NEW BOUNDS FOR FACIAL NONREPETITIVE COLOURING*

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ABSTRACT. We prove that the facial nonrepetitive chromatic number of any outerplanar graph is at most 11 and of any planar graph is at most 22.

1 Introduction

A sequence $S = s_1, s_2, \dots, s_{2r}, r \ge 1$, is a *repetition* if $s_i = s_{r+i}$ for each $i \in \{1, \dots, r\}$. For example, 1212 is a repetition, while 1213 is not. A *block* of a sequence *S* is any subsequence of consecutive terms in *S*. A sequence is *nonrepetitive* if for every non-empty block *B* of *S*, *B* is not a repetition. Otherwise *S* is *repetitive*. For example, 1312124 is repetitive as it contains the block 1212 which is a repetition, while 123213 is nonrepetitive as it contains no such block. No sequence of length greater than three using only two symbols can be nonrepetitive. A result by Axel Thue in 1906 states that nonrepetitive sequences of infinite length can be created using three symbols [25]. Thue's work is considered to have initiated the study of the combinatorics of words [1].

A graph colouring variation on this theme was proposed by Alon et al. [2]. A nonrepetitive (vertex) colouring of a graph *G* is an assignment of colours to the vertices of *G* such that, for every path *P* in *G*, the sequence of colours of vertices in *P* is not a repetition. The *nonrepetitive chromatic number* of *G*, denoted $\pi(G)$, is the minimum number of colours required to nonrepetitively colour *G*. In this setting, Thue's result states that, for all $n \ge 1$, the path P_n on vertices has $\pi(P_n) \le 3$. Since its introduction, nonrepetitive graph colouring has received much attention [3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24].

A well-known conjecture, due to Alon et al. [2], is that there exists a constant K such that, for every planar graph G, $\pi(G) \leq K$. The current best upper bound for *n*-vertex planar graphs is $O(\log n)$ [9]. No planar graph with nonrepetitive chromatic number greater than 11 is known (see Appendix A in [9]).

More is known about the *facial* version of the problem for embedded planar graphs. Harant and Jendrol [17] asked if every plane graph can be coloured with a constant number of colours such that every *facial* path¹ is nonrepetitively coloured. Barát and Czap [3]

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¹A facial path is a path that is a contiguous subsequence of a facial walk; see Section 2 for a more rigorous definition.

answered this question in the affirmative by showing that 24 colours are sufficient. We reduce this bound to 22 by proving a bound of 11 for facial nonrepetitive colouring of outerplane graphs.

1.1 Related Work

Nonrepetitive Colouring. It is known that some families of graphs have bounded nonrepetitive chromatic number. In their original work, Alon et al. [2] showed that $\pi(G) = O(\Delta^2)$ if G has maximum degree Δ and that there are graphs of maximum degree Δ with nonrepetitive chromatic number $\Omega(\Delta^2/\log \Delta)$. The constants in the $O(\Delta^2)$ upper bound have been steadily improved [10, 14, 15, 17].

Barát and Varjú [4] and Kündgen and Pelsmajer [20] independently showed that $\pi(G) \leq 12$ if *G* is outerplanar and, more generally, $\pi(G) \leq c^t$ if *G* has treewidth at most *t*. (Barát and Varjú proved the latter bound with c = 6 while Kündgen and Pelsmajer proved it with c = 4.) The bound of 4^t for *t*-trees is tight if t = 1 (trees), but it is not known if it is tight for other values of *t*. Even the upper bound of 12 for outerplanar graphs may not be tight, as no outerplanar graph with nonrepetitive chromatic number greater than 7 is known [4].

Facial Nonrepetitive Colouring. Facial nonrepetitive colouring was first considered by Havet et al [18], who studied the edge-colouring variant of the problem. In this setting, they were able to show that the edges of any plane graph can be 8-coloured so that every facial trail² is coloured nonrepetitively. For the list-colouring version of this problem, Przybyło [22] showed that lists with at least 12 colours are sufficient to colour the edges of any plane graph so that every facial trail is coloured nonrepetitively.

For the vertex-colouring version we study, Harant and Jendrol [17] proved that $\pi_f(G) = O(\log \Delta)$ if G is a plane graph of maximum degree Δ and that $\pi_f(G) \leq 16$ if G is a Hamiltonian plane graph. They also conjectured that $\pi_f(G) = O(1)$ when G is any plane graph. As mentioned above, this latter conjecture was confirmed by Barát and Czap [3], who showed that, for any plane graph G, $\pi_f(G) \leq 24$. The results of Barát and Czap [3] also extend to graphs embedded in surfaces. They show that a graph embedded on a surface of genus g can be facially nonrepetitively 24^{2g} -coloured. The best lower bounds for facial nonrepetitive chromatic numbers are 5 for plane graphs and 4 for outerplane graphs [3].

2 Preliminary Results and Definitions

We assume the reader is familiar with standard graph theory terminology as used by, e.g., Bondy and Murty [6]. All graphs we consider are undirected, but not necessarily simple; they may contain loops and parallel edges. For a graph, *G*, we use the notations V(G)and E(G) to denote *G*'s vertex and edge sets, respectively. For $S \subset V(G)$, *G*[*S*] denotes the subgraph of *G* induced by the vertices in *S* and $G - S = G[V(G) \setminus S]$.

