Large induced acyclic and outerplanar subgraphs of 2-outerplanar graph *

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Abstract

Albertson and Berman conjectured that every planar graph has an induced forest on half of its vertices. The best known lower bound, due to Borodin, is that every planar graph has an induced forest on two fifths of its vertices. In a related result, Chartran and Kronk, proved that the vertices of every planar graph can be partitioned into three sets, each of which induce a forest.

We show tighter results for 2-outerplanar graphs. We show that every 2-outerplanar graph has an induced forest on at least half the vertices by showing that its vertices can be partitioned into two sets, each of which induces a forest. We also show that every 2-outerplanar graph has an induced outerplanar graph on at least two-thirds of its vertices.

1 Introduction

For many optimization problems, finding subgraphs with certain properties is a key to developing algorithms with efficient running times or bounded approximation ratios. For example, balanced separator subgraphs support the design of divide-and-conquer algorithms for minor-closed graph families [18, 17, 12, 15] and large subgraphs of low-treewidth¹ support the design of approximation schemes, also for minor-closed graph families [4, 8, 11]. In the area of graph drawing, one often starts by drawing a subgraph that is somehow easier to draw than the entire graph (such as a planar graph or a tree) and then adds in remaining graph features [5]; the larger the subgraph, the bigger the head-start for drawing and the more structure the subgraph has, the easier the subgraph will be to draw.

In this paper we are concerned with finding large induced subgraphs, in particular large induced forests and large induced outerplanar subgraphs, of input planar graphs. We are motivated both by the intrigue of various conjectures in graph theory but also by the impact that graph theoretic results have on the design of efficient and accurate algorithms. In particular, many algorithms

^{*}This material is based upon work supported by the National Science Foundation under Grant Nos. CCF-1252833 and DMS-1359173. Melissa Sherman-Bennett performed this work while participating in the Summer 2015 REU program in Mathematics and Computer Science at Oregon State University.

¹Formal definitions of graph theoretic terms will be given at the end of this section.

that are specifically designed for planar graphs rely on deep graph theoretic properties of planar graphs [23, 24].

1.1 Large induced forests of planar graphs: known results

Albertson and Berman conjectured that every planar graph has an induced forest on at least half of its vertices [2]; K_4 illustrates that this would be the best possible lower bound. A proof of the Albertson-Berman Conjecture would, among other things, provide an alternative proof, avoiding the 4-Color Theorem, that every planar graph has an independent set with at least one-quarter of the vertices.

The best-known lower bound toward the Albertson-Berman Conjecture has stood for 40 years: Borodin showed that planar graphs are acyclically 5-colorable (i.e. have a 5-coloring, every two classes of which induce a forest), thus showing that every planar graph has an induced forest on at least two-fifths of its vertices [6]. This is the best lower bound achievable toward the Albertson-Berman Conjecture via acyclic colorings as there are planar graphs which do not have an acyclic 4-coloring (for example $K_{2,2,2}$ or the octahedron).

The Albertson-Berman Conjecture has been proven for certain subclasses of planar graphs. Shi and Xu [21] showed that the Albertson-Berman Conjecture holds when $m < \lfloor 7n/4 \rfloor$ where m and n are the number of edges and vertices of the graphs, respectively. Hosono showed that outerplanar graphs have induced forests on at least two-thirds of the vertices [14] and Salavatipour showed that every triangle-free planar graph on n vertices has an induced forest with at least $\frac{17n+24}{32}$ vertices [20], later improved to $\frac{6n+7}{11}$ by Dross, Montassier and Pinou [10] and to $\frac{5n}{9}$ by Le [16]. In bipartite planar graphs, the best bound on the size of the largest induced forest is $\frac{4n}{7}$ by Wang, Xie and Yu [22].

One direction toward proving the Albertson-Berman Conjecture is to partition the vertices of graph G into sets such that each set induces a forest; the minimum number, a(G), of such sets is the vertex arboricity of G. This implies that G has an induced forest with at least 1/a(G) of its vertices. Chartran and Kronk first proved that all planar graphs have vertex arboricity at most 3 [7]. Raspaud and Wang proved that $a(G) \leq 2$ if G is planar and either G has no 4-cycles, any two triangles of G are at distance at least 3, or G has at most 20 vertices; they also illustrated a 3-outerplanar graph on 21 vertices with vertex arboricity 3 [19]. Yang and Yuan [1] proved that $a(G) \leq 2$ if G is planar and has diameter at most 2.

1.2 Outline of our results

In this paper, we show that 2-outerplanar graphs have vertex arboricity at most 2, thus showing that they satisfy the Albertson-Berman Conjecture and closing the gap for planar graphs with vertex arboricity 2 versus 3 left by Raspaud and Wang's work (Section 2). We also show that every 2-outerplanar graph has an induced outerplanar graph on at least two-thirds of its vertices and propose a few related conjectures (Section 3).

