# Improved Strength Four Covering Arrays with Three Symbols 

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#### Abstract

A covering array $t-C A(n, k, g)$, of size $n$, strength $t$, degree $k$, and order $g$, is a $k \times n$ array on $g$ symbols such that every $t \times n$ sub-array contains every $t \times 1$ column on $g$ symbols at least once. Covering arrays have been studied for their applications to software testing, hardware testing, drug screening, and in areas where interactions of multiple parameters are to be tested. In this paper, we present an algebraic construction that improves many of the best known upper bounds on $n$ for covering arrays $4-C A(n, k, g)$ with $g=3$. The coverage measure $\mu_{t}(A)$ of a testing array $A$ is defined by the ratio between the number of distinct $t$-tuples contained in the column vectors of $A$ and the total number of $t$-tuples. A covering array is a testing array with full coverage. The covering arrays with budget constraints problem is the problem of constructing a testing array of size at most $n$ having largest possible coverage measure, given values of $k, g$ and $n$. This paper presents several strength four testing arrays with high coverage. The construction here is a generalisation of the construction methods used by Chateauneuf, Colbourn and Kreher, and Meagher and Stevens.


## 1 Introduction

This article focuses on constructing new strength-four covering arrays with $g=3$ and establishing improved bounds on the covering array numbers $4-C A N(k, 3)$. This article also presents solution to the covering arrays with budget constraints problem by constructing many strength four testing arrays with high coverage. A covering array $t-C A(n, k, g)$, of size $n$, strength $t$, degree $k$, and order $g$, is a $k \times n$ array on $g$ symbols such that every $t \times n$ sub-array contains every $t \times 1$ column on $g$ symbols at least once. It is desirable in most applications to minimise the size $n$ of covering arrays. The covering array number $t-C A N(k, g)$ is the smallest $n$ for which a $t-C A(n, k, g)$ exists. An obvious lower bound is

$$
g^{t} \leq t-C A N(k, g)
$$

In this paper, we describe a construction method which is an extension of the methods developed by Chateauneuf, Colbourn and Kreher [1] and Meagher and Stevens [13]. This method improves some of the best known upper bounds for strength four covering arrays with $g=3$. In the range of degrees considered in this paper, the best known results previously come from [5]; in that paper, covering arrays are also found by using a group action on the symbols (the affine or Frobenius group), but no group action on the rows is employed. While for $g=3$ the group that we employ on the symbols coincides with the affine group, we accelerate and improve the search by also exploiting a group action on the rows as in [1, 13], and develop a search method than can be applied effectively whenever $g \geq 3$ and $g-1$ is a prime power.

There is a large literature [1, 7] on covering arrays, and the problem of determining small covering arrays has been studied under many guises over the past thirty years. In [7], Hartman and Raskin discussed several generalizations motivated by their applications in the realm of software testing. When testing a software system with $k$ parameters, each of which must be tested with $g$ values, the total number of possible test cases is $g^{k}$. For instance, if there are 20 parameters and three values for each parameter then the number of input combinations or test cases of this system is $3^{20}=3486784401$. A fundamental problem with software testing is that testing under all combinations of inputs is not feasible, even with a simple product [9, 10]. Software developers cannot test everything, but they can use combinatorial test design to identify the minimum number of tests needed to get the coverage they want. The goal of most combinatorial testing research is to create test suites that find a large percentage of errors of a system while having a small number of tests required. Covering arrays prove useful in locating a large percentage of errors in software systems [3, 16]. The test cases are the columns of a covering array $t-C A(n, k, g)$. This is one of the five natural generalizations in [7]. Covering arrays with budget constraints: A practical limitation in the realm of testing is budget. In most software development environments, time, computing, and human resources needed to perform the testing of a component is strictly limited. To model this situation, we consider the problem of creating best possible test suite (covering the maximum number of $t$-tuples) within a fixed number of test cases. The coverage measure $\mu_{t}(A)$ of a testing array $A$ is defined by the ratio between the number of distinct t -tuples contained in the column vectors of $A$ and the total number of $t$-tuples given by $\binom{k}{t} g^{t}$. Our objective is to construct a testing array $A$ of size at most $n$ having largest possible coverage measure, given fixed values of $t, k, g$ and $n$. This problem is called covering arrays with budget constraints.

We summarize the results from group theory that we use. Let $\mathbb{F}_{q}$ be a Galois field $\operatorname{GF}(q)$ where $q=p^{m}$ and $p$ is prime. We adjoin to $\mathbb{F}_{q}$ the symbol $\infty$ : it may be helpful to think of the resulting set

$$
X=\mathbb{F}_{q} \cup\{\infty\}
$$

as the projective line consisting of $q+1$ points. Recall that the projective general
linear group of dimension 2 may be seen as the "fractional linear group":

$$
P G L(2, q)=\left\{\alpha: X \mapsto X \left\lvert\, x \alpha=\frac{a x+b}{c x+d}\right., \text { where } a, b, c, d \in \mathbb{F}_{q} \text { and } a d-b c \neq 0\right\}
$$

in which we define $\frac{1}{0}=\infty, \frac{1}{\infty}=0,1-\infty=\infty-1=\infty$, and $\frac{\infty}{\infty}=1$. It is known that $|P G L(2, q)|=\frac{\left(q^{2}-1\right)\left(q^{2}-q\right)}{(q-1)}=(q+1) q(q-1)$ and its action on $\mathbb{F}_{q} \cup\{\infty\}$ is sharply 3-transitive. For the undefined terms and more details see [15, Chapter 7].

Pair-wise or 2-way interaction testing and 3-way interaction testing are known to be effective for different types of software testing [3, 11, 12]. However, software failures may be caused by interactions of more than three parameters. A recent NIST study indicates that failures can be triggered by interactions up to 6 parameters [10]. Here we consider the problem of 4-way interaction testing of the parameters. The construction given in this paper improves many of the current best known upper bounds on $4-C A N(k, g)$ with $g=3$ and $21 \leq k \leq 74$. This paper also presents several strength four testing arrays with high coverage measures.

## 2 PGL Construction

Let $X=G F(g-1) \cup\{\infty\}$ be the set of $g$ symbols on which we are to construct a $4-C A(n, k, g)$. We choose $g$ so that $g-1$ is a prime or prime power.

