the *contraction* of x, y, z is defined as follows: We replace the vertices x, y, z by a new vertex s such that $N^b(s) = N^b_{G^c - \{x,y\}}(z)$, $N^r(s) = N^r_{G^c - \{x,z\}}(y)$ and $N^g(s) = N^g_{G^c - \{x,z\}}(y) \cap N^g_{G^c - \{x,y\}}(z)$.

Notice that if G''^c is the graph obtained from G^c by any of the contractions above, then any proper Hamiltonian path in G''^c can be easily transformed into a proper Hamiltonian one in G^c.

Theorem 4.4 *Let G^c be a c-edge-coloured multigraph on n vertices, $n \geq 2$ and $c \geq 3$. If $m \geq c\binom{n-1}{2} + 1$, then G^c has a proper Hamiltonian path.*

For the extremal case consider a rainbow complete multigraph on $n - 1$ vertices with c colours and add a new isolated vertex x. Although the resulting multigraph has $c\binom{n-1}{2}$ edges, it contains no proper Hamiltonian path since it is not connected.

Proof. By Lemma 4.1 we can assume that $c = 3$ and let $\{r, b, g\}$ be the set of colours. Assume $n \geq 6$ as cases $n \leq 5$ can be checked by exhaustive methods. The proof is by induction on n. We consider two cases depending on whether G^c contains a monochromatic vertex or not.

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Case A: *There exists a monochromatic vertex $x \in G^c$.* Assume without loss of generality that all the edges incident to x are on colour r. Suppose first that $d(x) \leq n-2$. Consider the multigraph G'^c obtained from G^c by contracting x and one of its neighbours, say y, to a vertex s as in Definition 4.2 considering r instead of b. By this, we delete at most $3n-6$ edges. This multigraph G'^c has $n-1$ vertices and at least $3\binom{n-2}{2}+1$ edges. Then by inductive hypothesis it has a proper Hamiltonian path. Since s is monochromatic, we easily extend the path with x to obtain a proper Hamiltonian path in G^c. Suppose next that $d(x) = n-1$. Then the multigraph $G^c - \{x\}$ has at least $3\binom{n-2}{2}+1$ edges and therefore by inductive hypothesis it has a proper Hamiltonian path $P = x_1 x_2 \ldots x_{n-1}$. Now if $c(x_1 x_2) \neq r$ or $c(x_{n-2} x_{n-1}) \neq r$, we are done. Otherwise, $c(x_1 x_2) = c(x_{n-2} x_{n-1}) = r$. If between x_1 and x_2 there exist the three possible edges then the path $xx_1 x_2 \ldots x_{n-1}$ is a proper Hamiltonian one by appropriately choosing the edge $x_1 x_2$ such that $c(x_1 x_2) \neq c(x_2 x_3)$ and $c(x_1 x_2) \neq c(xx_1)$. Otherwise the degree of x_1 in some colour different from r, say b is at most $n-3$. Then as before, we can make the contraction with x and x_1 removing the edges on colours b and r incident to x_1.

Case B: *There is no monochromatic vertex in G^c.* Suppose first that there exists a vertex x such that $|N(x)| = 1$. Let y be its unique neighbour. Now by contraction of x and y as in Definition 4.2 and by deleting edges incident to y in two appropriate colours we can complete the argument. Assume therefore that $|N(x)| \geq 2$ for all $x \in G^c$. Moreover we may suppose that there exists a vertex x such that $d(x) \leq 3n-6$. Otherwise, if for all $x \in G^c$, $d(x) \geq 3n-5$, then $d^i(x) \geq \lceil \frac{n}{2} \rceil \; \forall x \in G^c, i \in \{r, g, b\}$. Thus by a theorem in [1], G^c has a proper Hamiltonian cycle and so a proper Hamiltonian path. Consider now $G^c - \{x\}$. This multigraph has at least $3\binom{n-2}{2}+1$ edges, then by the inductive hypothesis it has a proper Hamiltonian path $P = x_1 x_2 \ldots x_{n-1}$. We try to add x to P in order to obtain a proper Hamiltonian path in G^c. If x is adjacent to either x_1 or x_{n-1} in any appropriate colour we are done. Otherwise there are four missing edges incidet to x. If there are at least five edges between x and some pair of vertices $\{x_i, x_{i+1}\}$, $i = 2, \ldots, n-2$, then by choosing the appropriate edges xx_i and xx_{i+1}, the path $x_1 \ldots x_i xx_{i+1} \ldots x_{n-1}$ is proper Hamiltonian one in G^c. Otherwise there are at most four edges between x and every pair of vertices $\{x_i, x_{i+1}\}$, for $i = 2, \ldots, n-2$. Therefore there are at least $n-3 \geq 3$ missing edges incident to x. It follows that the degree of x is at most $3(n-1) - 4 - (n-3) = 2n - 4 \leq 3n - 10$. Take now $y, z \in N(x)$ and suppose that $c(xy) = b$ and $c(xz) = r$. Contract x, y, z as in Definition 4.3. By this operation we remove at most $3n-10$ edges incident to x and at most $3n-6$ edges incident to y and z in $G^c - \{x\}$. It follows that the obtained multigraph