A graph is *k*-connected if it contains more than k vertices and has no vertex cut of size less than k. A *k*-connected component of a graph G is a maximal subset of vertices of G that induces a *k*-connected subgraph. A *bridge* in a graph G is an edge whose removal

²A *facial trail* is a contiguous subsequence of the edges traversed during the boundary walk of a face.

increases the number of connected components. A graph is *bridgeless* if it has no bridges.

A plane graph G is a fixed embedding of a graph in the plane such that its edges intersect only at their common endpoints. An *outerplane* graph G is a plane graph such that all the vertices of G are incident on the outer face of G. A *chord* in an outerplane graph is an edge that is not incident to the outer face. A *cactus graph* is an outerplane graph with no chords. An *ear* in an simple outerplane graph is an inner face that is incident to exactly one chord. An *ear* is *triangular* if it has exactly three vertices. The *weak dual* of a outerplane graph G is a forest whose vertices are the inner faces of G and that contains the edge fg if the face f and the face g have a chord in common. Note that the ears of G are leaves in the weak dual and that, if G is biconnected, then its weak dual is a tree.

A walk in a graph G is a sequence of vertices $v_0, ..., v_{\ell-1}$ such that, for every $i \in \{0, ..., \ell-2\}$ the edge $v_i v_{i+1}$ is in E(G). The walk is *closed* if $v_{\ell-1}v_0$ is also in E(G). A walk is a *path* if all its vertices are distinct. A *facial walk* in a plane graph G is a closed walk $v_0, ..., v_{\ell-1}$ such that, for every $i \in \{0, ..., \ell-1\}$, the edges $v_{(i-1) \mod \ell}v_i$ and $v_i v_{(i+1) \mod \ell}$ occur consecutively in the counterclockwise cyclic ordering of the edges incident to v_i in the embedding of G. A *facial path* is a contiguous subsequence of a facial walk that is also a path in G. A facial path is an *outer-facial path* if it appears in a facial walk of the outer face of G and it is an *inner-facial* path if it appears in a facial walk of some inner face of G.

Before proceeding with our results, we introduce a helper lemma due to Havet et al. [18] and two theorems that will be used throughout the paper. The helper lemma provides a way to interlace nonrepetitive sequences.

Lemma 1 (Havet et al. [18]). Let $B = B_1, B_2, ..., B_k$ be a nonrepetitive sequence over an alphabet \mathcal{B} in which each B_i has size at least 1. For each $i \in \{0, ..., k\}$, let A_i be a (possibly empty) nonrepetitive sequence over an alphabet \mathcal{A} with $\mathcal{B} \cap \mathcal{A} = \emptyset$. Then $S = A_0, B_1, A_1, ..., B_k, A_k$ is a nonrepetitive sequence.

We will require two results about the nonrepetitive chromatic number of trees and cycles:

Theorem 1 (Alon et al. [2]). *For every tree,* T*,* $\pi(T) \leq 4$.

Theorem 2 (Currie [8]). For every n > 2, the cycle C_n on n vertices has

$$\pi(C_n) = \begin{cases} 4 & if \ n \in \{5, 7, 9, 10, 14, 17\} \\ 3 & otherwise. \end{cases}$$

3 Outerplane Graphs

We begin with a simple lemma that allows us to focus, when convenient, on simple outerplane graphs.

Lemma 2. Let G be a simple outerplane graph and let G' be an outerplane graph obtained by adding parallel edge and/or loops to G. Then, any facially nonrepetitive colouring of G is also a facially nonrepetitive colouring of G', so $\pi_f(G') \leq \pi_f(G)$.



Figure 1: The blocking set, *B* (black vertices), of an outerplane graph, *G*, (with the edges of G - B shown in bold).

Proof. We argue that any facial path (described as a sequence of vertices) in G' is also a facial path in G. Therefore, by facially nonrepetitively colouring G, we obtain a facial nonrepetitive colouring of G'.

First, note that no facial path uses a loop, so the addition of loops does not introduce new facial paths in G'. When a (non-loop) edge e' is added parallel to an existing edge e of G, the union of the embeddings of e and e' form a Jordan curve that does not contain any vertices of G (since we require G' to be outerplane). This implies that any facial path in G' that uses the new edge e' exists in G as a facial path that uses the edge e.

Next, we introduce a definition that is crucial to the rest of the paper. Let *G* be an outerplane graph. A *blocking set* of *G* is a set of vertices $B \subseteq V(G)$ such that for each 2-connected component *H* of *G*, H - B is a tree and for each inner face *F*, $V(F) \setminus B \neq \emptyset$. See Figure 1 for an example of a blocking set.

The definition of a blocking set is subtle and implies some properties that we will use throughout.

Observation 1. For any blocking set B of G, B does not include both endpoints of any chord c of G.

Observation 2. For any blocking set B of G and any inner face F of G, the vertices of $V(F) \cap B$ occur consecutively on the boundary of F. In other words, F - B is a non-empty path.

