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Colouring of $(P_3 \cup P_2)$-free graphs

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Abstract

The class of $2K_2$-free graphs and its various subclasses have been studied in a variety of contexts. In this paper, we are concerned with the colouring of $(P_3 \cup P_2)$-free graphs, a super class of $2K_2$-free graphs. We derive a $O(\omega^3)$ upper bound for the chromatic number of $(P_3 \cup P_2)$-free graphs, and sharper bounds for $(P_3 \cup P_2$, diamond)-free graphs, where ω denotes the clique number. By applying similar proof techniques we obtain chromatic bounds for $(2K_2$, diamond)-free graphs. The last two classes are perfect if $\omega \geq 5$ and ≥ 4 respectively.

Keywords: Colouring, Chromatic number, Clique number, $2K_2$-free graphs, $(P_3 \cup P_2)$-free graphs, Diamond, Perfect graphs
2000 MSC: 05C15, 05C17

1. Introduction

A graph G is said to be H-free, if G does not contain an induced copy of H. More generally, a class of graphs $\mathscr{G}$ is said to be $(H_1, H_2, \cdots)$-free if every $G \in \mathscr{G}$ is H_i-free, for $i \geq 1$. The class of $2K_2$-free graphs and its subclasses are subject of research in various contexts: domination (El-Zahar and Erdös [10]), size (Bermond et al. [2], Chung et al. [9]), vertex colouring (Wagon [19], Nagy and Szentmiklossy [16], Gyárfás [12]), edge colouring (Erdös and Nesetril [11]) and algorithmic complexity (Blazsik et al. [3]). Here we are concerned with the colouring of $(P_3 \cup P_2)$-free graphs, a super class of $2K_2$-free graphs. A comprehensive result of Kral et al. [15] states that the decision problem of COLOURING H-free graphs is P-time solvable if H is an induced subgraph of P_4 or $P_3 \cup P_1$,and it is NP-complete for any other graph H. In particular, COLOURING $2K_2$-free graphs is NP-complete. However, there have been several studies to obtain tight upper and lower bounds for the chromatic number of $2K_2$-graphs. A problem of Gyárfás [12] asks for the smallest function $f(x)$ such that $\chi(G) \leq f(\omega(G)$, for every G belonging to the class of $2K_2$-free graphs, where $\chi(G)$ and $\omega(G)$ respectively denote the chromatic number and clique number of G. This problem is still open. In this respect, an often quoted result is due to Wagon [19]. It states that if a graph G is $2K_2$-free, then $\chi(G) \leq \binom{\omega(G)+1}{2}$. We look more closely at Wagon's proof and obtain a $O(\omega^3)$ upper bound for the chromatic number of $(P_3 \cup P_2)$-free graphs, and sharper bounds for $(P_3 \cup P_2$, diamond)-free graphs. By applying similar proof techniques we obtain chromatic bounds for $(2K_2$, diamond)-free graphs. The last two classes are perfect if the clique number is ≥ 5 and ≥ 4 respectively. The classes of $(H$, diamond)-free graphs and $(H_1, H_2$, diamond)-free graphs, for various graphs H, H_1 and H_2, have been studied

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in many papers; see Arbib and Mosca [1], Brandstädt [5], Choudum and Karthick [7], Karthick and Maffrey [14], Gyárfás [12], and Randerath and Schiermeyer [17]. See also a comprehensive book on problems of graph colourings by Jensen and Toft [13] and an extensive book of Brandtstädt et al. [6], for interesting subclasses and superclasses of $2K_2$-free graphs.