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on $n-2$ vertices has at least $c\binom{n-1}{2}+1-(3n-10)-(3n-6) \geq c\binom{n-3}{2}+1$ edges. Therefore, by the inductive hypothesis it has a proper Hamiltonian path P. Now it is easy to obtain from P a proper Hamiltonian path in G^c. $\square$

Notice that in the above theorem there is no condition guaranteeing the connectivity of the underlying graph. In view of Theorem 4.6 that adds this condition, we establish the following lemma.

Lemma 4.5 *Let G^c be a c-edge-coloured multigraph on n vertices fullfilling the conditions of Theorem 4.6 and $c \geq 4$. Then either G^c has a proper Hamiltonian path or G^c contains a connected $(c-1)$-edge-coloured multigraph G^{c-1} on n vertices with at least $(c-1)\binom{n-2}{2}+n$ edges such that if G^{c-1} has a proper Hamiltonian path then G^c has one too.*

Proof. Let c_i denote the colour i and E^i the set of edges of G^c on colour c_i, for $i = 1, \ldots, c$. Suppose first that there is a colour c_j such that $|E^j| \leq \binom{n-2}{2}$. Then, colour the edges on colour c_j with another colour, say c_l, and delete (if necessary) parallel edges with the same colour. Call this multigraph G^{c-1}. Clearly G^{c-1} is connected and it has at least $(c-1)\binom{n-2}{2}+n$ edges. Moreover if G^{c-1} has a proper Hamiltonian path, then it also does G^c. Suppose next that for every colour c_j, $|E^j| \geq \binom{n-2}{2}+1$. If we proceed as above and we obtain that the multigraph G^{c-1} has at least $(c-1)\binom{n-2}{2}+n$ edges, we are done. Otherwise, for each pair of colours c_j, c_l we have that $|E^j \cap E^l| \geq \binom{n-2}{2}+1$, that is, after colouring the edges on colour c_j with colour c_l, there are at least $\binom{n-2}{2}+1$ parallel edges on colour c_l. Now take any two colours c_j, c_l and consider the uncoloured simple graph G having same vertex set as G^c and for each pair of vertices x, y we add the uncoloured edge xy in G if and only if $xy \in E^j$ and $xy \in E^l$ in G^c. Clearly G has at least $\binom{n-2}{2}+1$ edges. We distinguish between two cases depending on the connectivity of G.

Suppose first that G is connected. Add a new vertex v to G and join it to all the vertices of G. Then $G+\{v\}$ has at least $m \geq \binom{n-1}{2}+3$ edges. Therefore by [6], $G + \{v\}$ is Hamiltonian-connected, that is, each pair of vertices in G is joined by a Hamiltonian path. In particular we have a Hamiltonian path P that starts at v. Therefore if we remove v from P and we take its edges on alternating colours c_j, c_l we obtain a proper Hamiltonian path in G^c.