Observations 1 and 2 are true because, otherwise, H - B would be disconnected for the 2-connected component containing *c* or *F*, respectively.

Lemma 3. For every biconnected outerplane graph, G, and any vertex $v \in V(G)$, there exists a blocking set B of G such that $v \in B$ and, for each inner face F of G, $|B \cap V(F)| = 1$.

Proof. The proof is by induction on the number of inner faces. If *G* has only one inner face, we take $B = \{v\}$ and we are done. Otherwise, select some ear, *F* of *G* whose chord is uw and such that $v \notin V(F) \setminus \{u, w\}$. Such an ear *F* always exists because *G* has at least two ears. Let $G' = G - (V(F) \setminus \{u, w\})$. The graph *G'* has one less inner face than *G* so, by induction, it has a blocking set *B'* that satisfies the conditions of the lemma. There are two cases to consider:

- 1. If one of *u* or *w* is in *B*' then we take B = B' to obtain a blocking set that satisifes the conditions of the lemma.
- 2. Otherwise, let *x* be any vertex in $V(F) \setminus \{u, w\}$ and take $B = B' \cup \{x\}$ to obtain a blocking set that satisifes the conditions of the lemma.

Lemma 3 allows us to prescribe that a particular vertex v be included in the blocking set, but it will also be convenient to exclude a particular vertex v by using Lemma 3 to force the inclusion of v's neighbour on the outer face (which is also on some inner face with v).

Corollary 1. For every biconnected outerplane graph, G, and any vertex $v \in V(G)$, there exists a blocking set B of G such that $v \notin B$ and, for each inner face F of G, $|B \cap V(F)| = 1$.

At this point we pause to sketch how Lemma 3 can already be used to give an upper-bound of 8 on the facial nonrepetitive chromatic number of biconnected outerplane graphs. For a biconnected outerplane graph, *G*, we take a blocking set *B* of *G* using Lemma 3. By Theorem 1, we can nonrepetitively 4-colour the tree T = G - B using the colours $\{1, 2, 3, 4\}$, so what remains is to assign colours to the vertices in *B*. To do this, we use Theorem 2 to nonrepetitively 4-colour the cycle, *C*, that contains the vertices of *B* in the order they appear on the outer face of *G* using the colours $\{5, 6, 7, 8\}$. We claim that the resulting 8-colouring of *G* is facially nonrepetitive. No facial path on an inner face is coloured repetitively since each such facial path is also either present in the tree *T* or it contains exactly one vertex of *B*. No facial path on the outer face is coloured repetitively since it is obtained by interleaving a nonrepetitive sequence of colours in *C* with nonrepetitive sequences taken from *T*; by Lemma 1, a sequence obtained in this way is nonrepetitive.

In Appendix A, we show that the preceding argument can be improved to give a bound of 7 on the facial nonrepetitive chromatic number of biconnected outerplane graphs. This is just a matter of adding vertices to the blocking set so that the cycle C does not have length in {5,7,9,10,14,17}, so that it can be nonrepetitively 3-coloured.

Finally, we remind the reader that, although Lemma 3 and Corollary 1 provide blocking sets that include only one vertex on each inner face, not all blocking sets have this property. It is helpful to keep this in mind in the next section.

3.1 The Blocking Graph

The *blocking graph* of *G* for a blocking set *B* is the graph, denoted by $block_B(G)$, whose vertex set is *B* and whose edges are defined as follows: Begin with the (closed) facial walk *W* on the outer face of *G*. Remove every vertex not in *B* from *W* to obtain a cyclic sequence *W'* of vertices in *B*. For each consecutive pair of vertices *uw* in *W'* we add an edge *uw* to $block_B(G)$. This naturally defines the embedding of $block_B(G)$. See Figure 2 for an example. Note that $block_B(G)$ is a plane graph that is not necessarily simple; it may contain parallel edges (cycles of length two) and self-loops (cycles of length one).

The fact that a blocking set does not contain both endpoints of any chord of G (Observation 1) implies the following observation:



Figure 2: The blocking graph (curved edges in red) associated with a blocking set.

Observation 3. For every outerplane graph G and any blocking set B of G, the blocking graph $block_B(G)$ is a bridgeless cactus graph.

Observation 4. For every outerplane graph G, any blocking set B of G, and any facial path P on the outer face of G, the subsequence of P containing only the vertices of B is a (outer) facial path in block_B(G).

Observation 5. For every outerplane graph G, any blocking set B of G and any inner face F of G, $G[V(F) \cap B]$ is a non-empty path that is a facial path (on some inner face) in block_B(G).

In the previous section, we sketched a proof of an upper bound of 8 on the facial chromatic number of biconnected outerplane graphs. This proof works by nonrepetitively 4-colouring the tree, G - B, obtained after removing the blocking set and then nonrepetitively 4-colouring a cycle, C, of vertices in the blocking set. This cycle, C, is actually the blocking graph, $block_B(G)$. The following lemma shows that this strategy generalizes to the situation where we can find a facial nonrepetitive colouring of $block_B(G)$ with few colours.