2. Terminology and Notation

We follow standard terminology of Bondy and Murty [4], and West [20]. All our graphs are simple and undirected. If u, v are two vertices of a graph $G(V,E)$, then their adjacency is denoted by $u \leftrightarrow v$, and non-adjacency by $u \not\leftrightarrow v$. P_n, C_n and K_n respectively denote the path, cycle and complete graph on n vertices. A chordless cycle of length ≥ 5 is called a *hole*. If $S \subseteq V(G)$, then $[S]$ denotes the subgraph induced by S. If S and T are two disjoint subsets of $V(G)$, then $[S,T]$ denotes the set of edges $\{st \in E(G) : s \in S$ and $t \in T\}$. A subset Q of $V(G)$ is called a *clique* if any two vertices in Q are adjacent. The *clique number* of G is defined to be $\max\{|Q| : Q$ is a clique in $G\}$; it is denoted by $\omega(G)$. A clique Q is called a *maximum clique* if $|Q| = \omega(G)$). A subset I of $V(G)$ is called an *independent set* if no two vertices in I are adjacent. A *k-vertex colouring* or a *k-colouring* or a *colouring* is a function $f : V(G) \to \{1,2,\cdots,k\}$ such that $f(u) \neq f(v)$, for any two adjacent vertices u, v in G. It is also referred to as a proper colouring of G for emphasis. The *chromatic number* $\chi(G)$ of G is defined to be $\min\{k : G$ admits a k-colouring$\}$. If $G_1, G_2, \cdots, G_k$ are vertex disjoint graphs, then $G_1 \cup G_2 \cup \cdots \cup G_k$ denotes the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. If $G_1 \simeq G_2 \simeq \cdots \simeq G_k \simeq H$, for some H, then $G_1 \cup G_2 \cup \cdots \cup G_k$ is denoted by kH. The three graphs which appear frequently in this paper are shown in Fig.1.

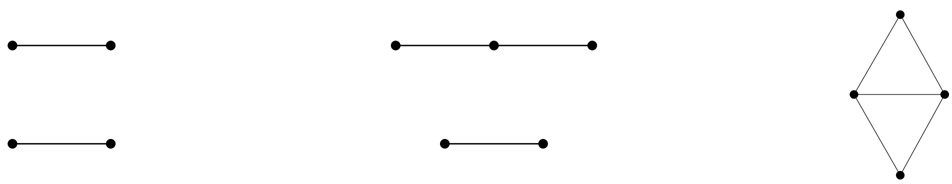

Figure 1: $2K_2$, $P_3 \cup P_2$, Diamond

3. A partition of the vertex set of a graph.

Throughout this paper we use a particular partition of the vertex set of a graph $G(V,E)$ and use its properties. Some of these properties are due to Wagon [19], but are restated for ready reference. In what follows, ω denotes the clique number of a graph under consideration.

Let A be a maximum clique in G with vertices $1,2,\cdots,\omega$. We iteratively define the sets C_{ij} in the lexicographic order of pairs of vertices i, j of A.

$C = \phi$
for i : 1 to ω
for j : $i+1$ to ω
$C_{ij} = \{v \in V(G) - C \,|\, v \not\leftrightarrow i$ and $v \not\leftrightarrow j\}$;
$C = C \cup C_{ij}$;

end
end

By definition, there are $\binom{\omega}{2}$ number of C_{ij} sets and these are pairwise disjoint. Also, every vertex in C_{ij} is adjacent to every vertex k of A, where $1 \leq k < j, k \neq i$. Moreover, every vertex in $V(G) - A$ which is non-adjacent to two or more vertices of A is in some C_{ij}. So, every vertex $v \in V(G) - (A \cup C)$ is adjacent to all the vertices of A or $|A| - 1$ vertices of A. The former case is impossible, since A is a maximum clique. Hence we define the following sets. For $a \in A$, let

$$I_a = \{v \in V(G) - (A \cup C) \mid v \leftrightarrow x, \forall x \in A - \{a\} \text{ and } v \not\leftrightarrow a\}.$$

By the above remarks, we conclude that $(A, \bigcup_{i,j} C_{ij}, \bigcup_{a \in A} I_a)$ is a partition of $V(G)$.

4. Colouring of $(P_3 \cup P_2)$-free graphs

We first observe a few properties of the sets C_{ij} and I_a, and then obtain an $O(\omega^3)$ upper bound for the chromatic number of a $(P_3 \cup P_2)$-free graph.

Theorem 1. *If a graph G is $(P_3 \cup P_2)$-free, then $\chi(G) \leq \frac{\omega(\omega+1)(\omega+2)}{6}$.*

Proof. Let A be a maximum clique in G. Let $(1, 2, 3, \cdots, \omega)$ be a vertex ordering of A. Since G is $(P_3 \cup P_2)$-free, the sets C_{ij} and I_a possess a few more properties, in addition to the ones stated in section 3.