Suppose next that G is disconnected. By a simple calculation on the number of edges of G we can see that G has two components, say A and B, such that either $|A| = 1$ and $|B| = n-1$, or $|A| = 2$ and $|B| = n-2$. If $|A| = 2$ and $|B| = n-2$, let v, w be the vertices of A. By the condition on

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the number of edges, both A and B are complete. Now, as G^c is connected there exists one edge between v (or w) and some vertex $u \in B$ on colour c_k. Therefore we obtain a proper Hamiltonian path in G^c starting with the edge wv on colour c_j (or c_l), then vu on colour c_k and following any Hamiltonian path in B alternating the colours c_j, c_l.

If $|A| = 1$ and $|B| = n - 1$, then let v be the unique vertex of A. Now by [6], B has a Hamiltonian cycle unless it is isomorphic to a complete graph on $n - 2$ vertices plus one vertex, say w, joint to exactly one vertex, say u, of the complete graph $B - \{w\}$. Now if B has a Hamiltonian cycle C, then as G^c is connected, there exists one edge between v and some vertex in B in some colour, say c_k. Therefore we obtain a proper Hamiltonian path in G^c starting at v taking this edge on colour c_k, then following C alternating the colours c_j, c_l. Alternatively, if B has no Hamiltonian cycle, then $B - \{w\}$ has a Hamiltonian path between every pair of vertices. As G^c is connected there exists one edge between v and some vertex $z \in B$ on some colour c_k. If $z \neq u, w$, then taking the edge vz on colour c_k, following a Hamiltonian path in $B - \{w\}$ that starts at z and ends at u alternating the colours c_j, c_l and taking the appropriate edge uw we obtain a proper Hamiltonian path in G^c. If $z = w$, take the edge vw on colour c_k, the edge wu on colour either c_j, c_l and then follow any Hamiltonian path in B starting at u, alternating the colours c_j, c_l, we obtain a proper Hamiltonian path in G^c. If none of the two above cases hold, then v has only one neighbour in B and $z = u$. Consider the following two cases.

Case A: *The edge vu exists on colour $c_k \neq c_j, c_l$.* Then, as G^c has at least $m \geq c\binom{n-2}{2} + n$ edges and $2c < n$, w has a neighbour, say x, in $B - \{u, w\}$ on some colour c_s. Then we obtain a proper Hamiltonian path in G^c as follows. Take the edge vu on colour c_k, continue with the edge uw on colour c_j or c_l (depending on the colour c_s) and the edge wx on colour c_s. Last, follow any Hamiltonian path in $B - \{u, w\}$ starting at x by appropriately alternating the colours c_j, c_l.

Case B: *The edge vu exists only on colour c_j or c_l, say c_j, but not both.* Now, by a similar argument as in case A, w has a neighbour, say x, in $B - \{u, w\}$ on some colour c_s. Let P be an alternating Hamiltonian path in $B - \{w\}$ from u to x such that its first edge is on colour c_l and its last edge has colour different of c_s (this is always possible because of the number of edges of G^c). Now we obtain a proper Hamiltonian path between v and w in G^c as follows. Add the edge vu on colour c_j to P and complete the path with the edge xw on colour c_s.

   This completes the argument and the proof. $\qquad\qquad \square$

Theorem 4.6 *Let G^c be a connected c-edge-coloured multigraph on n vertices, $n \geq 9$ and $3 \leq c < \frac{n}{2}$. If $m \geq c\binom{n-2}{2} + n$, then G^c has a proper Hamiltonian path.*

For the extremal example, $n \geq 9$, consider a rainbow complete multigraph on $n-2$ vertices with c colours and add two new vertices x and y. Now add the edge xy and all edges between y and the complete multigraph, all on the same colour. The resulting multigraph, although it has $c\binom{n-2}{2} + n - 1$ edges, it does not contain a proper Hamiltonian path as x cannot belong to such a path.

Proof. By Lemma 4.5 we can assume that $c = 3$. Let $\{r, b, g\}$ be the set of colours. The proof is by induction on n. For $n = 9, 10$ it can be shown by case analysis that the result holds. Now we have two cases depending on whether G^c contains a monochromatic vertex or not.

Case A: *There exists a monochromatic vertex $x \in G^c$.* Notice that among all neighbours of x there exists at least one, say y, that is not monochromatic, otherwise we would have a contradiction on the number of edges. Suppose that $c(xy) = b$. Now we will contract x, y to a new vertex s as in Definition 4.2. Here the resulting multigraph on $n-1$ vertices has to be connected (as we will show later) and we need to delete at most $3n - 8$ edges for the induction hypothesis to hold.

