On the pseudoachromatic index of the complete graph III *

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Abstract

Let Π_q be the projective plane of order q, let $\psi(m) := \psi(L(K_m))$ the pseudoachromatic number of the complete line graph of order m, let $a \in \{3, 4, \ldots, \frac{q}{2} + 1\}$ and $m_a = (q+1)^2 - a$. In this paper, we improve the upper bound of $\psi(m)$ given by Araujo-Pardo et al. [J Graph

Theory 66 (2011), 89–97] and Jamison [Discrete Math. 74 (1989), 99–115] in the following values: if $x \ge 2$ is an integer and $m \in \{4x^2 - x, \dots, 4x^2 + 3x - 3\}$ then $\psi(m) \le 2x(m - x - 1)$.

On the other hand, if q is even and there exists Π_q we give a complete edge-colouring of K_{m_a} with $(m_a - a)q$ colours. Moreover, using this colouring we extend the previous results for $a = \{-1, 0, 1, 2\}$ given by Araujo-Pardo et al. in [J Graph Theory 66 (2011), 89–97] and [Bol. Soc. Mat. Mex. (2014) 20:17–28] proving that $\psi(m_a) = (m_a - a)q$ for $a \in \{3, 4, \ldots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}$.

1 Introduction

The pseudoachromatic number of a graph $\psi(G)$, which is the number of colours in a maximum complete vertex-colouring of G, has attracted the attention of several researchers since its introduction by Gupta [8] in 1969 (see also [4, 7]). Being a hard parametre to calculate, it is in order to search for bounds in general classes of graphs. In this series of papers [1, 2], we endeavour to calculate the exact value of the *pseudoachromatic index* of the complete graph K_m , for a wide set of values of m (see also [3, 6, 10, 11, 12, 13]), which we denote by

$$\psi(m) := \psi(L(K_m))$$

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More precisely, on the one hand we study in detail the interaction of a couple of natural upper bounds for $\psi(m)$ in terms of the size of chromatic classes in a complete edge colouring of K_m ; namely, if x is the size of its smallest chromatic class, then (cf., [11])

$$\psi(m) \le \min\left\{2x(m-x-1)+1, \left\lfloor\frac{\binom{m}{2}}{x+1}\right\rfloor\right\}.$$

Each of these bounds by themselves does not give much information of the phenomena, but together they seem to dominate precisely what is going on. Therefore, we study those intervals of values of m where each of these bounds dominates the phenomena; viz., denoting by

$$g_m(x) := 2x(m-x-1) + 1$$
 and $f_m(x) := \left\lfloor \frac{\binom{m}{2}}{x+1} \right\rfloor$,

it will be proven the following

Theorem 1. If $x \ge 2$ is an integer, then

$$\psi(m) \leq \begin{cases} g_m(x) - 1 & m \in \{4x^2 - x, \dots, 4x^2 + 3x - 3\} \\ g_m(x) & m \in \{4x^2 + 3x - 2, 4x^2 + 3x - 1\} \\ f_m(x) & m \in \{4x^2 + 3x, \dots, 4(x+1)^2 - (x+1) - 1\} \end{cases}$$

On the other hand, using results given in [1, 2] and supported by the combinatorial structure of projective planes of even order, we exhibit optimal complete edge colourings of K_m ; we show that

Theorem 2. Let $q \ge 4$ be an even natural number, and let $n = q^2 + q + 1$, $a \in \{-1, 0, \dots, \frac{q}{2} + 1\}$ and $m_a = n + q - a$. If the projective plane Π_q of order q exists then

$$\psi(m_a) \ge (m_a - a)q.$$

With these results together, we obtain the following set of exact values for the pseudoachromatic index of the complete graph

Theorem 3. Let q > 4 be a power of 2, let $n = q^2 + q + 1$, $a \in \{-1, 0, \dots, \left\lceil \frac{1 + \sqrt{4q+9}}{2} \right\rceil - 1\}$ and $m_a = n + q - a$, then

$$\psi(m_a) = (m_a - a)q.$$

2 Proof of Theorem 1

In order to give the proof we will use the following proposition (see [11]).