Lemma 4. Let G be an outerplane graph and B be a blocking set of G. If there exists a facial nonrepetitive k-colouring of (the outer face of) $\operatorname{block}_B(G)$, then there exists a facial nonrepetitive (4 + k)-colouring of G.

Proof. By Theorem 1 we can colour G - B nonrepetitively using colours {1, 2, 3, 4} and, by assumption, we can facially nonrepetitively colour block_{*B*}(*G*) with colours {5,...,*k* + 4}. These two colourings define a colouring of *G* that we now show is facially nonrepetitive.

Let *P* be a facial path in *G*. If *P* is a facial path of $block_B(G)$ or a path in G - B then there is nothing to prove.

Otherwise consider first the case that *P* is a path on an inner face *F* of *G*. There are two cases to consider:

1. The colouring of *P* is of the form A_0, B_1, A_1 , where A_0 and A_1 are obtained from (possibly-empty) paths in G - B and B_1 is obtained from a non-empty outer-facial path in block_{*B*}(*G*) (by Observation 5). Lemma 1 therefore implies that the colour sequence A_0, B_1, A_1 is nonrepetitive.

2. The colouring of *P* is of the form B_0, A_1, B_1 , where A_1 is obtained from a non-empty path in *G* – *B* and B_0 and B_1 are (possibly empty) outer-facial paths in block_{*B*}(*G*) (by Observation 5). Again, Lemma 1 implies that the resulting colour sequence is nonrepetitive.

Finally, consider the case where *P* is a facial path on the outer face of *G*. In this case, the colour sequence obtained from *P* is of the form $A_0, B_1, A_1, \ldots, B_k, A_k$ where each A_i is obtained from a (possibly empty) path in *G* – *B* and B_1, \ldots, B_k is obtained from a (outer) facial path in block_{*B*}(*G*) (by Observation 4). Again, Lemma 1 implies that the resulting colour sequence is nonrepetitive.

3.2 Colouring Even Cactus Graphs

We now show how to colour the blocking graph—a cactus graph—of an outerplane graph. By Lemma 4, if we can find a facial nonrepetitive k-colouring of any cactus graph, we can get a facial nonrepetitive k + 4-colouring of any outerplane graph.

Recall that the best known upper bound for the facial Thue chromatic number of outerplane graphs is 12, which is the bound for the Thue chromatic number [4, 20]. Thus, to improve this bound, we need to find a facial nonrepetitive 7-colouring of the blocking graph. We have been unable to do this unless all cycles of the blocking graph are even. We will eventually address this limitation in Section 3.3 by proving the existence of a blocking set *B* such that $block_B(G)$ has no odd cycles.

For any graph, *G*, a *levelling* of *G* is a function $\lambda: V(G) \rightarrow \{0, 1, 2, ...\}$ such that for each $uv \in E(G)$, $|\lambda(u) - \lambda(v)| \leq 1$. The level pattern of a path $v_1, ..., v_k$ is the sequence $\lambda(v_1), \lambda(v_2), ..., \lambda(v_k)$.

Lemma 5 (Kündgen and Pelsmajer [20]). Let *G* be a graph and $\lambda: V(G) \rightarrow \{0, 1, 2, ...\}$ be a levelling of *G*. Let $S = s_0, s_1, ..., s_m$ be a nonrepetitive palindrome-free sequence on an alphabet \mathcal{A} with $m = \max\{\lambda(v) \mid v \in V(G)\}$ and $c: V(G) \rightarrow \mathcal{A}$ be a colouring of *G* defined as $c(v) = s_{\lambda(v)}$. If a path $P = P_1, P_2$ with $|P_1| = |P_2|$ in *G* is repetitively coloured under *c*, then P_1 and P_2 have the same level pattern.

Lemma 6. For every cactus graph, G, with no odd cycles (and therefore, for every blocking graph with no odd cycles), $\pi_f(G) \leq 7$.

Proof. By Lemma 2, we may assume that *G* is simple. We may also assume that *G* is connected as this does not affect its nonrepetitive chromatic number. Also, assume that *G* is neither a cycle nor a tree since $\pi_f(G) \le 4 < 7$ for both these classes of graphs. If there exists a vertex *v* of *G* such that $\deg_G(v) = 1$, then let the *root r* of *G* be *v*. Otherwise, let *r* be any vertex of *G* of degree at least 3. Let λ be a levelling of *G* where $\lambda(v)$ is the distance in *G* from *r* to *v*. Let *H* be a graph that contains all vertices $v \in V(G)$ such that

- 1. v is on a cycle C of G,
- 2. $\lambda(v) = \max_{u \in C} \lambda(u)$ and
- 3. $\deg_G(v) = 2$.



Figure 3: The graph H (black vertices and grey vertex) obtained from a cactus graph with no odd cycles. The vertex a is added in the final step so that H is not a cycle of length 5.

In other words, H contains the vertices of degree 2 that are on the deepest level of a cycle (see Figure 3). Notice that since every cycle of G is even, there is at most one vertex of H in each cycle of G.