Let us now consider $d^b(x)$. Observe that if $d^b(x) \leq n - 4$, we delete at most $3n - 8$ edges from x and any selected neighbour y of x and we are done. Further, from [1], if $d^i(z) \geq \lceil \frac{n}{2} \rceil$, $\forall z \in G^c - \{x\}, i \in \{r, g, b\}$, then $G^c - \{x\}$ has a proper Hamiltonian cycle. This would imply a proper Hamiltonian path in G^c. Thus, we may assume that there exists some vertex $w \in G^c - \{x\}$ such that $d^i(w) < \lceil \frac{n}{2} \rceil$ for some $i \in \{r, g, b\}$.

Subcase A1: $d^b(x) = n - 1$. Observe that $w \in N^b(x)$. In this case, considering w instead of y, the contraction process deletes $n - 1$ edges from x, and at most $n + \frac{n}{2} - 3$ from w, which is much less than $3n - 8$ for $n > 10$.

Subcase A2: $d^b(x) = n - 2$. If there is a vertex y adjacent to x such that $d^b_{G^c-\{x\}}(y) + d^r_{G^c-\{x\}}(y) \leq 2n - 6$ or $d^b_{G^c-\{x\}}(y) + d^g_{G^c-\{x\}} \leq 2n - 6$, then we just take x and y for the contraction process. Otherwise for all y adjacent to x we have $d^b_{G^c-\{x\}}(y) + d^r_{G^c-\{x\}}(y) \geq 2n - 5$ and $d^b_{G^c-\{x\}}(y) + d^g_{G^c-\{x\}}(y) \geq 2n - 5$. That implies $d^i(y) \geq \lceil \frac{n-2}{2} \rceil$, $\forall y \in G^c - \{x, z\}, i \in \{r, g, b\}$, where z is the unique non-neighbour of x. Then by [1], $G^c - \{x, z\}$ has a proper Hamiltonian cycle. Finally, we can add x and z to the cycle using the fact that x is adjacent to every vertex on it (as it is z) by the degree condition of the vertices of the cycle. By this we obtain a proper Hamiltonian path in G^c.

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Subcase A3: $d^b(x) = n - 3$. This case is similar to the previous one but finding a vertex y adjacent to x such that $d^b_{G^c-\{x\}}(y) + d^r_{G^c-\{x\}}(y) \leq 2n - 5$ or $d^b_{G^c-\{x\}}(y) + d^g_{G^c-\{x\}}(y) \leq 2n - 5$. Otherwise the multigraph $G^c - \{x\}$ is rainbow complete (except maybe for the three edges between the two non-neighbours of x), we easily find a proper Hamiltonian cycle in $G^c - \{x\}$ and then adding x, a proper Hamiltonian path in G^c.

Case B: *There is no monochromatic vertex in G^c.* If there exists a vertex x such that $|N(x)| = 1$ we proceed as in case B of Theorem 4.4. In what follows we assume that $|N(x)| \geq 2$ for all $x \in G^c$. Suppose now that there exists a vertex x such that $d(x) \leq 3n - 8$. Otherwise, if for all $x \in G^c$, $d(x) \geq 3n - 7$, then $m \geq \frac{n(3n-7)}{2} \geq 3\binom{n-1}{2} + 1$ and by Theorem 4.4 the result holds. Consider now $G^c - \{x\}$. This multigraph has at least $3\binom{n-3}{2} + n - 1$ edges and it is clearly connected. Then by the inductive hypothesis it has a proper Hamiltonian path P. Now we use the same argument as in Theorem 4.4 to add x to P. If we cannot add it, we obtain that $d(x) \leq 3n - 15$. Finally take $y, z \in N(x)$ such that $c(xy) = b$ and $c(xz) = r$. Contract x, y, z to a new vertex s as in Definition 4.3. By this we delete at most $6n - 21$ edges, that is, $3n - 15$ edges incident to x and $3n - 6$ edges incident to y and z in $G^c - \{x\}$. Since we can delete at most $6n - 19$ edges to use the inductive hypothesis, the result holds.