### 2.1 Case 1: Two starter vectors

Our construction involves selecting a group $G$ and finding vectors $u, v \in X^{k}$, called starter vectors. We use the vectors to form a $k \times 2 k$ matrix $M$.

$$
M=\left(\begin{array}{cccccccc}
u_{1} & u_{k} & \ldots & u_{2} & v_{1} & v_{k} & \ldots & v_{2} \\
u_{2} & u_{1} & \ldots & u_{3} & v_{2} & v_{1} & \ldots & v_{3} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
u_{k-1} & u_{k-2} & \ldots & u_{k} & v_{k-1} & v_{k-2} & \ldots & v_{k} \\
u_{k} & u_{k-1} & \ldots & u_{1} & v_{k} & v_{k-1} & \ldots & v_{1}
\end{array}\right)
$$

Let $G=P G L(2, g-1)$. For each $a \in P G L(2, g-1)$, let $M^{a}$ be the matrix formed by the action of $a$ on the elements of $M$. The matrix obtained by developing $M$ by $G$ is the $k \times 2 k|G|$ matrix $M^{G}=\left[M^{a}: a \in G\right]$. Let $C$ be the $k \times g$ matrix that has a constant column with each entry equal to $x$, for each $x \in X$. Vectors $u, v \in X^{k}$ are said to be starter vectors for a $4-C A(n, k, g)$ if any $4 \times 2 k$ subarray of the matrix $M$ has at least one representative from each non-constant orbit of $\operatorname{PGL}(2, g-1)$ acting on 4-tuples from $X$. Under this group action, there are precisely $g+11$ orbits of 4-tuples. These $g+11$ orbits are determined by the pattern of entries in their 4-tuples:

1. $\left\{[a, a, a, a]^{T}: a \in X\right\}$
2. $\left\{[a, a, a, b]^{T}: a, b \in X, a \neq b\right\}$
3. $\left\{[a, a, b, a]^{T}: a, b \in X, a \neq b\right\}$
4. $\left\{[a, b, a, a]^{T}: a, b \in X, a \neq b\right\}$
5. $\left\{[b, a, a, a]^{T}: a, b \in X, a \neq b\right\}$
6. $\left\{[a, a, b, b]^{T}: a, b \in X, a \neq b\right\}$
7. $\left\{[a, b, a, b]^{T}: a, b \in X, a \neq b\right\}$
8. $\left\{[a, b, b, a]^{T}: a, b \in X, a \neq b\right\}$
9. $\left\{[a, a, b, c]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
10. $\left\{[b, a, a, c]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
11. $\left\{[a, b, a, c]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
12. $\left\{[b, a, c, a]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
13. $\left\{[a, b, c, a]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
14. $\left\{[b, c, a, a]^{T}: a, b, c \in X, a \neq b \neq c\right\}$
15. $g-3$ orbits of patterns with four distinct entries. The reason is this. There are $g(g-1)(g-2)(g-3)$ 4-tuples with four distinct entries and each orbit contains $g(g-1)(g-2)$ 4-tuples as $|P G L(2, g-1)|=g(g-1)(g-2)$.

If starter vectors $u, v$ exist in $X^{k}$ (with respect to the group $G$ ) then there exists a $4-C A(2 k g(g-1)(g-2)+g, k, g)$. We give an example to explain the method.

Example 1. Let $g=3, k=30, X=G F(2) \cup\{\infty\}$ and $G=P G L(2,2)$. The action of $G$ on 4-tuples from $X$ has 14 orbits:

Orb 1: $[0000, \infty \infty \infty \infty, 1111]$
Orb 2: $[0001,000 \infty, \infty \infty \infty 0, \infty \infty \infty 1,1110,111 \infty]$
Orb 3: $[1 \infty \infty \infty, 1000,0111, \infty 000,0 \infty \infty \infty, \infty 111]$
Orb 4: $[0100, \infty 0 \infty \infty, 0 \infty 00, \infty 1 \infty \infty, 1011,1 \infty 11]$
Orb 5: $[11 \infty 1, \infty \infty 1 \infty, 0010,1101,00 \infty 0, \infty \infty 0 \infty]$
Orb 6: $[11 \infty \infty, \infty \infty 11,0011,1100,00 \infty \infty, \infty \infty 00]$
Orb 7: $[\infty 0 \infty 0,0101, \infty 1 \infty 1,0 \infty 0 \infty, 1010,1 \infty 1 \infty]$
Orb 8: $[\infty 11 \infty, 1 \infty \infty 1,1001,0110, \infty 00 \infty, 0 \infty \infty 0]$

Orb 9: $[11 \infty 0, \infty \infty 10,001 \infty, 110 \infty, 00 \infty 1, \infty \infty 01]$
Orb 10: $[\infty 0 \infty 1,010 \infty, \infty 1 \infty 0,0 \infty 01,101 \infty, 1 \infty 10]$
Orb 11: $[1 \infty 01,0 \infty 10, \infty 10 \infty, 01 \infty 0, \infty 01 \infty, 10 \infty 1]$
Orb 12: $[1 \infty 0 \infty, 0 \infty 1 \infty, \infty 101,01 \infty 1, \infty 010,10 \infty 0]$
Orb 13: $[1 \infty 00,0 \infty 11, \infty 100,01 \infty \infty, \infty 011,10 \infty \infty]$
Orb 14: $[1 \infty \infty 0,100 \infty, 011 \infty, \infty 001,0 \infty \infty 1, \infty 110]$
The following are starter vectors to construct $\left[M^{G}, C\right]$, a 4-CA $(363,30,3)$ :

$$
\begin{aligned}
& u=(011 \infty 11 \infty \infty \infty 001 \infty \infty \infty 1 \infty 10 \infty \infty 0 \infty 1100 \infty 01) \\
& v=(11 \infty \infty 01101000 \infty 101 \infty 1 \infty 0 \infty 000010 \infty \infty \infty) .
\end{aligned}
$$

We used computer search to find $u$ and $v$. One can check that on each set of 4 rows of $M$ there is a representative from each orbit $2-14$. Thus, $4-C A N(30,3) \leq 363$.