1.3 Definitions

We use standard graph theoretic notation [9]. In this paper, all graphs are assumed to be finite and simple (without loops or parallel edges). G[S] denotes the *induced subgraph* of graph G on vertex subset S: the graph having S as its vertices and having as edges every edge in G that has

both endpoints in S. Equivalently, G[S] may be constructed from G by deleting every vertex and incident edges that is not in S. We use $d_H(v)$ to denote degree of vertex v in graph H and |H| to denote the number of vertices of graph H.

Block-Cut Tree. A block of a graph G is a maximal two-connected component of G. A block-cut tree \mathcal{T} of a connected graph G is a tree where each vertex of \mathcal{T} corresponds to a block and there is an edge between two vertices X, Y of \mathcal{T} if two blocks X and Y share a common vertex or are incident to a common edge.

Planar graphs. A graph G is planar if it can be drawn (embedded) in the plane without any edge crossings. Although a planar graph may have many different embeddings, throughout this paper, we will assume that we are given a fixed embedding of the graph. A face of a planar graph is connected region of the complement of the image of the drawing. There is one infinite face, which we denote by f_{∞} . We denote the boundary of f_{∞} , which is the boundary of G, by ∂G . We say that a vertex v is enclosed by a cycle G if every curve from the image of G to an infinite point must cross the image of G.

Planar duality. Every planar graph G has a corresponding dual planar graph G^* : the vertices of G^* correspond to the faces of G and the faces of G^* correspond to the vertices of G; an edge of G^* connects two vertices of G^* if the corresponding faces of G share an edge (in this way the edges of the two graphs are in bijection).

Outerplanarity. A non-empty planar graph G with a given embedding is outerplanar (or 1-outerplanar) if all vertices are in ∂G . A planar graph is k-outerplanar for k > 1 if deleting the vertices in ∂G results in a (k-1)-outerplanar graph. A k-outerplanar graph has a natural partition of the vertices into k layers: L_1 is the set of vertices in ∂G ; L_i is the set of vertices in the boundary of $G \setminus \bigcup_{j < i} L_j$. We denote G(V, E) by $G(L_1, \ldots, L_k; E)$ if G is k-outerplanar. For a 2-outerplanar graph, we define the between degree of a vertex $v \in L_i$ to be the number of adjacent vertices in $L_j, j \neq i$.

Facial Block. Let C be the set of facial cycles bounding finite faces of $G[L_1]$. For each $C \in C$, let S_C be the set of vertices enclosed by C in G. Then we call the graph $G[C \cup S_C]$ a facial block of G.

2 2-outerplanar graphs have vertex-arboricity 2

In this section, we prove:

Theorem 1. If G is a 2-outerplanar graph, then the vertex arboricity of G is at most 2: $a(G) \le 2$.

We call a set of vertex-disjoint induced forests of G induced p-forests if their vertices partition the vertex set of G. We consider a counterexample graph G of minimal order. By studying the structure of this minimal counterexample, we will derive a contradiction. Let e be an edge that is not in G. We observe:

Observation 2. If $a(G \cup \{e\}) \leq 2$, then $a(G) \leq 2$.

Observation 2 allows us to assume w.l.o.g. that G is connected (by adding edges between components while maintaining 2-outerplanarity) and that G is a disk triangulation, i.e., that every face except the outer face of G is a triangle (by adding edges inside non-triangular faces while maintaining 2-outerplanarity). Let L_1, L_2 be the bipartition of the vertices of G into layers.

Observation 3. $G[L_1]$ is two-connected.

Proof. Suppose otherwise. Let v be a cut vertex of $G[L_1]$. Then v is also a cut vertex of G since L_1 is the outermost layer. Let B_1 , B_2 be two induced subgraphs of G that share the cut vertex v and $V(B_1) \cup V(B_2) = V$. Since G is minimal, we can partition each B_i into two induced forests F_{1i} and F_{2i} , $1 \le i \le 2$. W.l.o.g, we assume that $V(F_{11}) \cap V(F_{12}) = \{v\}$. Then, $F_{11} \cup F_{12}$ and $F_{21} \cup F_{22}$ are two induced p-forests of G, contradicting that G is a counter-example.

Claim 4. Every vertex in G has degree at least 4.

Proof. Suppose G has a vertex v of degree at most 3. Since G is a minimal order counterexample and G-v is a 2-outerplanar graph, $a(G-v) \leq 2$. Let F_0 and F_1 be two induced p-forests of G-v. Since v has at most 3 neighbors in G, one of F_0 or F_1 , w.l.o.g. say F_0 , contains at most one of these neighbors. Therefore $F_0 \cup \{v\}$ is a forest of G and $F_0 \cup \{v\}$, F_1 are two induced p-forests of G, contradicting that G is a counterexample.