In order to complete the proof, we will show that, either we can find two or three appropriate vertices to contract such that the obtained multigraph G'^c is connected or G^c has a proper Hamiltonian path.

Contraction of two vertices: Consider the above contraction of the vertices x, y to s and suppose by contradiction that G'^c is disconnected. It can be easily shown that G'^c has two components with one vertex, say z, and $n - 2$ vertices, respectively. Observe first that if $z = s$ then x and y are both monochromatic, a contradiction with the fact that y was chosen not monochromatic. Consequently $z \neq s$.

Suppose first that x is not monochromatic. In this case x has y as its unique neighbour. So, there are $3(n - 2)$ missing edges at x and $3(n - 3)$ missing edges at z since z is isolated in G'^c. This gives us a total of $6n - 15$ missing edges in G^c and this is greater than $|E(\overline{G^c})| = 5n - 9$ which is a contradiction.

Suppose next that x is monochromatic. In G^c there are at least $2(n - 1)$ missing edges at x since it is monochromatic and $3(n - 3)$ missing edges at z since z is isolated in G'^c. Further, there are two more missing edges between

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y and z since we have the choice of which colours to delete at y. This gives us a total of $5n - 9 = |E(\overline{G^c})|$ missing edges in G^c. Now z must be adjacent to x and y in colour b otherwise we obtain $5n - 8$ missing edges which is a contradiction. Therefore z is also monochromatic and $d(z) = 2$. We take then z and y for the contraction (instead of x, y) but in this case we delete just two edges at z which guarantees the connectivity of the contracted multigraph.

Contraction of three vertices: Suppose by contradiction that after the contraction of x, y, z to s, G''^c is disconnected. Then G''^c has exactly two components with one vertex, say u, and $n - 3$ vertices, respectively.

Suppose first that $u \neq s$. In G^c u must have at least two different neighbours in two different colours among the vertices x, y, z. Otherwise we would be in the case where either u is monochromatic or u has one unique neighbour. Let y' and z' be two neighbours of u among x, y, z such that $c(uy') \neq c(uz')$. Now we contract the vertices u, y', z' (instead of x, y, z). Observe that at u we delete at most six edges since u has only x, y, z as neighbours. In adittion the red edge uy, the blue edge uz and at least one green edge among uy, uz are missing. At y' and z' we delete $3n - 6$ edges as usual. With this contraction we delete at most $3n$ edges and therefore the contracted multigraph has at least $3\binom{n-3}{2} + n - 9$ edges which guarantees not only the inductive hypothesis but also the connectivity for $n \geq 10$.

Suppose next that $u = s$. Then there are no red edges between y and $G^c - \{x, y, z\}$ and no blue edges between z and $G^c - \{x, y, z\}$. Now, since we are not in the previous cases, y has at least two different neighbours y' and z' such that $c(yy') \neq c(yz')$. Then we contract the vertices y, y', z' (instead of x, y, z). In the contraction process we delete at most $2(n - 3)$ edges between y and $G^c - \{x, y, z\}$ (since there are no red edges), six between y and the vertices x, z, and $3n - 6$ at y' and z'. We obtain in total at most $5n - 6$ deleted edges. Now, this new contracted multigraph has $n - 2$ vertices and at least $3\binom{n-3}{2} - n - 3$ edges. Clearly, if the multigraph is connected we are done. Otherwise, as before, it has two components with one vertex and $n - 3$ vertices, respectively. We can suppose that the contracted vertex is the isolated one, otherwise we are done as above. Observe now that the component on $n - 3$ vertices has at least $3\binom{n-3}{2} - n - 3$, therefore it is almost rainbow complete. It is easy to prove by induction that it has a proper Hamiltonian cycle. Suppose now without losing generality that $c(yy') = b$ and $c(yz') = r$. Now, in the original multigraph if we cannot add y, y', z' to the proper cycle in order to obtain a proper Hamiltonian path (and also using the fact that the contracted multigraph is disconnected), we obtain that there are $n - 3$