### 2.2 Choice of starter vectors $u$ and $v$

The problem is to find two vectors $u, v \in X^{k}$ such that on each set of 4 rows of $M$ there is a representative from each orbit $2-15$. To determine which vectors work as starters, we define the sets $d[x, y, z]$ for positive integers $x, y$ and $z$ as follows:

$$
\begin{aligned}
& d[x, y, z]=\left\{\left(u_{i}, u_{i+x}, u_{i+x+y}, u_{i+x+y+z}\right): 0 \leq i \leq k-1\right\} \bigcup \\
& \left\{\left(v_{i}, v_{i+x}, v_{i+x+y}, v_{i+x+y+z}\right): 0 \leq i \leq k-1\right\}
\end{aligned}
$$

where the subscripts are taken modulo $k$. For computational convenience, we partition the collection of $\binom{k}{4}$ choices of four distinct rows from $k$ rows into disjoint equivalence classes.

Formally, let $S$ be the set of all $\binom{k}{4}$ 4-combinations of the set $\{1,2, \ldots, k\}$. Define a binary relation $R$ on $S$ by putting

$$
\begin{gathered}
\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\} R\left\{s_{1}^{\prime}, s_{2}^{\prime}, s_{3}^{\prime}, s_{4}^{\prime}\right\} \text { iff } \\
\left\{s_{1}+d, s_{2}+d, s_{3}+d, s_{4}+d\right\}=\left\{s_{1}^{\prime}, s_{2}^{\prime}, s_{3}^{\prime}, s_{4}^{\prime}\right\} \text { for some } d \in \mathbb{N}
\end{gathered}
$$

where all of the addition is modulo $k$. Because $R$ is an equivalence relation on $S, S$ can be partitioned into disjoint equivalence classes. The equivalence class determined by $\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\} \in S$ is given by

$$
\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]=\left\{\left\{s_{1}+d, s_{2}+d, s_{3}+d, s_{4}+d\right\} \mid 0 \leq d \leq k-1\right\}
$$

Without loss of generality, we may assume that $0=s_{1}<s_{2}<s_{3}<s_{4}$ for each equivalence class representative $\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]$. As an illustration, when $X=$ $\{0,1,2, \ldots, 7\} . S$ is partitioned into 10 disjoint equivalence classes:

$$
[\{0,1,2,3\}] \quad[\{0,1,2,4\}] \quad[\{0,1,2,5\}] \quad[\{0,1,2,6\}] \quad[\{0,1,3,4\}]
$$

$$
[\{0,1,3,5\}] \quad[\{0,1,3,6\}] \quad[\{0,1,4,5\}] \quad[\{0,1,4,6\}] \quad[\{0,2,4,6\}]
$$

A distance vector $(x, y, z, w)$ is associated with every equivalence class $\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]$ where $x=s_{2}-s_{1}, y=s_{3}-s_{2}, z=s_{4}-s_{3}, w=s_{1}-s_{4} \bmod k$. The fourth distance is redundant because $x+y+z+w=k$. We rewrite the equivalence class of 4 combinations $\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]$ as

$$
[x, y, z]=\{i, i+x, i+x+y, i+x+y+z\} \mid i=0,1,2, \ldots, k-1\}
$$

For $k=8,[1,1,1]=[\{0,1,2,3\}],[1,1,2]=[\{0,1,2,4\}],[1,1,3]=[\{0,1,2,5\}]$, $[1,1,4]=[\{0,1,2,6\}],[1,2,1]=[\{0,1,3,4\}],[1,2,2]=[\{0,1,3,5\}],[1,2,3]=$ $[\{0,1,3,6\}],[1,3,1]=[\{0,1,4,5\}],[1,3,2]=[\{0,1,4,6\}],[2,2,2]=[\{0,2,4,6\}]$.

Lemma 1. Let $S$ be the set of all 4 -combinations of $\{1,2,3, \ldots, k\}$. Then $S$ can be partitioned into disjoint equivalence classes

$$
[x, y, z]=\{i, i+x, i+x+y, i+x+y+z\} \mid i=0,1,2, \ldots, k-1\}
$$

where $x=1,2, \ldots,\left\lfloor\frac{k}{4}\right\rfloor, y=x, x+1, \ldots, k-1$ and $z=x, x+1, \ldots, k-1$ such that
(i) $2 x+y+z<k$
(ii) when $x=z, x \leq y \leq\left\lfloor\frac{k-2 x}{2}\right\rfloor$

There are no further classes distinct from these.
Before proving the result, we give an example. When $S$ is the set of all 4 combinations of $\{0,1,2,3,4,5,6,7\}, S$ can be partitioned into 10 disjoint classes: $[1,1,1],[1,1,2],[1,1,3],[1,1,4],[1,2,1],[1,2,2],[1,3,1],[1,3,2]$ and $[2,2,2]$.

Proof. Let $(x, y, z, w)$ be the distance vector corresponding to equivalence class $\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]$. Classes $\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right],[x, y, z],[y, z, w],[z, w, x]$ and $[w, x, y]$ are the same. Without loss of generality, we choose $[x, y, z]$ as class representative if $x \leq y, x \leq z$. Thus $1 \leq x \leq \frac{k}{4}, y=x, x+1, \ldots, k-1$ and $z=x, x+1, \ldots, k-1$. We consider three cases: (i) $x=w$, (ii) $x=z$, (iii) $x=y$. If $w=x$, then the classes $[x, y, z]$ and $[x, x, y]$ obtained from distance vector $(x, y, z, x)$ are the same equivalence class. The classes of the form $[x, x, y]$ are generated under case (iii) as well. In order to avoid repetition, $w$ has to be strictly greater than $x$. That is, $w=k-x-y-z>x$ which implies $2 x+y+z<k$. If $z=x$, then the classes $[x, y, z]$ and $[x, w, x]$ are the same where $y+w=k-2 x$. Thus it is sufficient to consider the classes of the form $[x, y, x]$ for $y \leq\left\lfloor\frac{k-2 x}{2}\right\rfloor$ only. Hence the lemma follows.