Claim 1: Each induced subgraph $[C_{ij}]$ of G is P_3-free and hence it is a disjoint union of cliques.
If some C_{ij} contains an induced $P_3 = (x, y, z)$, then $[\{x, y, z\} \cup \{i, j\}] \simeq P_3 \cup P_2$, a contradiction.

Claim 2: Each I_a is an independent set.
If some I_a contains an edge vw, then $A \cup \{v, w\} - \{a\}$ is a clique of size $\omega + 1$, a contradiction to the maximality of $|A|$.

Claim 3: $\omega\left([C_{ij}]\right) \leq \omega - (j-2)$, where $j \geq 2$
Let B be a maximum clique in $[C_{ij}]$. Every vertex in B is adjacent to every vertex in $K = \{1, 2, \cdots, j-1\} - \{i\} \subseteq A$, by the definition of C_{ij}. So, $B \cup K$ is a clique of G. Hence, $\omega(G) \geq |B \cup K| = \omega([C_{ij}]) + |K| = \omega([C_{ij}]) + j - 2$. Hence the claim.

Table 1 gives the the number of sets C_{ij}, for a fixed j, where $i < j$ and $2 \leq j \leq \omega$. The entries of the last column, follow by Claim 3.

We now properly colour G as follows:

(1) Colour the vertices $1, 2, \cdots, \omega$ of A with colours $1, 2, \cdots, \omega$ respectively.

(2) Colour the vertices of C_{ij} with $\omega([C_{ij}])$ new colours, $1 \leq i < j \leq \omega$. By Claim 1, $[C_{ij}]$ is a disjoint union of cliques and hence one can properly colour $[C_{ij}]$ with $\omega([C_{ij}])$ colours. Note also that one requires at most $\omega - (j-2)$ colours, by Claim 3.

(3) Each vertex in I_a is given the colour of $a \in A$.

Table 1: Clique size of each $[C_{ij}]$

j	C_{ij}'s	Number of C_{ij}'s	$\omega([C_{ij}]) \leq$
2	C_{12}	1	ω
3	C_{13}, C_{23}	2	$\omega - 1$
4	C_{14}, C_{24}, C_{34}	3	$\omega - 2$
.	.	.	.
.	.	.	.
.	.	.	.
j	$C_{1j}, C_{2j}, \ldots, C_{j-1\,j}$	$j-1$	$\omega - (j-2)$
.	.	.	.
.	.	.	.
.	.	.	.
ω	$C_{1\,\omega}, C_{2\,\omega}, \ldots, C_{\omega-1\,\omega}$	$\omega - 1$	2

It is a proper colouring of G by Claims 1, 2 and 3. We first estimate the number of colours used in step (2) to colour the vertices of C (see Table 1) and then estimate the total number of colours used to colour G entirely.

$$\begin{aligned}
\chi([C]) &\leq 1(\omega) + 2(\omega - 1) + 3(\omega - 2) + \ldots + (\omega - 1)2 \\
&= \sum_{k=1}^{\omega-1} k(\omega + 1 - k) \\
&= \sum_{k=1}^{\omega-1} k(\omega + 1) - \sum_{k=1}^{\omega-1} k^2 \\
&= (\omega + 1)\frac{(\omega - 1)(\omega)}{2} - \frac{(\omega - 1)(\omega)(2\omega - 2 + 1)}{6} \\
&= \frac{\omega(\omega - 1)(\omega + 4)}{6}
\end{aligned}$$

Hence,

$$\begin{aligned}
\chi(G) &\leq |A| + \chi([C]) \\
&= \omega + \frac{\omega(\omega - 1)(\omega + 4)}{6} \\
&= \frac{\omega(\omega + 1)(\omega + 2)}{6}
\end{aligned}$$

□

Theorem 2. *If a graph G is $(P_4 \cup P_2)$-free, then $\chi(G) \leq \frac{\omega(\omega+1)(\omega+2)}{6}$.*

Proof. The bound for the chromatic number of $(P_3 \cup P_2)$-free graphs holds for $(P_4 \cup P_2)$-free graphs too. In this case, each $[C_{ij}]$ is P_4-free and hence perfect, by a result of Seinsche [18]. So, we can properly colour each $[C_{ij}]$ with at most $\omega(C_{ij}) \leq \omega - (j-2)$ colours, and the entire G with at most $\frac{\omega(\omega+1)(\omega+2)}{6}$ colours , as in the proof of Theorem 1. □

We next consider $(P_3 \cup P_2$, diamond)-free graphs and obtain sharper bounds for the chromatic number. If $\omega = 1$, then obviously chromatic number is 1. So in the following, all graphs have $\omega \geq 2$.