By Observation 3, $\partial G[L_1]$ is a simple cycle. Thus, the graph, say H_1^* , of $G[L_1]$ obtained from the dual graph of $G[L_1]$ by removing the dual vertex corresponding to the infinite face of $G[L_1]$ is a tree. Let B be a facial block of G that has the boundary cycle corresponding to a leaf of H_1^* . Then, either ∂B has exactly one edge not in ∂G or $B \equiv G$. In the former case, let e_B be the shared edge; in the later case, let e_B be any edge of B. Denote $L_2^B = L_2 \cap V(B)$. We have:

Claim 5. $|L_2^B| \ge 2$.

Proof. If $|L_2^B| = 0$, then B is a triangle since G is a disk-triangulation and vertices have degree at least 4. Then, the vertex of B that is not an endpoint of e_B has degree 2 in G, contradicting Claim 4. If $L_2^B = \{v\}$, by Claim 4, v has at least four neighbors in L_1 and thus, at least one neighbor u of v in L_1 is not an endpoint of e_B . Then the degree of u in G is 3, contradicting Claim 4.

Claim 6. Let v be a vertex in L_2^B that has between degree at least 3. Then, either v is a cut vertex of $G[L_2^B]$ or v is adjacent to both endpoints of e_B .

Proof. Let v_1, v_2, v_3 be neighbors of v in ∂B in clockwise order around v. Let $\partial B[v_i, v_j]$ be the clockwise segment of ∂B from v_i to v_j , $i \neq j$. We define $C_{ij} = \partial B[v_i, v_j] \cup \{vv_i, vv_j\}$, which is a cycle of B. Assume v is not a cut vertex, at most one cycle of $\{C_{12}, C_{23}, C_{31}\}$ encloses a vertex of L_2^B , say C_{31} . Thus, v_2 is only adjacent to v and two other neighbors, say v'_1, v'_3 , of ∂B . Since C_{12} and C_{23} enclose no vertex of L_2^B , vv'_1v_2 and $vv_2v'_3$ are faces of G. If neither $v'_1v_2 = e_B$ nor $v_2v'_3 = e_B$, then $d_G(v_2) = 3$, contradicting Claim 4.

Suppose $v \in L_2^B$ is such that $d_{G[L_2^B]}(v) = 1$. By Claim 4, $d_G(v) \ge 4$ so v has between degree at least 3. Thus, by Claim 6, we have:

Observation 7. If there exists $v \in L_2^B$ such that $d_{G[L_2^B]}(v) = 1$, then v must be adjacent to both endpoints of e_B .

Let x_B, y_B be the endpoints of e_B . Since G is a triangulation, there is a vertex $v \in L_2^B$ such that vx_By_B is a face of G. We call v the separating vertex of B.

Claim 8. If $v' \neq v$ is a vertex in L_2^B that is adjacent to both endpoints of e_B , then, v' is a cut vertex of L_2^B .

Proof. We will prove that v' has at least one neighbor in L_2^B inside the triangle $v'x_By_B$ and at least one neighbor in L_2^B outside the triangle $v'x_By_B$; thus v' is a cut vertex of L_2^B .

By planarity, the triangle $v'x_By_B$ encloses v. Let $C_{vv'} = \{v, x_B, v', y_B\}$ which is a cycle of G. Since G is a disk triangulation and the edge x_B, y_B is embedded outside $C_{vv'}$, there must be an edge or a path inside $C_{vv'}$ connecting v and v'. Thus, v' has at least one neighbor in L_2^B inside the triangle $v'x_By_B$.

Suppose that the cycle $C_{v'} = \{\partial B \setminus e_B\} \cup \{v'x_B, v'y_B\}$ does not enclose any vertex of L_2^B . Since B is a facial block that only has e_B as a possible edge not in ∂G , every vertex in $C_{v'} \setminus \{v'\}$ must have v' as a neighbor and has degree 3, contradicting Claim 4. Thus, $C_{v'}$ must enclose at least one vertex of L_2^B . That implies v' has at least one neighbor in L_B^2 outside the triangle $v'x_By_B$ as desired.

Since every cut vertex of L_2^B has degree at least 2 in $G[L_2^B]$, by Claim 8 and Observation 7, we have:

Observation 9. Only the separating vertex v of B can have $d_{G[L_2]}(v) = 1$.

If the block-cut tree of $G[L_2^B]$ has at least two vertices, let K be a leaf block of $G[L_2^B]$ that does not contain the separating vertex of B. In this case, by Observation 9, $|K| \geq 3$. Otherwise, let $K = G[L_2^B]$. We refer to the cut vertex of K in the former case and the separating vertex of K in the latter case as the *separating vertex* of K. By Claim 6, we have:

Observation 10. Non-separating vertices of K have between degree at most 2.