All the equivalence classes are enumerated by the following algorithm.
Equivalence-Classes( $k$ )
Input: $k$
Output: All $[x, y, z]$ classes.
for $\mathbf{x} \leftarrow 1$ to $\frac{k}{4}$ do

```
        for \(\mathbf{y} \leftarrow x\) to \(k-1\) do
        if \(y>\frac{k-2 x}{2}\) then
            for \(\mathbf{z} \leftarrow x+1\) to \(k-2 x-y-1\) do
                add \([x, y, z]\)
        end for
        else
            if \(y==\frac{k-2 x}{2}\) and \(x==\frac{k-2 x}{2}\) then
            add \(\left[\frac{k}{4}, \frac{k}{4}, \frac{k}{4}\right]\)
        else
            for \(\mathbf{z} \leftarrow x\) to \(k-2 x-y-1\) do
                add \([x, y, z]\)
            end for
        end if
        end if
    end for
end for
```

Theorem 1. Let $X=G F(g-1) \cup\{\infty\}$ and $G=P G L(2, g-1)$. If there exists $a$ pair of vectors $u, v \in X^{k}$ such that each $d[x, y, z]$ has a representative from each of the orbits $2-15$, then there exists a $4-\operatorname{CA}(2 k g(g-1)(g-2)+g, k, g)$ covering array.

Proof. Let $u, v \in X^{k}$ be vectors such that each $d[x, y, z]$ has a representation from each of the orbits $2-15$. Using $u, v$, we create the matrix $\left[M^{G}, C\right]$. Let $\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}$ be a member in $S$. By Lemma 1, there exists three positive integers $x_{0}, y_{0}$ and $z_{0}$ such that $\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\} \in\left[x_{0}, y_{0}, z_{0}\right]$. It is given that $d\left[x_{0}, y_{0}, z_{0}\right]$ has a representative from each of the orbits 2-15. In other words, if we look at the rows $s_{1}, s_{2}, s_{3}, s_{4}$ of $M$, we see representative from each of the $g+11$ orbits. Consequently, because $\operatorname{PGL}(2, g-1)$ is 3-transitive on $X,\left[M^{G}, C\right]$ is a 4-CA $(2 k g(g-1)(g-2)+g, k, g)$.

At this stage, we make a few remarks about the size of equivalence classes defined by above choices of $x, y$ and $z$.

1. $k \not \equiv 0 \bmod 2$ :

If $k$ is an odd integer, each class contains exactly $k$ distinct choices from the collection of $\binom{k}{4}$ choices and hence there are $l=\frac{(k-1)(k-2)(k-3)}{24}$ distinct classes of size $k$.
2. $k \equiv 0 \bmod 2$ :

If $k$ is an even integer, $\frac{k}{2}$ can be written as sum of two positive integers $a$ and
$b$ where $a \leq b$ in $\left\lfloor\frac{k}{4}\right\rfloor$ different ways.
Case 1: If $k \not \equiv 0 \bmod 4$, a class of the form $[a, b, a]$ contains only $\frac{k}{2}$ distinct choices. There are total $\left\lfloor\frac{k}{4}\right\rfloor$ equivalence classes of the form $[a, b, a]$ with size $\frac{k}{2}$ and the remaining classes are of size $k$.
Case 2: If $k \equiv 0 \bmod 4$, a class of the form $[a, b, a]$ contains only $\frac{k}{2}$ distinct choices and a class of the form $[a, a, a]$ where $a=\frac{k}{4}$ contains only $\frac{k}{4}$ distinct choices. Here we get total $\frac{k}{4}-1$ equivalence classes of size $\frac{k}{2}$, exactly one class of size $\frac{k}{4}$ and the remaining classes are of size $k$.

For $k=8$, there are 10 equivalence classes. The classes $[1,3,1]$ and $[2,2,2]$ are of size 4 and 4 respectively and the remaining 8 classes are of size 8 each. Thus $8 \times 8+4+2=\binom{8}{4}$.

### 2.3 Case 2: Two vectors $u, v$ and a matrix $C_{1}$

If we do not find vectors $u$ and $v$ such that each $d[x, y, z]$ contains a representative from each of the orbits $2-15$, we look for vectors that produce an array with maximum possible coverage. In order to complete the covering conditions, we add a small matrix $C_{1}$. We give an example below to illustrate the technique.

Example 2. Let $k=21$ and $g=3$. Here we do not find vectors $u$ and $v$ such that each $d[x, y, z]$ contains a representative from each of the orbits $2-15$. For $k=$ 21, there are $285[x, y, z]$ classes. All classes $[x, y, z]$ are obtained by the algorithm Equivalence-Classes. One can check that for the vectors

$$
\begin{aligned}
& u=00001010 \propto 1 \infty \infty 10 \infty \infty 001 \infty 1 \\
& v=0000100 \infty 00 \infty 10001 \infty 111 \infty
\end{aligned}
$$

there is a representative from each orbit $2-15$ on 276 of the $d[x, y, z]$ classes. Table 1 shows nine classes which do not have representative from all the orbits:

Table 1: List of classes not having representative from all the orbits

| Class | Missing orbits |
| :---: | :---: |
| $d[1,2,2]$ | 10 |
| $d[1,5,6]$ | 2 |
| $d[1,6,12]$ | 5 |
| $d[1,13,5]$ | 9 |
| $d[2,3,8]$ | 6 |
| $d[2,7,3]$ | 10 |
| $d[2,12,3]$ | 13 |
| $d[3,6,8]$ | 6 |
| $d[3,7,7]$ | 10 |

In order to complete the covering conditions, we add a small matrix $C_{1}$.

$$
C_{1}=\left(\begin{array}{ccccccccc}
\infty & 0 & 1 & 1 & 0 & \infty & \infty & \infty & 1 \\
\infty & 1 & 1 & \infty & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & \infty & 1 & 1 & 0 & 1 & \infty & 0 \\
0 & \infty & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \infty & 1 & \infty & 0 \\
\infty & 0 & 0 & \infty & 0 & \infty & \infty & \infty & 1 \\
\infty & \infty & \infty & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & \infty & 1 & \infty & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & \infty & 1 & \infty & 0 \\
\infty & \infty & \infty & \infty & 0 & 0 & 1 & \infty & 0 \\
\infty & 1 & \infty & 1 & 0 & 1 & \infty & \infty & \infty \\
0 & 1 & 1 & 0 & \infty & 1 & \infty & 1 & 0 \\
0 & \infty & 1 & 0 & \infty & \infty & 0 & 0 & 0 \\
0 & 0 & 0 & \infty & 1 & \infty & 1 & 0 & 0 \\
\infty & 0 & 0 & 1 & \infty & 0 & 0 & 0 & \infty \\
\infty & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & \infty & \infty & 0 & 1 & 0 & 1 & \infty & 1 \\
0 & \infty & \infty & \infty & \infty & \infty & 1 & 0 & 0 \\
\infty & 0 & 0 & 1 & \infty & 0 & \infty & \infty & 1
\end{array}\right) .
$$