Theorem 3. *If a graph G is $(P_3 \cup P_2$, diamond$)$-free, then*

$$\chi(G) \leq \begin{cases} \omega+2 & \text{if } \omega = 2 \\ \omega+3 & \text{if } \omega = 3 \\ \omega+1 & \text{if } \omega = 4 \end{cases}$$

and G is perfect if $\omega \geq 5$.

Proof. We continue to use the terminology and notation of sections 2 and 3. In particular, we use the sets A, C_{ij}, I_a, and Claims 1, 2 and 3.

Claim 4: If G is C_5-free, then it is a perfect graph.
Clearly, every hole $C_{2k+1}, k \geq 3$ contains an induced $P_3 \cup P_2$, and the complement $\overline{C}_{2k+1}, k \geq 3$ of the hole contains an induced diamond. So G is $(C_{2k+1}, \overline{C}_{2k+1})$-free for all $k \geq 3$. Hence if G is C_5-free, then G is perfect, by the Strong Perfect Graph Theorem [8].

Claim 5: $C_{ij} = \emptyset$, for every $j \geq 4$.
On the contrary, let $x \in C_{ij}$, for some $j \geq 4$. Then by the definition of C_{ij}, there exist two distinct vertices $p, q \in \{1,2,3\} \subseteq A$ such that $x \leftrightarrow p$ and $x \leftrightarrow q$. But then $[\{x, j, p, q\}] \simeq$ diamond, a contradiction.

So, we conclude that $C = C_{12} \cup C_{13} \cup C_{23}$, for $j \geq 4$.

Claim 6: If $a \in A$, then I_a is an empty set if $\omega \geq 3$, and it is an independent set if $\omega = 2$.
If $\omega \geq 3$, and $x \in I_a$, for some $a \in A - \{1,2\}$, then $[\{x, a, 1, 2\}] \simeq$ diamond, a contradiction; if a = 1 or 2, then [{x,1,2,3}] is a diamond. If $\omega = 2$, then the assertion follows by Claim 2.

Therefore, $V(G) = A \cup C_{12} \cup C_{13} \cup C_{23}$, if $\omega \geq 3$.

Recall that by Claim 3, $\omega([C_{13}]) \leq \omega - 1$, and $\omega([C_{23}]) \leq \omega - 1$. But $[C_{12}]$ may contain an ω-clique. However, we have the following claim.

Claim 7: $\omega([C_{12}]) \leq \omega - 1$, if $\omega(G) \geq 3$, and $C_{23} \neq \emptyset$ or $C_{13} \neq \emptyset$
On the contrary suppose $[C_{12}]$ contains an ω-clique Q, and for definiteness suppose $C_{23} \neq \emptyset$ (if $C_{13} \neq \emptyset$, proof is similar). Let $x \in C_{23}$. If x is adjacent to all the vertices of Q or $|Q| - 1$ vertices of Q, then we have an $(\omega+1)$-clique or a diamond in G, both impossible. Else, there exist two vertices u and v in Q such that $x \not\leftrightarrow u$ and $x \not\leftrightarrow v$. Then $[\{x, 1, 2\} \cup \{u, v\}] \simeq P_3 \cup P_2$, a contradiction. Hence the claim.

Claim 8: $[C_{13}, A - \{2\}] = \emptyset$, and $[C_{23}, A - \{1\}] = \emptyset$.
If there exists an edge $xi \in [C_{13}, A - \{2\}]$, then $[\{x, i, 1, 2\}] \simeq$ diamond, a contradiction. Similarly, $[C_{23}, A - \{1\}] = \emptyset$

We now prove the theorem for different values of ω, by making the cases as stated in the theorem.

- $\omega = 2$; so $A = \{1, 2\}$.