We call a triangle abc of K a *critical triangle* with top c if $d_K(c) = 2$ and c is non-separating. By Observation 10 and Claim 4, c has exactly two neighbors in L_1 , that we denote by d, e (see Figure 1). Since G is a disk triangulation, two edges da and eb are edges of G.

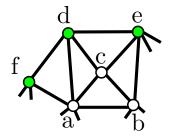


Figure 1: The critical triangle abc and two neighbors d, e of c in L_1 . Hollow vertices are in L_2 .

Claim 11. Vertices d and e have degree at least 5.

Proof. Neither d nor e has degree less than 4 by Claim 4. For contradiction, w.l.o.g, we assume that $d_G(d) = 4$. Recall a, c, e are three neighbors of d. Let f be the only other neighbor of d. Since

 $a, c \in L_2$ and ∂G is a simple cycle (Observation 3), f must be in L_1 (see Figure 1). Since G is a disk triangulation, $af \in E(G)$. Let G' be the graph obtained from G by contracting fd and dc and removing parallel edges. Then G' is a minor of G (and so is 2-outerplanar) with fewer vertices. Let F_0, F_1 be two induced p-forests of G' that exist by the minimality of G. Without loss of generality, we assume that $f \in F_0$. We have two cases:

- 1. If $b \in F_0$, then $a, e \in F_1$. If $bf \notin G$, adding c, d to F_0 does not destroy the acyclicity of F_0 in G. Thus, $F_0 \cup \{c, d\}$, F_1 are two induced p-forests of G. If $bf \in G$, the cycle $\{b, f, d, c\}$ separates a from e so a and e are in different trees in F_1 . Thus, $F_0 \cup \{c\}$, $F_1 \cup \{d\}$ are two induced p-forests of G.
- 2. Otherwise, $b \in F_1$. We have three subcases:
 - (a) If a, e are both in F_0 , then $F_0, F_1 \cup \{c, d\}$ are two induced p-forests of G.
 - (b) If $a, e \in F_1$, then, $F_0 \cup \{c, d\}$, F_1 are two induced p-forests of G.
 - (c) If a, e are in different induced p-forests of G', then, $F_0 \cup \{c\}, F_1 \cup \{d\}$ are two induced p-forests of G.

In each case, the resulting p-forests contradict that G is a minimal order counter example. \Box

Claim 12. $|K| \ge 4$.

Proof. If $K \neq G[L_2^B]$, as noted in the definition of K, $|K| \geq 3$. If $K = G[L_2^B]$, then by Claim 5, $|K| \geq 2$ and by Observation 9, $|K| \geq 3$. Suppose that |K| = 3. Then, K is a triangle. Let u, w be two neighbors of the separating vertex v in K. Then, wuv is a critical triangle with top u (or w). By Claim 4 and Observation 10, u and w both have between degree 2. Thus, u and w have a common neighbor on L_1 which therefore has degree 4, contradicting Claim 11.

Suppose that a and b of a critical triangle abc with top c of K have a common neighbor f in L_2 . We have:

Claim 13. If fa (resp. fb) is in $\partial G[L_2^B]$, then a (resp. b) must be the separating vertex.

Proof. For a contradiction (and w.l.o.g), we assume that $fa \in \partial G[L_2^B]$ and a is non-separating. See Figure 2. Let G' be the graph obtained from G by contracting ac and ce and removing parallel edges. Then, G' is a minor of G (and so is 2-outerplanar) with fewer vertices. Let F_0, F_1 be two induced p-forests of G', which are guaranteed to exist by the minimality of G. Without loss of generality, we assume that $f \in F_0$. We consider two cases:

- 1. If $e \in F_0$, then d, b are in F_1 . If edge $f e \notin G$, then, $F_0 \cup \{a, c\}$, F_1 are two induced p-forests of G. If $f e \in G$, cycle $\{f, a, c, e\}$ separates d from b so d and b are in different trees of F_1 . Thus, $F_1 \cup \{c\}$, $F_0 \cup \{a\}$ are two induced p-forests of G.
- 2. Otherwise, $e \in F_1$. We have three subcases:
 - (a) If b, d are both in F_0 , then $F_0, F_1 \cup \{a, c\}$ are two induced p-forests of G.
 - (b) If b, d are both in F_1 , then $F_0 \cup \{a, c\}$, F_1 are two induced p-forests of G.
 - (c) If b, d are in different forests of G', then, $F_0 \cup \{c\}, F_1 \cup \{a\}$ are two induced p-forests of G.

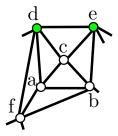


Figure 2: The critical triangle abc with edge $fa \in \partial G[L_2]$. Hollow vertices are in L_2 .

In each case, the resulting p-forests contradicts that G is a minimal order counter example. \Box

If the edge fb is shared with another critical triangle fbg with top g, then we call $\{abc, bfg\}$ a pair of critical triangles. See Figure 3. Note that we are assuming that f is a common neighbor of a and b in L_2 .