red missing edges and $n - 3$ green missing ones at y', $n - 3$ blue and $n - 3$ green at z', and $n - 3$ red at y. We obtain a total of $5n - 15$ missing edges. If we have any of the edges r, b or g between y' and z', either y has no green edges at all to $G^c - \{y, y', z'\}$ leading us to a contradiction on the number of edges, or a proper Hamiltonian path can be found. So, these three edges are missing. Similar arguments can be used if we have the edge yy' or yz' in colour g. Therefore, two more missing edges. Now if we have the edges yy' in r and yz' in b, we can do the contraction using these colours instead of the originals. Then, either the contracted multigraph is connected and thus we obtain a proper Hamiltonian path, or we obtain a contradiction on the number of edges. We can conclude that at least one between these two edges is missing obtaining a total of $5n - 9 = |E(\overline{G^c})|$. That implies that $G^c - \{y, y', z'\}$ is rainbow complete and we have all of the green and blue edges between y and $G^c - \{y, y', z'\}$, all of the blue between y' and $G^c - \{y, y', z'\}$, and all of the red between z' and $G^c - \{y, y', z'\}$. In this last case, it is easy to obtain a proper Hamiltonian path in G^c. □

In view of Theorem 4.8 we prove the following lemma.

Lemma 4.7 *Let G^c be a c-edge-coloured multigraph on n vertices fullfilling the conditions of Theorem 4.8. Then either G^c has a proper Hamiltonian path or there exists a vertex $x \in G^c$ such that $d(x) \le 2n - 6$.*

Proof. Let E^i be the set of edges of colour i, $i \in \{r, g, b\}$, and suppose without loss of generality that $|E^b| \ge |E^r|, |E^g|$. Then, as the subgraph G^b has minimum degree one and $|E^b| \ge \binom{n-2}{2} + 3$, it can be easily checked that it is connected. Thus by Lemma 2.1 there is a matching M^b such that $|M^b| = \frac{n}{2}$ for n even and $|M^b| = \frac{n-1}{2}$ for n odd. Let $P = x_1 y_1 x_2 y_2 \ldots x_p y_p$ be the longest proper path compatible with M^b.

Suppose first that n is odd. By Lemma 2.5, if there is a proper cycle C such that $V(C) = V(P)$, then $|P| \ge n - 5$. Else, if such a cycle does not exist then $|P| \ge n - 7$. Otherwise in both cases we obtain a contradiction on the number of edges. Let us consider here the case $|P| = n - 1$ (the other cases are easier to handle, refer to [8] for more details). Now observe that if there is a proper cycle C such that $V(C) = V(P)$, then the result easily follows as the unique vertex of $G^c - C$ can be appropriately joint to C in order to obtain a proper Hamiltonian path. Assume therefore that there is no proper cycle C such that $V(C) = V(P)$. Let x be the unique vertex of $G^c - P$. Clearly we cannot have either the edge xx_1 on colours r or g, or the edge xy_p on colours r or g, otherwise we easily obtain a proper Hamiltonian path in G^c. Now, if there are at least three edges on colours r, g between x and some