We use computer search to find matrix $C_{1}$. This matrix has the property that every choice of four rows in $[1,2,2],[2,7,3]$ and $[3,7,7]$ contains at least one representative from orbit 10 ; every choice of four rows in $[2,3,8]$ and $[3,6,8]$ contains at least one representative from orbit 6 ; each choice of four rows in $[1,5,6],[1,6,12]$, $[1,13,5]$ and $[2,12,3]$ contains at least one representative from orbit $2,5,9$ and 13 respectively. We also need to use the following matrix

$$
C=\left(\begin{array}{ccc}
0 & 1 & \infty \\
0 & 1 & \infty \\
\vdots & \vdots & \vdots \\
0 & 1 & \infty
\end{array}\right)
$$

to ensure the coverage of all identical 4-tuples. Therefore, $\left[M^{G}, C_{1}^{G}, C\right]$ is a 4$C A(315,21,3)$.

### 2.4 Case 3: One vector $u$ and a matrix $C_{1}$

For $k=37$ to 58 , we use one starter vector and a $C_{1}$ matrix of order $k \times \ell$ with $\ell<k$. Tables 2 3 4 4 and 5give a list of starter vectors and matrix $C_{1}$ that improves the best known bounds. When the new bound is marked with an asterisk, post-optimization has been applied (see Section 3.2).

## 3 Improving the solutions

We examine two methods to obtain small improvements on the computational results obtained.

### 3.1 Extending a solution

Until this point, starter vectors have been developed by applying a cyclic rotation of the starter vectors in addition to the action of PGL on the symbols. As in [13], one can also consider fixing one row, and developing the remaining $k-1$ cyclically. This can be viewed as first finding a solution of the type already described on $k-1$ rows, but requiring an additional property. For the 4 -subsets of $\{0, \ldots, k-2\}$, equivalence classes are defined as before, with arithmetic modulo $k-1$ :

$$
\left[\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right]=\left\{\left\{s_{1}+d, s_{2}+d, s_{3}+d, s_{4}+d\right\} \mid 0 \leq d \leq k-2\right\}
$$

For 3-subsets $\left\{t_{1}, t_{2}, t_{3}\right\}$ of $\{0, \ldots, k-2\}$ we define further equivalence classes as

$$
\left[\left\{t_{1}, t_{2}, t_{3}, k-1\right\}\right]=\left\{\left\{t_{1}+d, t_{2}+d, t_{3}+d, k-1\right\} \mid 0 \leq d \leq k-2\right\}
$$

If we can place an entry in position $k-1$ to extend the length of each starter vector so that every one of the (old and new) equivalence classes represents each of the orbits $2-15$, we obtain a 4-CA of degree $k$.

The potential advantage of this approach is that a solution for degree $k-1$ can sometimes be extended to one of degree $k$ without increasing the size of the covering array produced. Indeed we found that the solutions for $k-1 \in\{32,34,35\}$ do ensure that the new equivalence classes also represent each of the orbits $2-15$. Hence we obtain the following improvements. Old indicates the bound obtained by applying our methods to $k$; Improved gives the bound by applying the method to $k-1$ and ensuring that the new equivalence classes represent all orbits:

| $k$ | Old | Improved | $k$ | Old | Improved | $k$ | Old | Improved |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 33 | 399 | 387 | 35 | 423 | 411 | 36 | 435 | 423 |

### 3.2 Randomized Post-optimization

Nayeri, Colbourn, and Konjevod [14] describe a post-optimization strategy which, when applied to a covering array, exploits flexibility of symbols in an attempt to reduce its size. We applied their method to the arrays provided here, and to arrays obtained by removing one or more rows. Because the method is described in detail elsewhere, we simply report improvements for eight values of $k$. Basic gives the bound from starter vectors, Improved gives the bound on $4-\operatorname{CAN}(k, 3)$ after postoptimization:

| $k$ | Basic | Improved | $k$ | Basic | Improved | $k$ | Basic | Improved |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 19 | 309 | 300 | 20 | 309 | 303 | 21 | 309 | 305 |
| 22 | 309 | 307 | 27 | 351 | 345 | 28 | 363 | 360 |
| 34 | 411 | 410 | 37 | 435 | 433 |  |  |  |

## 4 Covering arrays with budget constraints problem

In this section we present several strength four testing arrays with high coverage measure for $g \geq 3$. The coverage measure $\mu_{4}(A)$ of a strength four testing array $A$ is defined by the ratio between the number of distinct 4 -tuples contained in the column vectors of $A$ and the total number of 4 -tuples given by $\binom{k}{4} g^{4}$. Note that the coverage measure of a covering array is always one. For computational convenience, we rewrite the coverage measure in terms of equivalence classes $[x, y, z]$ and $d[x, y, z]$ as follows:

$$
\mu_{4}(A)=\frac{\sum_{x, y, z}|[x, y, z]| \times \text { number of distinct 4-tuples covered by } d[x, y, z]}{\binom{k}{4} g^{4}} .
$$

We search by computer to find vectors $v$ with very high coverage measures. Tables 6 and 7 show vectors with high coverage, the number of test cases ( $n$ ) generated by our technique, and the best known size with full coverage. Comparison of our construction with best known covering array sizes shows that our construction produces significantly smaller testing arrays with very high coverage measures.

## 5 Conclusions

In this paper, we present a construction method of strength four covering arrays with three symbols that combines an algebraic technique with computer search. This method improves the current best known upper bounds on 4-CAN $(k, g)$ for $21 \leq k \leq 74$ and $g=3$. We have also proposed a construction of strength four covering arrays with budget constraints. In order to test software with 25 parameters each having three values, our construction can generate a test suite with 153 test cases that ensure with probability 0.93 that software failure cannot be caused due to interactions of two, three or four parameters whereas the best known covering array in [4] requires 363 test cases for full coverage. The results show that the proposed method could reduce the number of test cases significantly while compromising only slightly on the coverage.