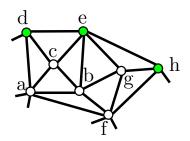


Figure 3: A pair of critical triangles abc and bfg. Hollow vertices are in L_2

Claim 14. If there exists a pair of critical triangles abc and bfg in K, then b must be the separating vertex of K.

Proof. Note that neither c nor g can be the separating vertex by definition of critical triangles. Suppose for contradiction that b is non-separating. Let d, e be two neighbors of c as defined above and i and h be the neighbors of g in L_1 . We first argue that $i \equiv e$. Suppose otherwise. Since G is a disk triangulation, ec, eb, ig, ih, ib are edges of G. Let P be the subpath of ∂G between e and e that does not contain e and e . Note that e could simply be edge ei. Since e is a facial block that shares at most one edge with other facial blocks and e is non-separating, e has exactly one neighbor on e. That implies e would have degree e, contradicting Claim 11.

We also note that $h \neq d$ (for otherwise, e would not be in L_1) and $hf \in E(G)$. See Figure 3. Let G' be the graph obtained from G by contracting ec, eb, eg and eh and removing parallel edges. Thus, G' is a minor of G with fewer vertices. By minimality, G' has two induced p-forests F_0, F_1 . Without loss of generality, we assume that $a \in F_0$. We will reconstruct two induced p-forests of G by considering two cases:

1. If $h \in F_0$, then $d, f \in F_1$. If edge $ah \in G$, then, by planarity, d and f are in different trees of F_1 . Thus, $F_1 \cup \{e, b\}$ has no cycle which implies $F_1 \cup \{e, b\}$, $F_0 \cup \{c, g\}$ are two induced p-forests in G. Otherwise, $F_0 \cup \{b, e\}$ has no cycle. Thus, $F_0 \cup \{b, e\}$, $F_1 \cup \{c, g\}$ are two induced p-forests in G.

- 2. Otherwise, $h \in F_1$. We have four subcases:
 - (a) If d, f are both in F_1 , then $F_0 \cup \{c, e, g\}, F_1 \cup \{b\}$ are two induced p-forests of G.
 - (b) If d, f are both in F_0 , then $F_0 \cup \{e\}, F_1 \cup \{b, c, g\}$ are two induced p-forests of G.
 - (c) If $d \in F_0$, $f \in F_1$, then $F_0 \cup \{e, g\}$, $F_1 \cup \{b, c\}$ are two induced p-forests of G.
 - (d) If $d \in F_1$, $f \in F_0$, then $F_0 \cup \{c, g\}$, $F_1 \cup \{b, e\}$ are two induced p-forests of G.

Thus, in all cases, the resulting induced p-forests contradict that G is a counterexample. \Box

We are now ready to complete the proof of Theorem 1 by considering a triangle of K of $G[L_2^B]$, say uvw, containing the separating vertex v of K and has the most edges in common with ∂K . Since v is separating, uvw contains at least one edge in ∂K . We note that $K^* \setminus (\partial K)^*$ where $(\partial K)^*$ is the dual vertex of the infinite face of K, is a tree that we denote by T_K^* . Recall that K is a block of $G[L_2^B]$. We root T_K^* at the vertex corresponding to the triangle uvw. Consider the deepest leaf $x^* \in T_K^*$ and its parent y^* . Let abc be the triangle corresponding to x^* such that the dual edge of ab is x^*y^* . Then $d_K(c) = 2$. Since $K \geq 4$, $abc \not\equiv uvw$ and thus, it is a critical triangle with top c. Let abf be the triangle that corresponds to y^* . Note here it may be that $abf \equiv uvw$. We have three cases:

- 1. If $d_{T_K^*}(y^*) = 1$, then $abf \equiv uvw$. Thus, two edges fa, fb are both in ∂K but only one of the two vertices a, b can be the separating vertex of K. This contradicts Claim 13.
- 2. If $d_{T_K^*}(y^*) = 2$, then exactly one of two edges $af, bf \in \partial K$; w.l.o.g, we assume that $bf \in \partial K$. Then, by Claim 13, b must be the separating vertex of K. Thus, only two triangles abc and abf contain the separating vertex. Since uvw is the triangle containing the separating vertex with most edges in ∂K , $uvw \equiv abc$, contradicting our choice of triangle abc.
- 3. Otherwise, we have $d_{T_K^*}(y^*) = 3$. Then, none of $\{ab, bf, af\}$ is in ∂K , so $abf \not\equiv uvw$. Let z^* and t^* be the other two neighbors of y^* in T_K^* with t^* as the parent of y^* . Then, x^* and z^* have the same depth. By our choice of x^* , z^* must also be a leaf. Thus, the triangle, say bfg, corresponding to z^* is critical. Thus $\{abc, bfg\}$ is a pair of critical triangles. Since t^* is the parent of y^* , b cannot be the separating vertex of K, contradicting Claim 14.