If $\deg_G(r) \neq 1$, there must exist at least one face F^* of G such that exactly one vertex v of F^* has degree greater than two. From our choice of r, it follows that $\lambda(v)$ is the minimum over all vertices in $V(F^*)$. Since $|V(F^*)| \geq 4$, F^* has three consecutive degree-2 vertices a, b, and c, such that $b \in V(H)$. If $|V(H)| \in \{5,7,10,14,17\}$, we add a to V(H). If |V(H)| = 9, we add both a and c to V(H). Notice that now, either $\deg_G(r) = 1$ or $|V(H)| \notin \{5,7,9,10,14,17\}$.

We now define the edge set E(H) of H. For each $u, v \in V(H)$, we add the edge uv to E(H) if there is a facial path on the outer face of G with endpoints u and v that does not contain any other vertices in V(H). Note that H is either a cycle or a forest of paths. It can only be a cycle if G has no vertices of degree 1, in which case, $\deg_G(r) \neq 1$. In this case, our choice of V(H) ensures that the length of this cycle is not in $\{5, 7, 9, 10, 14, 17\}$. This implies that H can be nonrepetitively coloured using the colour set $\mathcal{B} = \{1, 2, 3\}$, either by using the result of Thue [25] or Currie (Theorem 2).

To colour the remaining vertices of *G*, let $h = \max_{v \in V(G)} \lambda(v)$ and $S = s_0, s_1, ..., s_h$ be a palindrome-free nonrepetitive sequence on $\mathcal{A} = \{4, 5, 6, 7\}$. (A nonrepetitive palindromefree sequence can be constructed from any ternary nonrepetitive sequence by adding a fourth symbol between blocks of size 2 [7].) Then, each vertex $v \in V(G) \setminus V(H)$ is assigned the colour $s_{\lambda(v)}$.

We will now show that the resulting 7-colouring of *G* is a facial nonrepetitive colouring. Suppose that this is not the case. Thus, there exists a path $P = P_1, P_2$ such that the colour sequence *S* corresponding to vertices of *P* is a repetition. Let us first suppose that *P* is on the outer face of *G*. We will need the following claim:

Claim 1. Let P be a path on the outer face of G such that $V(P) \cap V(H) = \emptyset$. The level sequence L corresponding to vertices of P must be strictly decreasing, strictly increasing, or strictly decreasing then strictly increasing.

Proof of Claim 1. Suppose that this is not the case. Then *L* cannot contain two consecutive elements of the form *i*, *i* as this can only correspond to an odd cycle of *G*, but all cycles of *G* are even. Thus, *L* must contain a block of the form *i*, *i* + 1, *i*. Since *P* is on the outer face, we must have that the vertex *v* corresponding to *i* + 1 is the highest numbered vertex on some cycle *C* and that deg_{*G*}(*v*) = 2. But in this case, *v* must be in *H*, which is a contradiction. \Box

By Lemma 5, P_1 and P_2 have the same level pattern. However, if $V(P) \cap V(H) = \emptyset$ this is incompatible with Claim 1. Thus, P must contain vertices of H. Let $P_H = p_1, p_2, ..., p_k$ be the sequence of vertices of $V(P) \cap V(H)$ in the same order as in P. Notice that P_H is a path in H. Therefore, the colour sequence corresponding to P_H is nonrepetitive. Now, observe that the colour sequence formed by P is of the form $A_0, B_1, A_1, B_\ell, A_\ell$ where $B_1, ..., B_\ell$ is a nonrepetitive sequence of colours from $\mathcal{B} = \{1, 2, 3\}$ and each A_i is a non-repetitive sequence of colours from $\mathcal{A} = \{4, 5, 6, 7\}$. Therefore, by Lemma 1, P is coloured nonrepetitively.

Thus, *P* must be a facial path on some inner face *F* of *G*. If $V(P) \cap V(H) = \emptyset$ then, by Lemma 5, P_1 and P_2 have the same level pattern. No path on an even cycle has such a pattern using the levelling λ . Therefore, $V(P) \cap V(H) \neq \emptyset$, so *P* contains 1, 2, or 3 vertices of *H*. If *P* is a repetition it must contain exactly 2 vertices of *H*, thus *P* is a facial path on *F*^{*} (since every other inner face contributes at most one vertex to V(H)). Reusing the notation above, *P* cannot contain *b* since *b* has a unique colour in $V(F^*)$. Therefore, *P* must contain *a* and *c* and, in fact, these are the endpoints of *P*. The colour sequence of *P* must therefore be of the form *xAx* where $x \in \{1, 2, 3\}$ and *A* is a non-empty sequence over the alphabet $\{5, 6, 7, 8\}$. Such a sequence is not a repetition.

3.3 Making an Even Blocking Graph

Lemma 7. For every biconnected outerplane graph, G, and any vertex $v \in V(G)$:

- *G* has a blocking set *B* such that $block_B(G)$ is an even cycle and $v \in B$; and
- *G* has a blocking set *B* such that $block_B(G)$ is an even cycle and $v \notin B$.

Proof. We first obtain a blocking set B' that contains or does not contain v, as appropriate, by applying Lemma 3 or Corollary 1. Recall that B' contains exactly one vertex on each inner face of G. It is simple to verify that $block_{B'}(G)$ is a cycle; if it is an even cycle, then we are done, so we may assume that $block_{B'}(G)$ is an odd cycle.