Colouring G with four colours is easy in this case, since $V(G) = A \cup C_{12} \cup I_1 \cup I_2$, $\omega([C_{12}]) \leq$

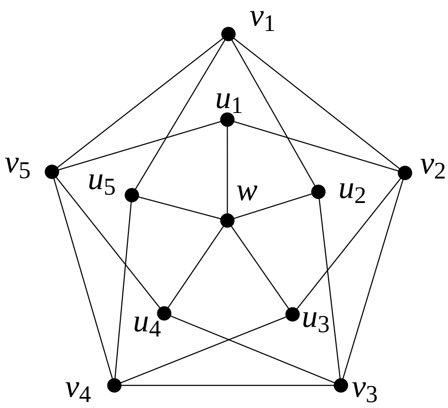

Figure 2: Mycielski-Grötzsch graph

$\omega = 2$, and I_1, I_2 are independent sets, by Claim 6. Moreover, $\omega[C_{12}] \leq \omega(G) = 2$. The following is a proper 4-colouring of G:

(1) Colour the vertices 1 and 2 of A with colours 1 and 2 respectively.

(2) Colour $[I_1]$ with colour 1.

(3) Colour $[I_2]$ with colour 2.

(4) Colour $[C_{12}]$ with two new colours.

An extremal $(P_3 \cup P_2$, diamond)-free graph G with $\omega(G) = 2$, and $\chi(G) = 4$ is the Mycielski-Grötzsch graph; see Fig. 2. It is well known that this graph has clique number 2 and chromatic number 4. The graph is clearly diamond free since it is triangle free. It can be observed that this graph is $(P_3 \cup P_2)$-free by selecting every edge P_2 and then verifying that the second neighborhood of P_2, is P_3-free. There are not too many cases for such a verification because of the symmetry of edges; we need to choose only three kinds of edges: v_1v_2, v_1u_2 and u_1w.

- $\omega = 3$; so $A = \{1,2,3\}$.

At the outset, recall that every $I_a = \emptyset$, by Claim 6. So, $V(G) = A \cup C_{12} \cup C_{23} \cup C_{13}$. Moreover, $\omega[C_{12}] \leq 2$, $\omega[C_{13}] \leq 2$, $\omega[C_{23}] \leq 2$, by Claims 7 and 3. We colour G with six colours as follows:

(1) Colour the vertices 1, 2, 3 of A with colours 1, 2, 3 respectively.

(2) Colour $[C_{12}]$ with colours 1 and 2.

(3) Colour $[C_{23}]$ with colours 3 and 4.

(4) Colour $[C_{13}]$ with colours 5 and 6.

It is a proper colouring by the above observations.

Remarks:

(i) If some C_{ij} is empty, we may not require all the six colours.

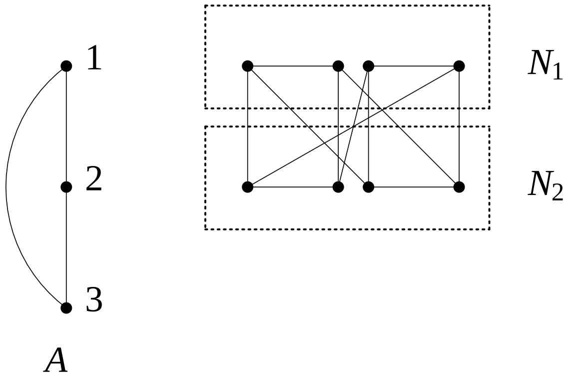

Figure 3: ($P_3 \cup P_2$, diamond)-free graph with $\omega = 3$ and $\chi = 4$

(ii) We do not have extremal graphs with chromatic number 6.

(iii) However, we do have a graph with chromatic number 4 (see Fig. 3). In this figure, A is an ω-clique and $N_i \subseteq V(G)$ such that every vertex of N_i is adjacent to i and only i of A, $i \in \{1,2\}$.

- $\omega = 4$; so $A = \{1,2,3,4\}$.

We colour G with five colours by considering two cases.

Case 1: $[C_{23}, C_{13}] \neq \emptyset$; let $ab \in [C_{23}, C_{13}]$.
Clearly, $[\{a,b,2\}] \simeq P_3$.