This completes the proof of Theorem 1.

3 2-outerplanar graphs have large induced outerplanar graphs

In this section, we prove:

Theorem 15. Let G be a 2-outerplanar graph on n vertices. G has an induced outerplanar subgraph on at least $\frac{2n}{3}$ vertices whose outerplanar embedding is induced from G.

Let L_1, L_2 be the partition of G into layers. Note that $\partial G[L_i]$ is a cactus graph (every edge is in at most 1 cycle). As in Section 2, we assume w.l.o.g that G is connected and a disk triangulation. This gives us:

Observation 16. If uv is an edge in $\partial G[L_2]$, then there exists $w \in L_1$ such that uvw is a face.

Observation 17. The between degree of every vertex in L_2 is at least 1.

Lemma 18. If $v \in L_2$ has between degree 1, then it is incident to exactly two edges in $\partial G[L_2]$.

Proof. Let u be v's neighbor in L_1 . Since G is a disk triangulation, there exist two triangular faces, say xuv and yuv, containing the edge uv. As the between degree of v is 1, x and y are in L_2 , and the edges xv and yv are in $\partial G[L_2]$. Therefore, v is incident to at least two edges in $\partial G[L_2]$.

Suppose for the sake of contradiction that v is incident to more than two edges in $\partial G[L_2]$. Let w be a neighbor of v such that $w \notin \{x, y\}$. Then by Observation 16, there exists $s \in L_1$ such that vws is a face. Since $w \notin \{x, y\}$ and G is simple, $s \neq u$. This implies v has between degree at least 2; contradicting that v's between degree is 1.

Lemma 19. If a facial block B contains a vertex in L_2 , then endpoints of any edge $uv \in \partial B$ are adjacent to a common vertex in L_2^B .

Proof. Since G is a triangulation, there is a vertex $w \in B$ such that uvw is a triangular face; thus uvw contains no vertex of L_2 . Suppose that $w \in \partial B$, then uvw is an induced cycle of $G[L_1]$. Thus, $uvw \equiv \partial B$; contradicting that B contains a vertex in L_2 .

Lemma 20. There exists a matching $M \subseteq \partial G[L_2]$ with the following property:

If
$$v \in L_2 \setminus V(M)$$
 then v has between degree at least 2. (1)

Proof. Let L be the set of vertices of between degree 1 in L_2 . We proceed by strong induction on |L|. If |L| = 0, any matching $M \subseteq \partial G[L_2]$ has property (1).

Let $v \in L$ and $u \in L_2 \setminus L$ be vertices such that $uv \in \partial G[L_2]$; v exists by Lemma 19. By Lemma 18, v has exactly one other neighbor w such that $vw \in \partial G[L_2]$. Contract uv and wv to v and delete parallel edges and loops; let the resulting graph be G'. Since v now has between degree at least 2, the number of vertices of between degree 1 in G' is strictly less than |L|. Therefore, by the inductive hypothesis, there exists a matching $M' \subseteq \partial G'[L_2]$ with property (1). Now, consider M' as a matching in $\partial G[L_2]$.

If v is not covered by M' in G', then u, v, w are not covered by M' in G. Then, $M' \cup \{vw\}$ is a matching and has property (1), since u has between degree at least 2 as argued above.

If $vx \in M'$ for some $x \in G'$, then either $ux \in \partial G[L_2]$ or $wx \in \partial G[L_2]$. In the first case, let $M = (M' \setminus \{vx\}) \cup \{ux, vw\}$; in the second, let $M = (M' \setminus \{vx\}) \cup \{wx, uw\}$. In both cases, M is a matching of $\partial G[L_2]$ with property (1).

We are now ready to prove Theorem 15. To find the vertices inducing a large outerplanar graph in G, we delete vertices in L_1 until all vertices in L_2 are "exposed" to the external face. To ensure that the resulting outerplanar graph is sufficiently large, we delete vertices in L_1 that expose 2 vertices in L_2 or otherwise ensure 2 vertices will be included in the outerplanar graph.

Let M be a matching as guaranteed by Lemma 20. We create a list K of triples such that each vertex in L_2 occurs in exactly one triple. For each $u \in L_2$ not covered by M, u has between degree at least 2, and we add $\{u, v, w\}$ to K, where v, w are neighbors of u in L_1 . For each edge $xy \in M$, by Observation 16, there exists $z \in L_1$ such that xyz is a face, and we add $\{x, y, z\}$ to K.

We then delete vertices from L_1 as follows:

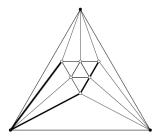


Figure 4: The outerplanar induced subgraph of this graph found by the algorithm in Theorem 15 is induced by the white vertices. Every induced forest on at least half of the vertices of this graph (an example is shown by the bolded edges) includes vertices not in this outerplanar subgraph.