If *G* has only one inner face, then *B*' contains one vertex, *u*, on this face and $block_{B'}(G)$ is an odd cycle (of length 1). In this case, we can select the neighbour, *w*, of *u* such that $w \neq v$ and let $B = B' \cup \{w\}$. It is easy to verify that *B* is a blocking set, and either includes or excludes *v*, as appropriate, and that $block_B(G)$ is a cycle of length 2.

Thus, we may assume that *G* has at least two inner faces and we consider several cases:

1. If *G* contains an ear, *F*, with four or more vertices such that either $v \notin V(F)$ or *v* is one of the endpoints of the chord of *F*. There is exactly one vertex $x \in B'$ on the face *F*. Let *y* be a neighbour of *x* on *F* such that *y* is not on the chord of *F* (so *y* has degree



Figure 4: The face F''_{uw} in the proof of Lemma 7.

2). Such a y exists since F has at least four vertices. Set $B = B' \cup \{y\}$. Now |B| is even, so block_{*B*}(*G*) is an even cycle. Furthermore, G - B' is a tree and y is a leaf in this tree, so G - B is also a tree. Finally, by choice, B contains v if and only if B' contains v, so B satisifies the conditions of the lemma.

- 2. Next, consider the case where *G* contains a triangular ear, *F*, such that one of the endpoints of the chord of *F* is in *B'* and *v* is not the degree 2 vertex, *y*, of *F*. By the same argument as above, $B = B' \cup \{y\}$ satisfies the conditions of the lemma.
- 3. Refer to Figure 4. For an edge $uw \in E(G)$, let V'_{uw} and V''_{uw} be the two (possibly empty) sets of vertices in the (at most) two connected components of $G \{u, w\}$. Let $G'_{uw} = G[V'_{uw} \cup \{u, w\}]$ and $G''_{uw} = G[V''_{uw} \cup \{u, w\}]$. If neither of the two previous cases applies, then there exists an edge uw of G such that $v \notin V'_{uw}$ and the weak dual of G'_{uw} is a star whose central vertex is the face, F_{uw} incident on uw.

We now argue why such an edge uw exists. Recall that the weak dual, G° , of G is a tree whose vertices are the faces of G. Select some face, R, of G that has v on its boundary and root G° at R. This tree has a height, h, and some vertex F of depth h-1(recall that G has at least two inner faces). The face F will be the face F_{uw} described above. We now show how to choose the edge uw.

If F = R (because h = 1), then we take uw to be an edge of F, one of whose endpoints is v. Such an edge uw exists since v is on F. (Note that, in this case, uw may be a chord or may be on the outer face.)

If $F \neq R$, then we take uw to be the chord of F that separates it from R. (In this case, v may still be one of u or w.) In either case, the edge uw and the face $F = F_{uw}$ satisfy the condition described above. In particular, the dual of G'_{uw} is a star because F had height h-1 and $v \notin V'_{uw}$ by our choice of uw.

Now that we have established the existence of uw and F_{uw} , we will now show that we can select another vertex, y, from $V'_{uw} \setminus B'$ so that $B = B' \cup \{y\}$ is a blocking set. This is sufficient since |B| is even so $block_B(G)$ is an even cycle.

(*) By choice, G'_{uw} has at least 2 faces and each of them, other than F_{uw} , is a triangular ear incident to F_{uw} and whose degree-2 vertex is in B' (otherwise, one of

those faces would have been handled by Case 1 or 2).

Let x be the unique vertex of B' on F_{uw} . The vertex x has two neighbours on F_{uw} . We claim that one of these is not in $\{u, w\}$ and we take this vertex to be y. This claim is valid because otherwise, F_{uw} is a triangle, xuw, with $x \notin \{u, w\}$. This case is not possible because by (\star) at least one of xu or xw is incident on both F_{uw} and a triangular ear E of G'_{uw} . Both x and the degree 2 vertex of E are in B'. This contradicts the fact that B' includes at most one vertex from each face of G, including E.

Let $B = B' \cup \{y\}$. We claim that *B* is a blocking set of *G*. First, note that *B* contains at most two vertices from each face, *F*, of *G*, so $V(F) \setminus B' \neq \emptyset$. We now show that *y* is a leaf in the tree G - B', so that G - B is also a tree.

First, observe that xy is not a chord of G since, by (\star), the face incident to xy other than F_{uw} would have two of its vertices in B'. Thus, in addition to x, y has at most two neighbours in G. One of these, z, is on F_{uw} and $z \neq x$ so $z \notin B'$. Finally, y may have one additional neighbour, which is a degree-2 vertex of a triangular ear incident on yz. In this case, by (\star), this degree-2 vertex is in B'. Thus, yz is the only edge incident to y in the tree G - B' so y is a leaf in this tree.