Claim 9: a is an isolated vertex in $[C_{23}]$*, and b is an isolated vertex in* $[C_{13}]$.
Suppose, $a \leftrightarrow c$, for some $c \in C_{23}$. If $c \leftrightarrow b$, then $[\{a,b,c,1\}] \simeq$ diamond, a contradiction. If $c \not\leftrightarrow b$, then $[\{a,b,c\} \cup \{3,4\}] \simeq P_3 \cup P_2$, since no vertex of $C_{23} \cup C_{13}$ is adjacent to the vertex $4 \in A$, by Claim 8. Hence, we conclude that a is an isolated vertex in C_{23}. Similarly, b is an isolated vertex in C_{13}.

Claim 10: C_{23} *and* C_{13} *are independent sets.*
Suppose there exists an edge cd in $[C_{23}]$, where $c \neq a$ and $d \neq a$, by Claim 9. If $c \not\leftrightarrow b$ and $d \not\leftrightarrow b$, then $[\{a,b,2\} \cup \{c,d\}] \simeq P_3 \cup P_2$. Next, without loss of generality, suppose that $c \leftrightarrow b$. Then $[\{a,b,c\} \cup \{3,4\}] \simeq P_3 \cup P_2$, by Claim 8 and by the definition of C_{ij}'s, a contradiction. Hence, C_{23} is independent. Similarly C_{13} is independent.

We now colour G with five colours as follows:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.

(2) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.

(3) Colour $[C_{13}]$ with colour 3.

(4) Colour $[C_{23}]$ with colour 4.

It is a proper colouring by Claims 8, 7 and 10.

Case 2: $[C_{23}, C_{13}] = \emptyset$.
If both C_{23} and C_{13} are empty sets, then G is C_5-free, since $[C_{12}]$ is P_3-free and any 5-cycle contains at most two vertices of A. So, G is perfect, by Claim 4. If one of the sets C_{23} or C_{13} is nonempty, then we have the following assertion.

Claim 11: If C_{23} or C_{13} is non empty, then the other is independent.
Suppose $C_{23} \neq \emptyset$ and $x \in C_{23}$. If uv is an edge in $[C_{13}]$, then $[\{x,1,3\} \cup \{u,v\}] \simeq P_3 \cup P_2$, a contradiction. Hence C_{13} is independent. Similarly, C_{23} is independent if $C_{13} \neq \emptyset$.

Without loss of generality, we henceforth assume that $C_{23} \neq \emptyset$. Since C_{13} is nonempty or empty, we consider two subcases.

Subcase 2.1: C_{13} is nonempty.
This implies that both C_{23} and C_{13} are independent sets, by Claim 11.

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.

(2) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.

(3) Colour $[C_{13}]$ with colour 3.

(4) Colour $[C_{23}]$ with colour 3.

It is a proper 5-colouring by Claims 7, 11 and the fact that $[C_{23}, C_{13}] = \emptyset$.

Subcase 2.2: C_{13} is empty.
We now examine this subcase based on number of components in C_{23} and the maximum cliques in C_{12}.
Case 2.2.a: C_{23} has exactly one component.
Recall that every component of C_{23} is K_1, K_2 or K_3, by Claim 3. If the component is K_1, then colour G with five colours as follows:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.

(2) Colour $[C_{23}]$ with colour 3.

(3) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.

It is a proper 5-colouring by Claim 7 and by our assumptions.
If the component is K_2 or K_3, let cd be an edge in $[C_{23}]$ (see Fig. 4). We claim that C_{12} is independent. Else, there is an edge ab in $[C_{12}]$. If c is neither adjacent to a nor adjacent to b, then $[\{c,1,2\} \cup \{a,b\}] \simeq P_3 \cup P_2$, a contradiction. Without loss of generality, assume that $a \leftrightarrow c$. But then $a \not\leftrightarrow d$; else, $[\{a,c,d,1\}] \simeq$ diamond. By definition of C_{12} and C_{23}, no vertex in $\{a,c,d\}$ is adjacent to vertex 2 of A. By Claim 8, a is adjacent to at most one vertex of $A - \{1,2\}$, namely 3 or 4. So $[\{a,c,d\} \cup \{2,3\}] \simeq P_3 \cup P_2$ or $[\{a,c,d\} \cup \{2,4\}] \simeq P_3 \cup P_2$, a contradiction. Hence, C_{12} is independent. Recall that $\omega([C_{23}]) \leq 3$, by Claim 3.
We colour G with four colours:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.