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Table 2: Improved strength four covering arrays for $g=3$.

| $k$ | Starter vectors and matrix $C_{1}$ | New bound | Old bound |
| :---: | :---: | :---: | :---: |
| 21 |  | 305* | 315 |
| 22 |  | 307* | 315 |
| 27 |  | 345* | 378 |

Table 3: Improved strength four covering arrays for $g=3$ (continued).

| $k$ | Starter vectors and matrix $C_{1}$ | New bound | Old bound |
| :---: | :---: | :---: | :---: |
| 28 |  | 360* | 383 |
|  | $u=(011 \infty 11 \infty \infty \infty 001 \infty \infty \infty 1 \infty 10 \infty \infty 0 \infty 1100 \infty 01)$ |  |  |
| 30 | $v=(11 \infty \infty 01101000 \infty 101 \infty 1 \infty 0 \infty 000010 \infty \infty \infty)$ | 363 | 393 |
|  | $u=(\infty 1100010 \infty 111 \infty 1 \infty 010 \infty \infty 0100 \infty \infty 0 \infty \infty 010)$ |  |  |
| 32 | $v=(\infty 000 \infty 1 \infty \infty 0 \infty 000110 \infty \infty 100 \infty 0 \infty 11 \infty 11111)$ | 387 | 409 |
| 33 | Obtained from $C A(387,32,3)$ | 387 | 417 |
|  | $u=(00 \infty 101 \infty \infty \infty 1001 \infty 010 \infty \infty 0 \infty 0 \infty 01 \infty \infty 0 \infty 11111)$ |  |  |
| 34 | $v=(1100 \infty 1 \infty 01 \infty 10110 \infty \infty 0 \infty \infty 011 \infty 101001 \infty 000)$ | 410* | 423 |
| 35 | Obtained from $C A(411,34,3)$ | 411 | 429 |
|  | $u=01 \infty 0 \infty \infty 1000 \infty 01 \infty \infty 0 \infty 1 \infty 111 \infty \infty \infty 01 \infty 01000 \infty 1$ |  |  |
| 35 | $v=0 \infty 00111 \infty 0 \infty 110 \infty 11 \infty 110 \infty 010010000 \infty 1 \infty \infty 0$ | 423 | 429 |

Table 4: Improved strength four covering arrays for $g=3$ (continued).

| $k$ | Starter vectors and matrix $C_{1}$ | New bound | Old bound |
| :---: | :---: | :---: | :---: |
| 36 | Obtained from $C A(423,35,3)$ | 423 | 441 |
| 37 | $\begin{aligned} & u=(001 \propto 10 \infty 1 \infty 01000 \infty 1100 \infty 101111 \infty 001 \infty \infty \infty \infty 00 \infty) \\ & C_{1}: 37 \times 35 \text { matrix } \end{aligned}$ | 433* | 441 |
| 39 | $\begin{aligned} & u=(001 \infty \infty 11 \infty 11 \propto 0001 \infty 11 \infty 101 \infty \infty \infty 1 \infty 0 \infty 0010 \infty 00 \infty \infty 0) \\ & C_{1}: 39 \times 34 \text { matrix } \end{aligned}$ | 441 | 453 |
| 41 | $\begin{aligned} & u=(\infty 001 \infty 010 \infty \infty 0 \infty 0101111 \infty \infty 011 \infty \infty 10000 \infty 0 \infty \infty 10 \infty 0 \infty 1) \\ & C_{1}: 41 \times 34 \text { matrix } \end{aligned}$ | 453 | 465 |
| 42 | $\begin{aligned} & u=(\infty 0111 \propto 1 \infty \infty 100 \infty 101 \infty 01000 \infty 011 \infty 1010011 \propto 00 \infty 1 \infty \infty \infty) \\ & C_{1}: 42 \times 35 \text { matrix } \end{aligned}$ | 465 | 471 |
| 46 | $\begin{aligned} & u=(\infty 00000 \infty 1100010 \infty 101 \infty \infty 1 \infty 01 \infty 00110 \infty \infty \infty \infty 11 \infty 1101 \infty 101 \infty) \\ & C_{1}: 46 \times 33 \text { matrix } \end{aligned}$ | 477 | 483 |
| 47 | $\begin{aligned} & u=(\infty 0011 \propto 1101 \propto 1 \infty 000 \infty 1 \infty 01 \infty 00 \infty 111010 \infty 00 \infty \infty \infty 10 \infty \infty 1 \infty \infty 1 \infty \infty) \\ & C_{1}: 47 \times 33 \text { matrix } \end{aligned}$ | 483 | 489 |
| 48 | $\begin{aligned} & u=(01 \propto \infty \infty 11 \infty 01 \infty 1010111 \infty \infty 001 \infty \infty \infty 0 \infty 110010 \infty 0 \infty \infty 000100 \infty 00 \infty) \\ & C_{1}: 48 \times 33 \text { matrix } \end{aligned}$ | 489 | 495 |
| 51 | $\begin{aligned} & u=(\infty 0 \infty \infty 101011 \infty 000 \infty \infty 11 \infty 1 \infty 1001 \infty \infty \infty \infty \infty 11 \\ & \infty 0 \infty 1 \infty 01111001001 \infty 00) \\ & C_{1}: 51 \times 32 \text { matrix } \end{aligned}$ | 501 | 507 |

Table 5: Improved strength four covering arrays for $g=3$ (continued).