Remark 1. The proof of Lemma 7 can be modified to prove something stronger than just requiring the inclusion or exclusion of $v \in B$. We can specify an edge ab on the outer face of G and obtain a blocking set B such that $block_B(G)$ is an even cycle, $a \notin B$ and $b \in B$. The only difference in the proof is ensuring that a is not included in B. The resulting proof has the same three cases. Case 1 applies as long as ab is not on the boundary of the ear F. Case 2 applies as long as ab is not on the boundary of the ear E. Otherwise, in Case 3, the edge ab is in the subgraph G''_{uw} , so there is no chance of including a in B. This stronger version of Lemma 7 is used in Appendix A.

Lemma 8. Every simple bridgeless outerplane graph G has a blocking set B such that all cycles in $block_B(G)$ are even.

Proof. The proof is by induction on the number of 2-connected components of *G*. If *G* has no 2-connected components, then we take *B* to be the empty blocking set. If *G* has only one 2-connected component, then we apply Lemma 7. Otherwise, select a 2-connected component, *C*, that shares exactly one vertex, *v*, with the rest of *G*. Let $G' = G - (V(C) \setminus \{v\})$ and apply induction on *G'* to obtain a blocking set, *B'*, of *G'* such that $\text{block}_{B'}(G')$ has only even cycles. There are two cases to consider:

- 1. If B' contains v, then we apply the first part of Lemma 7 to obtain a blocking set B" of C such that $block_{B''}(C)$ is an even cycle and $v \in B$ ". We take $B = B' \cup B$ ", which clearly forms a blocking set of G. The blocking graph $block_B(G)$ is simply the union of the two blocking graphs $block_{B'}(G)$ and $block_{B''}(C)$, which have only the vertex v in common. Thus, every cycle in $block_B(G)$ is also a cycle in one of these two graphs, so it has even length.
- 2. If B' does not contain v, then we apply the second part of Lemma 7 to obtain a blocking set B'' of C such that $block_{B''}(C)$ is an even cycle and $v \notin B''$. We take $B = B' \cup B''$.



Figure 5: Case 2 in the proof of Lemma 8.

Refer to Figure 5. Starting at some appropriate vertex in $V(G) \setminus V(C)$ in the facial walk on the outer face of *G*, there is a last vertex, $x \in V(G') \cap B'$, encountered before the walk encounters the first vertex $u \in V(C) \cap B''$ and there is a last vertex $w \in V(C) \cap B''$ encountered before the walk returns to the next vertex $y \in V(G') \cap B'$. The edge xy is in block_{B'}(G') and the edge vw is in block_{B''}(C).

Since every blocking graph is a bridgeless cactus graph (Observation 3), each of these edges is part of one even cycle in its respective graph. In $block_B(G)$ these two cycles are merged by removing the edges xy and vw and adding the edges xv and yw. The resulting cycle is even. Every other cycle in $block_B(G)$ is also a cycle in one of $block_{B'}(G')$ or $block_{B''}(C)$ so it has even length. \Box

Lemma 9. Every simple outerplane graph G has a blocking set B such that all cycles in $block_B(G)$ are even.

Proof. The proof is by induction on the number of bridges of *G*. If *G* has no bridges, then we apply Lemma 8. Otherwise, select some bridge uw of *G* and contract it to obtain a graph *G'* in which uw corresponds to a single vertex *v*. By induction, we obtain a blocking set *B'* of *G'* such that $block_{B'}(G')$ has only even cycles (or is empty). There are two cases to consider:

1. If $v \in B'$, then we take $B = B' \cup \{u, w\} \setminus \{v\}$. This introduces exactly one new cycle in block_{*B*}(*G*) that is not present in block_{*B*}(*G*') and this cycle has length 2.

2. If $v \notin B'$, then we take B = B', so $block_B(G) = block_{B'}(G')$.

Finally, have all the tools to prove our main result on outerplane graphs:

Theorem 3. For every outerplane graph, G, $\pi_f(G) \le 11$.

Proof. By Lemma 2, we may assume that *G* is simple. From Lemma 9, *G* has a blocking set *B* such that $block_B(G)$ has no odd cycles. Therefore, by Lemma 6, $\pi_f(block_B(G)) \leq 7$. Using this with Lemma 4 implies that $\pi_f(G) \leq 11$.

4 Plane Graphs

In this section, we show how to reuse the ideas from Barát and Czap [3] to facially non-repetitively 22-colour every plane graph. Some modifications are needed because Barát and Czap use a nonrepetitive 12-colouring of outerplanar graphs whereas our Theorem 3 provides an *facial* nonrepetitive 11-colouring of *outerplane* graphs.

Theorem 4. Let $r = \max{\pi_f(H) : H \text{ is outerplane}}$. Then, for every plane graph G, $\pi_f(G) \le 2r$.

Proof. For any plane graph, H, the *peeling layering* of H is a partition of V(H) into sets as follows. Let $V_0(H)$ be the vertices on the outer face of H and let $V_i(H)$, $i \ge 1$, be the vertices on the outer face of $H - (V_0(H) \cup \cdots \cup V_{i-1}(H))$.