(2) Colour C_{23} with colours 2, 3 and 4.

(3) Colour C_{12} with colour 1.

It is a proper 4-colouring by Claims 3 and 8.

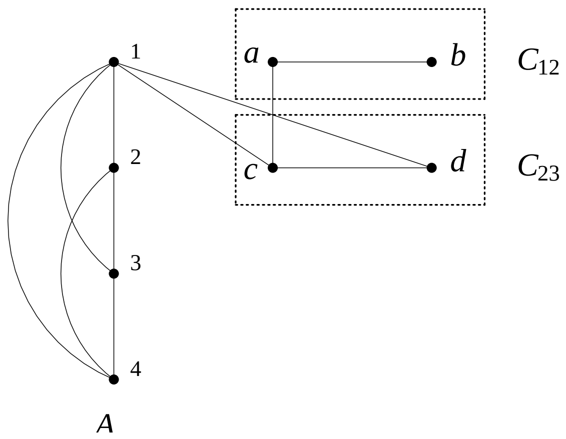

Figure 4: $[C_{23}]$ has one component

Case 2.2.b: C_{23} has ≥ 2 components; let x and y be vertices of two distinct components (see Fig. 5).
Our first claim is that $\omega([C_{12}]) \leq 2$. On the contrary suppose that $[\{a,b,c\}]$ is a triangle in $[C_{12}]$. Since $\{x,1,2\}$ induces a P_3, x is adjacent to every vertex of the triangle; else we have an induced diamond or $P_3 \cup P_2$ in G. Similarly y is adjacent to every vertex of the triangle. Then $[\{a,b,x,y\}] \simeq$ diamond. Hence, $\omega([C_{12}]) \leq 2$. So we can colour G with 4 colours as follows:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.

(2) Colour C_{23} with colours 3 and 4.

(3) Colour C_{12} with colour 1 and 2.

It is a proper 4-colouring by the above observations and Claim 8.

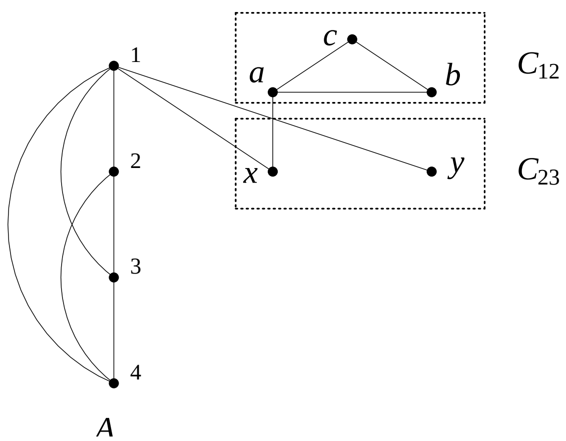

Figure 5: $[C_{23}]$ has more than one component

- $\omega \geq 5$.

It is enough to show that G is C_5-free, in view of Claim 4.4. On the contrary, suppose that G contains an induced C_5. As before, $V(G) - A = C = C_{12} \cup C_{13} \cup C_{23}$. Since at most two vertices of C_5 can belong to the clique A, a $P_3 = (a,b,c)$ is an induced subgraph of $[C]$. Since each C_{ij} is P_3-free, either (*i*) two vertices are in one C_{ij}, and the third vertex is in one of the other two C_{ij}'s, or (*ii*) each C_{ij} contains a vertex.

Claim 12: A vertex of C_{12} is adjacent to at most one vertex of A.
The claim is obvious for $\omega = 2,3$. Next, assume that $\omega \geq 4$. If some vertex $x \in C_{12}$ is adjacent to two distinct vertices say, i and j of $A - \{1,2\}$, then $[\{1,x,i,j\}] \simeq$ diamond, a contradiction.