| $k$ | Starter vectors and matrix $C_{1}$ | New bound | Old bound |
| :---: | :---: | :---: | :---: |
| 55 | $\begin{aligned} & u=(1 \infty \infty 1 \infty 1 \infty 0 \infty 111 \infty \infty 1 \infty 0010 \infty 00 \infty 0011011 \infty 1 \infty 0 \\ & 00 \infty 11 \infty \infty 0101 \infty 001110 \infty \infty) \\ & C_{1}: 55 \times 30 \text { matrix } \end{aligned}$ | 513 | 519 |
| 57 | $\begin{aligned} & u=(\infty 10 \infty \infty \infty 0011 \infty 01 \infty 10 \infty 11001 \infty 1 \infty \infty 0011 \infty \infty 110 \\ & 110111010 \infty \infty 1 \infty 0 \infty 0000 \infty 01) \\ & C_{1}: 57 \times 29 \text { matrix } \end{aligned}$ | 519 | 531 |
| 58 | $\begin{aligned} & u=(\infty 0 \infty \infty 00101 \infty 0010 \infty 0 \infty 1 \infty 1000 \infty 0 \infty 11001 \infty 00010 \infty 111 \\ & \infty \infty \infty 11011011 \infty \infty 0 \infty 0 \infty) \\ & C_{1}: 58 \times 29 \text { matrix } \end{aligned}$ | 525 | 531 |
| 63 | $\begin{aligned} & u=(1101 \infty 10 \infty 100 \infty \infty \infty 00101 \infty \infty 0 \infty 0 \infty \infty 1 \infty 010 \infty 11 \infty \infty \infty 01 \\ & 10 \infty 10110001 \infty 0 \infty 11 \infty \infty 0 \infty 0 \infty 11) \\ & C_{1}: 63 \times 26 \end{aligned}$ | 537 | 549 |
| 67 | $\begin{aligned} & u=(010101 \infty 1100 \infty 100 \infty 11 \infty \infty \infty \infty 0110 \infty 01111 \infty \infty 1011 \infty 0 \infty \\ & 1101 \infty 0 \infty \infty 0 \infty 101 \infty \infty 1 \infty \infty 10000 \infty 00) \\ & C_{1}: 67 \times 25 \end{aligned}$ | 555 | 561 |
| 70 | $\begin{aligned} & u=(1 \infty 001 \infty 11 \infty 1 \infty \infty \infty 0 \infty 11 \infty 0 \infty 0 \infty 1 \infty 00011 \infty 0 \infty \infty \infty \infty 111 \\ & \infty 0101001 \infty 010011 \infty \infty 010000 \infty 10 \infty \infty 1100) \\ & C_{1}: 70 \times 24 \end{aligned}$ | 567 | 573 |
| 72 | $\begin{aligned} & u=(\infty \infty 000 \infty 1010 \infty \infty \infty \infty \infty 010111000 \infty 11011 \infty 011101 \infty 0 \infty \infty 1 \infty 00 \\ & \infty 1 \infty 1 \infty \infty 010 \infty 101100 \infty 01 \infty \infty \infty 1 \infty \infty 0 \infty) \\ & C_{1}: 72 \times 24 \end{aligned}$ | 573 | 579 |
| 74 | $\begin{aligned} & u=(1 \infty 0010 \infty \infty 01 \infty \infty \infty 111 \infty \infty 1 \infty \infty 0100 \infty \infty \infty \infty 10 \infty 1011011 \infty \\ & 001100001 \infty \infty 0 \infty 0 \infty 0 \infty \infty 101100 \infty 1 \infty 01 \infty 111 \infty) \\ & C_{1}: 74 \times 24 \end{aligned}$ | 585 | 591 |

Table 6: A comparison of the number of test cases $(n)$ produced by our construction with high coverage measure and best known $n$ for full coverage. For $g=5$, the elements of $G F(4)$ are represented as $\mathbf{0 , 1}, \mathbf{2}$, and 3 ; here 2 stands for $x$ and 3 stands for $x+1$.

| $(\mathrm{g}, \mathrm{k})$ | Vector $v$ with good coverage | $\begin{gathered} \text { Our Results } \\ n(\mu) \end{gathered}$ | $\begin{gathered} \text { Best known } \\ n \text { [4] } \end{gathered}$ |
| :---: | :---: | :---: | :---: |
| $(3,16)$ | $00001001 \infty \infty 011 \infty 1 \infty$ | 99 (0.828) | 237 |
| $(3,17)$ | $0000010 \infty \infty 101 \infty 01 \infty 1$ | 105 (0.851) | 282 |
| $(3,18)$ | $00010 \infty 0 \infty 1001 \infty 111 \infty \infty$ | 111 (0.864) | 293 |
| $(3,19)$ | $000010010 \propto 01 \infty 0 \infty 111 \infty$ | 117 (0.883) | 305 |
| $(3,20)$ | $0000110101 \infty 0 \infty 10 \infty \infty 11 \infty$ | 123 (0.892) | 314 |
| $(3,21)$ | $00001010 \infty 1 \infty \infty 10 \infty \infty 001 \infty 1$ | 129 (0.906) | 315 |
| $(3,22)$ | $0000011 \infty 0 \infty 0110 \infty 1 \infty \infty \infty 01 \infty$ | 135 (0.913) | 315 |
| $(3,23)$ | $0000001 \infty \infty 0101 \propto 10 \propto 10 \infty \infty \infty 1$ | 141 (0.923) | 315 |
| $(3,24)$ | $00000001 \infty \infty 0101 \propto 10 \infty 101 \infty \infty 1$ | 147 (0.924) | 315 |
| $(3,25)$ | $0000000011 \infty 0 \infty 011 \infty 01 \infty 0 \infty 11 \infty$ | 153 (0.930) | 363 |
| $(3,28)$ | $1 \infty \infty 00 \infty \infty 1 \infty 01101111 \infty 0 \infty 0101 \infty \infty \infty 1$ | 171 (0.957) | 383 |
| $(3,29)$ | $010 \propto 00 \propto 1 \infty 0 \infty \infty \infty 101 \propto 00 \propto 000111 \infty 10$ | 177 (0.961) | 392 |
| $(3,30)$ | $011 \infty 11 \infty \infty \infty 001 \infty \infty \infty 1 \infty 10 \infty \infty 0 \infty 1100 \infty 01$ | 163 (0.969) | 393 |
| $(3,35)$ | $01 \infty 0 \infty \infty 1000 \infty 01 \infty \infty 0 \infty 1 \infty 111 \infty \infty \infty 01 \infty 01000 \infty 1$ | 213 (0.979) | 429 |
| $(3,36)$ | $11 \infty 0110 \infty \infty 00 \sim 111101011 \infty 001 \infty \infty \infty \infty \infty \infty 100 \infty 0 \infty$ | 219 (0.981) | 441 |
| $(3,38)$ | $1 \infty 1 \infty 111 \infty \infty 010 \infty 10 \infty \infty 00010 \infty \infty 0 \infty \infty \infty 1101 \infty \infty 100 \infty$ | 231 (0.985) | 447 |
| $(3,39)$ | $001 \infty \infty 11 \infty 11 \infty 0001 \infty 11 \infty 101 \infty \infty \infty 1 \infty 0 \infty 0010 \infty 00 \infty \infty 0$ | 237 (0.986) | 453 |
| $(3,40)$ | $100 \infty \infty 00001 \infty \infty 1 \infty 10 \infty 000 \infty \infty \infty 0 \infty 10 \infty \infty 1 \infty 1 \infty 0111 \infty 01$ | 243 (0.988) | 465 |
| $(4,18)$ | $00010021 \infty \infty \infty \sim 21020 \infty 2$ | 436 (0.851) | 760 |
| $(4,19)$ | $0000121011 \infty 01 \infty 0 \infty 221$ | 460 (0.866) | 760 |
| $(4,20)$ | $0000112101202 \infty 0221 \infty 2$ | 484 (0.878) | 760 |
| $(4,21)$ | $0000011021010 \propto 2 \infty 0221 \infty$ | 508 (0.887) | 1012 |
| $(4,22)$ | $0000001102 \infty 02021 \infty \infty 01 \infty 1$ | 532 (0.894) | 1012 |
| $(4,23)$ | $00000001210210 \infty \infty 20112 \infty 1$ | 556 (0.898) | 1012 |
| $(4,24)$ | $00000000121 \infty 011 \infty 02 \infty 0 \infty 112$ | 580 (0.899) | 1012 |
| $(4,25)$ | $000000000121220 \infty 011 \infty 2012 \infty$ | 604 (0.901) | 1012 |
| $(4,26)$ | $00100 \propto 2221110102 \propto 0022 \propto 020 \infty 2$ | 628 (0.921) | 1012 |
| $(4,27)$ | $0100 \propto 2221110102 \infty 0022 \infty 020 \infty 2$ | 652 (0.928) | 1012 |
| $(4,28)$ | $01110 \infty 0102 \infty 021110022001 \infty 1001$ | 676 (0.933) | 1012 |
| $(4,29)$ | $0 \infty \infty 122101 \infty 000220200221220 \infty 02$ | 702 (0.937) | 1012 |
| $(4,30)$ | $10 \propto 20 \propto 020 \propto 2 \infty 2 \infty 01 \propto 2222 \propto 022002 \infty 1$ | 726 (0.943) | 1012 |