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pair of vertices $\{y_i, x_{i+1}\}$, $i = 2, \ldots, p-1$, then by choosing the appropriate edges xy_i and xx_{i+1}, the path $x_1 \ldots y_i xx_{i+1} \ldots y_p$ is a proper Hamiltonian one in G^c. Otherwise there are at most two edges on colours r, g between x and every pair of vertices $\{y_i, x_{i+1}\}$, for $i = 2, \ldots, p-1$. Therefore $d^{r,g}(x) \leq n - 3$ and clearly $d(x) \leq 2n - 4$ as $d^b(x) \leq n - 1$. In addition, if we have two more missing edges incident to x we would obtain that $d(x) \leq 2n - 6$ as claimed. Now, we can assume the worst case, that is, for each edge $y_i x_{i+1}$ in the path, $i = 2, \ldots, p-1$, we have both edges xy_i, xx_{i+1} on the same colour of $y_i x_{i+1}$ (that is, r or g). Otherwise, if we suppose without losing generality that $c(xy_i) = r$ and $c(y_i x_{i+1}) = g$ then we cannot have the blue edge xx_i (or we would obtain the proper Hamiltonian path $x_1 \ldots x_i xy_i \ldots y_p$). Therefore, there would be one more missing edge at x. Consider now x_1. Suppose that we have any edge $x_1 y_i$ on colour r or g that is different of the colour of $y_i x_{i+1}$, for $i = 1, \ldots, p-1$. Taking the blue edge xx_i we obtain the proper Hamiltonian path in G^c, $xx_i \ldots x_1 y_i x_{i+1} \ldots y_p$. Otherwise we obtain at least $p - 1 = \frac{n-3}{2}$ missing edges $x_1 y_i$ on colours r or g. Suppose that we have any edge $x_1 x_i$ on at least one colour r or g, for $i = 2, \ldots, p$. Therefore taking the edge xy_{i-1} on colour r or g (one of both is supposed to exist) we obtain the proper Hamiltonian path $xy_{i-1} \ldots x_1 x_i y_i \ldots y_p$. Otherwise the edges $x_1 x_i$ on colours r and g are missing for all $i = 2, \ldots, p$, that is, $2(p-1) = n - 3$ additional missing edges at x_1. Finally, summing up and considering that we cannot have the edge $x_1 y_p$ on colours r or g (or P would also be a proper cycle), we obtain that $d(x) \leq 3(n-1) - \frac{n-3}{2} - (n-3) - 2 \leq 2n - 6$ as claimed.

Suppose next that n is even. If there is a proper cycle C such that $V(C) = V(P)$, then by Lemma 2.5 $|P| \geq n - 2$. This case is easy since either P is a proper Hamiltonian path or we can connect the unique edge of $M^b - E(P)$ to C in order to obtain a proper Hamiltonian path. Assume therefore that there is no proper cycle C such that $V(C) = V(P)$. It follows by Lemma 2.5 that $|P| \geq n - 4$ otherwise we obtain a contradiction on the number of edges. Let us consider just the case $|P| = n - 2$ ($|P| = n - 4$ is easier, refer to [8] for full details). Let $e = xy$ be the edge of $M^b - E(P)$. Now by similar arguments as in the odd case above, we can prove that, either the edge e can be added to P in order to obtain a proper Hamiltonian path in G^c, or one of the vertices x, y, x_1, y_p has degree at most $2n - 6$ as required. $\qquad\square$

Theorem 4.8 *Let G^c be a c-edge-coloured multigraph on n vertices, $n \geq 11$ and $c \geq 3$. If $rd(G^c) = c$ and $m \geq c\binom{n-2}{2} + 2c + 1$, then G^c has a proper Hamiltonian path.*

For the extremal example, $n \geq 11$, consider a rainbow complete multi-

graph, say A, on $n - 2$ vertices. Add two new vertices v_1, v_2 and join them to a vertex v of A with all possible colours. The resulting c-edge-coloured multigraph has $c\binom{n-2}{2} + 2c$ edges and clearly has no proper Hamiltonian path.

Proof. By Lemma 4.1 it is enough to prove the theorem for $c = 3$. Let $\{r, b, g\}$ be the set of colours. As $m \geq 3\binom{n-2}{2} + 7$ then $|E(\overline{G^c})| \leq 6n - 16$. The proof will be done either by construction of a proper Hamiltonian path or using Theorem 4.6. We will do this by contracting two or three vertices depending on if there exists a vertex x in G^c such that $|N(x)| = 1$ or not.

If there exists a vertex $x \in G^c$ such that $|N(x)| = 1$ we contract x and its unique neighbour y to a new vertex s as in Definition 4.2. By this we delete at most $2n - 1$ edges and the resulting multigraph is still connected. Thus the conclusion follows from Theorem 4.6.

Suppose next that there is no vertex $x \in G^c$ such that $|N(x)| = 1$. It follows that for any vertex x there are two distinct neighbours y, z in G^c such that $c(xy) = b$ and $c(xz) = r$. Now by Lemma 4.7 consider a vertex x such that $d(x) \leq 2n - 6$. Then contract x, y, z to a new vertex s as in Definition 4.3. Let G''^c be the resulting multigraph. In this case, as we delete at most $5n - 12$ ($= 2n - 6 + 3(n - 3) + 3$) edges, it is enough to prove that G''^c is connected to apply Theorem 4.6.