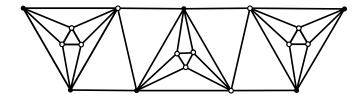


Figure 5: A 2-outerplanar graph whose largest induced outerplanar subgraph is on $\frac{2}{3}$ of its vertices. The white vertices induce such a subgraph, found by the algorithm in Theorem 15.

- 1. While there exists $\{u, v, w\} \in K$ such that $\{u, v, w\} \cap L_1 = \{v\}$ delete v from G and delete all triples containing v from K;
- 2. While there exists $v \in L_1$ such that v is in two or more distinct triples of K delete v from G and delete all triples containing v from K;
- 3. While $\{u, v, w\} \in K$ delete $v \in L_1$ from G and delete $\{u, v, w\}$ from K.

Note that if $v \in L_1$ is deleted from G, all L_2 vertices in a triple with v are exposed. Therefore, the undeleted vertices induce an outerplanar subgraph of G.

In the first two steps, at least two L_2 vertices were exposed for every deleted L_1 vertex. In the final step, all triples are disjoint, so each deletion of an L_1 vertex exposes one L_2 vertex and ensures that one L_1 vertex will not be deleted; again, 2 vertices are included in the induced outerplanar subgraph for every deleted vertex. This means that the subgraph contains at least two thirds of the vertices of G. This complete the proof of Theorem 15.

This result is tight, as the disjoint union of multiple octahedrons (see Figure 4) is 2-outerplanar, and its largest induced outerplanar subgraph is on $\frac{2}{3}$ of its vertices. The result is also tight for arbitrarily large connected 2-outerplanar graphs, as the same property holds for graphs constructed by connecting disjoint octahedrons as shown in Figure 5.

Theorem 15 has an immediate corollary for k-outerplanar graphs.

Corollary 21. Let G be a k-outerplanar graph on n vertices. Then G has an induced $\lceil \frac{k}{2} \rceil$ -outerplanar subgraph on at least $\frac{2n}{3}$ vertices.

Proof. We apply Theorem 15 to pairs of successive layers $\{L_1, L_2\}, \{L_3, L_4\}, \ldots$ in G, finding large induced outerplanar subgraphs $H_i \subseteq G[L_i \cup L_{i-1}]$ for $i = 1, 3, 5, \ldots, k-1$ (if k is even; if k is odd, we end at i = k-2). Let $V' := \bigcup_i V(H_i)$. G[V'] is $\lceil \frac{k}{2} \rceil$ -outerplanar, as $L_i(G[V']) = V(H_i)$, and $|V'| \ge \frac{2n}{3}$, as $|V(H_i)| \ge \frac{2}{3} |L_i \cup L_{i-1}|$.

4 Future directions

We define an induced outerplane graph of a planar graph G is an induced subgraph of G whose embedding inherited from G is an outerplanar embedding. We point out that our last result implies an improvement to a graph drawing result of Angelini, Evans, Frati, and Gudmundsson [3] for the class of 2-outerplanar graphs. A simultaneous embedding with fixed edges and without mapping (SEFENOMAP) of two planar graphs G_1 and G_2 of the same size n is a pair of planar drawings of G_1 and G_2 that maps any vertex of G_1 into any vertex of G_2 such that: (i) vertices of both graphs are mapped to the same point set in a plane and (ii) every edge that belongs to both G_1 and G_2 must be represented by the same curve in the drawing of two graphs. The OPTSEFENOMAP problem asks for the maximum $k \leq n$ such that: given any two planar graphs G_1 and G_2 of size n and k, respectively, there exists an induced subgraph G'_1 of G_1 such that G'_1 and G_2 have a SEFENOMAP where the drawing of G'_1 is inherited from a planar drawing of G. The result of Gritzmann et al. [13], implies that k can be as large as the size of any induced outerplane graph of G_1 . Angelini, Evans, Frati, and Gudmundsson (Theorem 1 [3]) showed that any planar graph G of size n has an induced outerplane graph of size at least $\lceil n/2 \rceil$ which implies $k \geq \lceil n/2 \rceil$ by the result of Gritzmann et al. [13]. Our Theorem 15 implies the following corollary, which is an improvement of the result of Angelini, Evans, Frati, and Gudmundsson for the class of 2-outerplanar graphs.

Corollary 22. Every n-vertex 2-outerplanar graph and every $\lceil 2n/3 \rceil$ -vertex planar graph have a SEFENOMAP.

Based on Theorem 15, we conjecture that:

Conjecture 23. Any 3-outerplanar graph on n vertices contains an induced outerplane graph of size at least $\frac{2n}{3}$.