Table 7: A comparison of the number of test cases $(n)$ produced by our construction with high coverage measure and best known $n$ for full coverage (continued).

| $(g, k)$ | Vector $v$ with good coverage | Our Results | Best known |
| :---: | :---: | :---: | :---: |
|  |  | $n(\mu)$ | $n$ [4] |
| $(5,21)$ | $110131300 \infty 30010 \infty \infty 3203$ | $1265(0.834)$ | 1865 |
| $(5,22)$ | $3 \infty 32011200 \infty \infty 00 \infty 0 \infty 10010$ | $1325(0.842)$ | 1865 |
| $(5,23)$ | $0002 \infty 03100 \infty 203021332320$ | $1385(0.854)$ | 1865 |
| $(5,24)$ | $003 \infty 21022212300032302310$ | $1445(0.860)$ | 1865 |
| $(5,25)$ | $\infty 200 \infty 0 \infty \infty 31020 \infty 300303 \infty \infty 33$ | $1505(0.869)$ | 2485 |
| $(5,26)$ | $202002211000 \infty 0121031 \infty \infty 2300$ | $1565(0.873)$ | 2485 |
| $(5,27)$ | $\infty \infty 03002030 \infty 000 \infty 11 \infty 0031301 \infty 3$ | $1625(0.880)$ | 2485 |
| $(5,28)$ | $013333130320 \infty 1 \infty 1003200310300$ | $1685(0.883)$ | 2485 |
| $(5,29)$ | $00012212 \infty 010 \infty 3110031020031010$ | $1745(0.891)$ | 2485 |
| $(5,30)$ | $33001 \infty 0 \infty 000330 \infty \infty 010012 \infty 1313001$ | $1805(0.894)$ | 2485 |
| $(5,31)$ | $033 \infty 21333010313 \infty 303320030012020$ | $1865(0.895)$ | 2485 |
| $(5,32)$ | $310031000 \infty 330130321 \infty \infty 03031111310$ | $1925(0.897)$ | 2485 |
| $(5,33)$ | $\infty 0010 \infty \infty 3 \infty 0 \infty 2 \infty 01 \infty 00 \infty 12222 \infty \infty 03 \infty 020 \infty$ | $1985(0.904)$ | 2485 |
| $(5,34)$ | $\infty \infty 3 \infty 00101001 \infty 0 \infty 001 \infty 002 \infty 01110231112$ | $2045(0.906)$ | 2485 |
| $(5,35)$ | $1203003303 \infty 0 \infty 013233310 \infty 032020003220$ | $2105(0.906)$ | 2485 |
| $(5,36)$ | $12022 \infty 3203230023223220001010200 \infty 2230$ | $2165(0.912)$ | 2485 |
|  |  |  |  |
| $(6,25)$ | $000403014003033404320 \infty 1 \infty \infty$ | $3006(0.811)$ | 6325 |
| $(6,26)$ | $\infty 0 \infty 40021404010013010011444$ | $3126(0.819)$ | 6456 |
| $(6,27)$ | $433 \infty \infty 01 \infty \infty 20 \infty 03020 \infty \infty 0 \infty 00401 \infty$ | $3246(0.826)$ | 6606 |
| $(6,28)$ | $4023031100232200 \infty 21 \infty \infty 2020020$ | $3366(0.829)$ | 6714 |
| $(6,29)$ | $00 \infty 40023103301343401230334400$ | $3486(0.834)$ | 6852 |
| $(6,30)$ | $1 \infty \infty \infty 42 \infty 4040004 \infty 104 \infty 03034 \infty \infty 0300$ | $3606(0.836)$ | 6966 |
| $(6,31)$ | $44122002 \infty 2000020202031 \infty 42044001$ | $3726(0.838)$ | 7092 |
| $(6,32)$ | $44441341 \infty 424000 \infty \infty 040004410103400$ | $3846(0.846)$ | 7200 |
| $(6,33)$ | $0330344 \infty 0232133100313000030 \infty 4303 \infty$ | $3966(0.855)$ | 7320 |