Hence by the above claim, for any two vertices $x,y \in C_{12}$, there is a vertex, say 5, in A which is neither adjacent to x nor y. Also, by Claim 8, $[C_{13} \cup C_{23}, \{3,4,5\}] = \emptyset$. So, whether (*i*) or (*ii*) holds, there exists an edge ij in $[A]$ such that $[\{a,b,c\} \cup \{i,j\}] \simeq P_3 \cup P_2$, a contradiction. For the choice of an appropriate edge ij, it is enough if we consider the following four cases:

(a) If P_3 is an induced subgraph of $[\{C_{12} \cup C_{13}\}]$, then $[\{a,b,c,1,5\}] \simeq P_3 \cup P_2$.

(b) If P_3 is an induced subgraph of $[\{C_{12} \cup C_{23}\}]$, then $[\{a,b,c,2,5\}] \simeq P_3 \cup P_2$.

(c) If P_3 is an induced subgraph of $[\{C_{13} \cup C_{23}\}]$, then $[\{a,b,c,4,5\}] \simeq P_3 \cup P_2$.

(d) If (*ii*) holds, then $[\{a,b,c,4,5\}] \simeq P_3 \cup P_2$, where without loss of generality we assume that the vertex of (a,b,c) that is in C_{12} is adjacent to the vertex $3 \in A$.

□

5. $(2K_2$, diamond)-free graphs

The Claims of Section 4 are valid for ($2K_2$, diamond)-free graphs too. So we continue to use the Claims made in Sections 3 and 4. In what follows, we assume that graphs have clique number at least 2, as before.

Theorem 4. *If a graph G is $(2K_2$, diamond)-free, then*

$$\chi(G) \leq \begin{cases} \omega+1 & \text{if } \omega = 2 \\ \omega & \text{if } \omega \geq 3 \end{cases}$$

and G is perfect if $\omega \geq 4$.

Proof. Since the proof is similar to the proof of Theorem 3, we give an outline. As before, consider the partition $(A, \bigcup C_{ij}, \bigcup I_a)$ of $V(G)$. In this case, every C_{ij} is K_2-free, and so it is an independent set.
If $\omega = 2$, then $V(G) = A \cup C_{12} \cup I_1 \cup I_2$. So one can easily colour G with three colours.
Next suppose $\omega \geq 3$. If $j \in A$, then $I_j = \emptyset$. Else, some $x \in I_j$. So, if $a,b \in A - \{j\}$, then $[\{x,j,a,b\}] \simeq$ diamond, a contradiction. Also, $C_{ij} = \emptyset$, if $j \geq 4$; else G contains an induced diamond. Hence $V(G) = C_{12} \cup C_{13} \cup C_{23}$. An ω-colouring of G is obtained as follows:

(1) Colour the vertices $1,2,\cdots,\omega$ of A, by colours $1,2,\cdots,\omega$.

(2) Colour every vertex of C_{12} with colour 1, colour every vertex of C_{13} with colour 3, colour every vertex of C_{23} with colour 2.

Remark: There exist $(2K_2$, diamond)-free graphs with $\omega = 3$, which are not perfect. See Fig. 6, where each circled vertex is multiplied by an independent set.

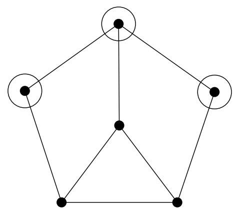

Figure 6: Graphs that are not perfect and have $\chi(G) = \omega(G)$

Now we prove perfectness for $\omega \geq 4$.
It is similar to the proof of Theorem 3, Case $\omega = 5$. By Claim 4.4, it is enough if we show that G is C_5-free. On the contrary, if G contains an induced 5-cycle, then $C(= C_{12} \cup C_{13} \cup C_{23})$ contains an edge xy of the 5-cycle. Since C_{ij}'s are independent, no $[C_{ij}]$ contains xy. We use Claims 8 and 12 and arrive at a contradiction:

(a) If $xy \in [C_{12}, C_{13}]$, then $[\{x,y,1,3\}] = 2K_2$ or $[\{x,y,1,4\}] = 2K_2$.

(b) If $xy \in [C_{12}, C_{23}]$, then $[\{x,y,2,3\}] = 2K_2$ or $[\{x,y,2,4\}] = 2K_2$.

(c) If $xy \in [C_{13}, C_{23}]$, then $[\{x,y,1,3\}] = 2K_2$ or $[\{x,y,1,4\}] = 2K_2$.

So, G is C_5-free and hence it is perfect. $\square$

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