If this conjecture is true, it would, by Hosono's result [14], imply that the largest induced forest of 3-outerplanar graphs on n vertices has size at least $\frac{4n}{9}$, that is an improvement over Borodin's result. It also improves the result of Angelini, Evans, Frati, and Gudmundsson for 3-outerplanar graphs. We note that in the proof of Theorem 15, we only need to delete vertices in L_1 , and leave L_2 untouched, to get a large induced outerplane graph of 2-outerplanar graphs. For 3-outerplanar graph, one may need to delete vertices in L_3 as shown by Figure 6.

We also believe that following conjecture, which is also mentioned in in [3], is true:

Conjecture 24. A planar graph on n vertices contains an induced outerplane graph of size at least $\frac{2n}{3}$.

If this conjecture is true, it would imply an improvement of Borodin's result and Angelini, Evans, Frati, and Gudmundsson' result for general planar graphs.

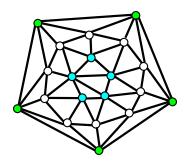


Figure 6: A 3-outerplanar graph with 20 vertices. Filled vertices are in odd layers and hollow vertices are in even layers. To obtain an induced outerplane graph of size at least 14 ($\lceil 2n/3 \rceil$) vertices, one needs to delete at least one vertex in the innermost layer

Acknowledgments: The authors thank conversations with Bigong Zheng and Kai Lei. We thank anonymous reviewers for comments that help improving the presentation of this paper.

References

[1] Y. Aifeng and Y. Jinjiang. On the vertex arboricity of planar graphs of diameter two. *Discrete Mathematics*, 307(19-20):2438–2447, 2007.

[2] M. O. Albertson and D. M. Berman. A conjecture on planar graphs. *Graph Theory and Related Topics*, page 357, 1979.

[3] P. Angelini, W. Evans, F. Frati, and J. Gudmundsson. SEFE with no mapping via large induced outerplane graphs in plane graphs. *Journal of Graph Theory*, 82(1):45–64, 2016.

- [4] B. S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal* of the ACM (JACM), 41(1):153–180, 1994.
- [5] G. Di Battista, P. Eades, I. G. Tollis, and R. Tamassia. *Graph Drawing: Algorithms for the Visualization of Graphs*. Prentice Hall, 1st edition, 1998.
- [6] O. V. Borodin. On acyclic colorings of planar graphs. Discrete Mathematics, 25(3):211–236, 1979.
- [7] G. Chartrand and H. V. Kronk. The point-arboricity of planar graphs. *Journal of the London Mathematical Society*, 44:612–616, 1969.
- [8] E. D. Demaine, M. Hajiaghayi, and B. Mohar. Approximation algorithms via contraction decomposition. In *Proc. of 18th SODA*, pages 278–287, 2007.
- [9] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, third edition, 2005.
- [10] F. Dross, M. Montassier, and A. Pinlou. Large induced forests in planar graphs with girth 4 or 5. arXiv preprint arXiv:1409.1348, 2014.

- [11] D. Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3):275–291, 2000.
- [12] J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan. A separator theorem for graphs of bounded genus. *Journal of Algorithms*, 5(3):391–407, 1984.
- [13] P. Gritzmann, B. Mohar, J. Pach, and R. Pollack. Embedding a planar triangulation with vertices at specified points. *American Mathematical Monthly*, 98(2):165–166, 1991.
- [14] K. Hosono. Induced forests in trees and outerplanar graphs. *Proceedings of the Faculty of Science of Tokai University*, 25:27–29, 1990.
- [15] K. Kawarabayashi and B. Reed. A separator theorem in minor-closed classes. In *In Proc. of* 51st FOCS, pages 153–162, 2010.
- [16] H. Le. A better bound on the largest induced forests in triangle-free planar graphs. arXiv Preprint, 2016.
- [17] R. J. Lipton and R. E. Tarjan. Applications of a planar separator theorem. SIAM Journal on Computing, 9(3):615–627, 1980.
- [18] R.J. Lipton and R.E. Tarjan. A separator theorem for planar graphs. SIAM Journal on Applied Mathematics, 36(2):177–189, 1979.
- [19] A. Raspaud and W. Wang. On the vertex-arboricity of planar graphs. *European Journal of Combinatorics*, 29:1064–1075, 2008.
- [20] M. R. Salavatipour. Large induced forests in triangle-free planar graphs. *Graphs and Combinatorics*, 22:113–126, 2006.
- [21] L Shi and H Xu. Large induced forests in graphs. Journal of Graph Theory, 2016.
- [22] Y. Wang, Q. Xie, and X. Yu. Induced forests in bipartite planar graphs. Preprint, 2016.
- [23] H. Whitney. Non-separable and planar graphs. Transactions of the American Mathematical Society, 34(2):339–362, 1932.
- [24] H. Whitney. Planar graphs. Fundamenta Mathematicae, 21(1):73–84, 1933.