We augment *G* to obtain a plane graph G^+ in the following way. For each inner face *F* of *G*, let *W* be the facial walk of *F*. The walk *W* contains only vertices from $V_i(G)$ and $V_{i+1}(G)$ for some *i*. Remove from *W* all vertices in $V_{i+1}(G)$ to obtain a cyclic sequence *W'* of vertices from $V_i(G)$. For any two consecutive vertices u, w in *W'*, we add the edge uw to G^+ and embed it inside the face *F*. This construction has the following implications: (a) The resulting graph G^+ is still plane (though not necessarily simple) and, for all *i*, $V_i(G) = V_i(G^+)$. From this point onward, we use the notation $V_i = V_i(G) = V_i(G^+)$. (b) The cyclic sequence *W'* defined above is a facial walk in G^+ and, since it contains only vertices in V_i , it is also a facial walk in $G^+[V_i]$.

For each i, $G^+[V_i]$ is outerplane. To colour G we use Theorem 3 to facially nonreptively colour $G^+[V_i]$ with $\{1,...,11\}$ if i is even or $\{12,...,22\}$ if i is odd. This defines a colouring of G that we now prove is facially nonrepetitive.

Let *P* be a facial path, *F*, in *G*. The graphs $G^+[V_0]$ and *G* have the same outer face so, if *P* is on the outer face, then the colour sequence of *P* is not a repetition since our colouring is facially nonrepetitive for $G^+[V_0]$.

Therefore, *F* is an inner face and all vertices on *F* are in $V_i \cup V_{i+1}$ for some *i*. Write *P* as $P_0, Q_1, P_1, \ldots, Q_k, P_k$, where each Q_j consists of vertices from V_i and each P_j consists of vertices from V_{i+1} . Notice that, for each $j \in \{1, \ldots, k-1\}$, $G^+[V_i]$ contains an edge joining the last vertex in Q_j to the first vertex in Q_{j+1} . Indeed, Q_1, \ldots, Q_k is a facial path in $G^+[V_i]$ (It is a contiguous subsequence of the sequence *W'* defined above.) Next, observe that each P_j is a facial path on the outer face of $G^+[V_{i+1}]$. Therefore, the colour sequence determined by *P* is of the form $A_0, B_1, A_1, \ldots, B_k, A_k$ (with A_j corresponding to P_j and B_j corresponding to Q_j). The sequence B_1, \ldots, B_k is nonrepetitive and each sequence A_j is non-repetitive. Therefore, by Lemma 1, the colour sequence determined by *P* is nonrepetitive.

5 Concluding Remarks

We note that the proofs in this paper lead to straightforward linear-time algorithms. After the appropriate decomposition steps, there are essentially two subproblems: (1) finding an appropriate blocking set in a biconnected outerplane graph (Lemma 7) and (2) colouring cactus graphs with no odd cycles (Lemma 6). The proof of Lemma 6 is easily made into a linear-time algorithm. The proof of Lemma 7 can be implemented by a recursive earcutting algorithm that implements Lemma 3 followed by a traversal of the dual tree in order to find the face F_{uw} used in the proof of Lemma 7.

It seems unlikely that our upper bound of 11 for the facial nonrepetitive chromatic number of outerplane graphs (and hence the bound of 22 for plane graphs) is tight. (Recall that the best known lower bounds are 4 and 5, respectively [3].) Thus, an obvious direction for future work is to improve these bounds. Our proof of Lemma 4 uses a nonrepetitive 4-colouring of trees, but a facial nonrepetitive colouring would also be sufficient. This leads naturally to the following problem:

Open Problem 1. Is $\pi_f(T) \le 3$ for every tree, *T*?

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A Biconnected Outerplane Graphs

Lemma 10. If G is a biconnected outerplane graph, then G has a blocking set B such that $|B| \notin \{5,7,9,10,14,17\}$.

Proof. If *G* is a cycle, take *B* to be two consecutive vertices on *G* and we are done. Otherwise, select an ear, *E* of *G* and let uv be *E*'s chord. Then apply the stronger version of Lemma 9 discussed in Remark 1 to the graph $G' = G - (V(E) \setminus \{u, v\})$ to obtain a blocking set *B*' of even size that contains *v* and not *u*.

Note that, since $v \in B'$ and v is on E, G-B' has no cycles. Furthermore, since $u \notin B'$, G-B' is connected, so B' is a blocking set of G. Since |B'| is even, $|B'| \notin \{5,7,9,17\}$, so if $|B'| \notin \{10,14\}$, then we are done with B = B'. Otherwise, let y be v's degree-2 neighbour on E and take $B = B' \cup \{y\}$.

Corollary 2. If G is an outerplane graph with at most one 2-connected component, then $\pi_f(G) \leq 7$.

Proof. If *G* is a tree, then $\pi_f(G) \le 4$, so we may assume *G* contains exactly one 2-connected component, *G'*. Apply Lemma 10 to *G'* to obtain a blocking set *B* with $|B| \notin \{5, 7, 9, 10, 14, 17\}$ and observe that *B* is also a blocking set of *G*. The blocking graph block_{*B*}(*G'*) is a cycle, *C*, that has a nonrepetitive 3-colouring. The blocking graph block_{*B*}(*G*) consists of *C* and possibly some self loops so, by Lemma 2, block_{*B*}(*G*) has a facial nonrepetitive 3-colouring. Therefore, by Lemma 4, $\pi_f(G) \le 7$. □