Suppose therefore by contradiction that G''^c is disconnected. Then it has exactly two components with one vertex, say u, and $n - 3$ vertices, respectively, otherwise we arrive to a contradiction on the number of edges.

Assume first that $u \neq s$. Then, as in the equivalent case of Theorem 4.6, instead of x, y, z, we may find three other vertices u, y', z' to contract to a vertex s' just deleting $3n$ edges. This new obtained multigraph has at least $3\binom{n-3}{2} - 2$ edges. Then, if it is connected we are done, otherwise there is a component with one vertex, say u', and another one on $n - 3$ vertices with at least $3\binom{n-3}{2} - 2$ edges, i.e., almost rainbow complete. Therefore, the biggest component contains a proper Hamiltonian cycle and then we can easily add either the isolated vertex u' (if $u' \neq s'$) or the three u, y', z' (if $u' = s'$) vertices to the cycle to obtain a proper Hamiltonian path in G^c.

Assume next $u = s$. If $d(x) \leq n + 1$, then the contraction process deletes $4n - 5$ edges instead of $5n - 12$. Furthermore as G''^c is disconnected by hypothesis, its component on $n - 3$ vertices has at least $3\binom{n-3}{2} - n + 3$ edges. As in Theorem 4.6, this component is almost rainbow complete and thus it contains a proper Hamiltonian cycle C. This allows us to easily add x, y, z to C in order to obtain a proper Hamiltonian path in G^c. In the sequel, we may suppose that $d(x) \geq n + 2$. Then x has two different neighbours y' and z' with

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parallel edges. Consider the next two cases:

Assume first that the parallel edges are on the same two colours, say $c(xy') = c(xz') = \{b, r\}$ (cases with other two colours are similar). Here we can consider two possible contractions: 1) x, y', z' with $c(xy') = b$, $c(xz') = r$ and 2) x, y', z' with $c(xy') = r$, $c(xz') = b$. Now, suppose that in both contractions the multigraph is disconnected and the contracted vertex is always the isolated one, otherwise we are finished. We can observe that G^c has $n+3$ missing edges incident to x (since $d(x) \leq 2n - 6$), $n - 3$ green edges and $4(n - 3)$ blue and red edges incident to y' and z' (since in both contractions the multigraph is disconnected). By this we obtain a total of $6n - 12 > 6n - 16$ missing edges, which is a contradiction.

Assume next that the parallel edges are not on the same two colours, that is, $c(xy') = \{b, r\}$ and $c(xz') = \{b, g\}$ (cases with other combinations are similar). Now since we are not in the previous case, we do not have either the green edge xy' or the red one xz'. Try any of the three possible contractions: 1) x, y', z' with $c(xy') = b$, $c(xz') = g$, 2) x, y', z' with $c(xy') = r$, $c(xz') = g$ and 3) x, y', z' with $c(xy') = r$, $c(xz') = b$. Then, after each of these contractions the multigraph is still disconnected and the contracted vertex is always the isolated one. We can observe that there can exist just the red edges between y' and $G^c - \{x, y', z'\}$ and the green edges between z' and $G^c - \{x, y', z'\}$. Now as $rd(G^c) = 3$ there must exist the green edge $y'z'$ and the red edge $y'z'$. Since we are not in the previous case, the blue edge $y'z'$ is not present. We find us in the situation that $c(xy') = \{b, r\}$, $c(xz') = \{b, g\}$ and $c(y'z') = \{r, g\}$. Now, we have nine different contractions to try, three for each triplet x, y', z', y', x, z' and z', x, y'. If in all of them we are in this same situation (the contracted multigraph is disconnected and the isolated vertex is the contracted one) we can conclude that in G^c there can exist just the blue edges between x and $G^c - \{x, y', z'\}$, the red edges between y' and $G^c - \{x, y', z'\}$, and the green edges between z' and $G^c - \{x, y', z'\}$. This gives a total of $6(n - 3)$ missing edges in G^c. Finally, adding the three missing edges xy' in green, xz' in red and $y'z'$ in blue, we obtain $6n - 15$ missing edges which is a contradiction. $\square$

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