Proposition 1. If $x \ge 2$ is an integer, then

$$\psi(m) \leq \begin{cases} g_m(x) & m \in \{4x^2 - x, \dots, 4x^2 + 3x - 1\} \\ f_m(x) & m \in \{4x^2 + 3x, \dots, 4(x+1)^2 - (x+1) - 1\}. \end{cases}$$

Proof of Theorem 1. Let $x \ge 2$ an integer and let $n \in \{4x^2 - x, \ldots, 4x^2 + 3x - 3\}$. By Proposition 1, we already know that $\psi(n) \le g_n(x)$. We will prove that $\psi(n) \le g_n(x) - 1$. To do this we suppose that $\psi(n) = g_n(x)$ and finally arrive to a contradiction. Let $\varsigma \colon V \to \{1, \ldots, g_n(x)\}$ be a complete colouring. First of all we will prove that any class of colour can not have less than x edges. Suppose there exists a colour class C with s edges such that s < x, then C will be adjacent to at most $\binom{s}{2} - s + 2s(n-2s) = 2s(n-s-1)$ edges, but 2s(n-s-1) < 2x(n-x-1), in consequence C could not be adjacent to all other colour classes. Then, each colour class has at least x edges. Suppose now that there exists a colour class C with exactly x edges. Then it is clear that C is adjacent to exactly 2x(n-x-1) other edges and also they must all have different colours, otherwise C does not meet all the other colour classes –note that the only way to get this is when C is a matching. Since each colour class has at least x edges, then the number of colour classes with more than x edges is at most $\binom{n}{2} - xg_n(x)$, hence, there are at least $g_n(x) - \{\binom{n}{2} - xg_n(x)\}$ colour classes with x edges. Now we will see that there are at least two colour classes of size x. For this just observe that

$$2 \le g_n(x) - \left\{\binom{n}{2} - x(g_n(x))\right\} \text{ if and only if } n^2 - (4x^2 + 4x + 1)n + 4x^3 + 8x^2 + 2x + 2 \le 0,$$

i.e., $\left(n - \frac{4x^2 + 4x + 1 - \sqrt{D_1}}{2}\right) \left(n - \frac{4x^2 + 4x + 1 + \sqrt{D_1}}{2}\right) \le 0$

where $4x^2 + 2x - 3/2 < \sqrt{D_1} = \sqrt{16x^4 + 16x^3 - 8x^2 - 7} < 4x^2 + 2x - 1$, which is equivalent to $\sqrt{D_1} = 4x^2 + 2x - 3/2 + \epsilon$ for some $0 < \epsilon < 1/2$ and then

$$n \in \left[x + \frac{5}{4} - \frac{\epsilon}{2}, 4x^2 + 3x - \frac{1}{4} + \frac{\epsilon}{2}\right] \cap \{4x^2 - x, \dots, 4x^2 + 3x - 3\} = \{4x^2 - x, \dots, 4x^2 + 3x - 3\}$$

Let C be a colour class of size x. This class is a matching with 2x vertices and $\binom{2x}{2} - x$ edges in the induced subgraph $\langle C \rangle$ that are not in C. Therefore, each one of these edges has a different colour and each one of these colours is in a class with more than x edges because they are adjacent to two edges of C.



Figure 1: K_n

Let C' be another colour class of size x. C' meets C in a vertex u. In Fig 1 we give a description of K_n . The rest of the 2x - 2 edges meets C' through u in only one vertex and they have different colours and also their colour classes are larger than x because they meet two vertices of C'. As

before there are at least $g_n(x) - \{\binom{n}{2} - x(g_n(x))\}$ colour classes of size x then there are at most $(2x-2)(g_n(x) - \binom{n}{2} - x(g_n(x))\} - 1)$ chromatic classes of size greater than x and, hence, we have the following:

$$\binom{n}{2} - x(g_n(x)) \ge \binom{2x}{2} - x + (2x - 2)(g_n(x) - \{\binom{n}{2} - x(g_n(x))\} - 1)$$

Therefore,

$$(2x-1)n^2 - (8x^3 + 4x^2 - 6x - 1)n + 8x^4 + 12x^3 - 12x^2 - 2x \ge 0$$

i.e.,
$$\left(n - \frac{8x^3 + 4x^2 - 6x - 1 - \sqrt{D_2}}{4x - 2}\right) \left(n - \frac{8x^3 + 4x^2 - 6x - 1 + \sqrt{D_2}}{4x - 2}\right) \ge 0$$

where $\sqrt{D_2} = \sqrt{64x^6 - 144x^4 + 80x^3 - 4x^2 + 4x + 1} = 8x^3 - 9x + 5 + r_2$ and then

$$\left(n - \left(x + \frac{5}{4} + r_3\right)\right) \left(n - \left(4x^2 + 3x - \frac{9}{4} + r_4\right)\right) \ge 0$$

i.e.,
$$n \in \left[4x^2 + 3x - \frac{9}{4} + r_4, \infty\right) \cap \{4x^2 - x, \dots, 4x^2 + 3x - 3\} = \emptyset$$

Then we have a contradiction and we conclude that if $x \ge 2$ is an integer and $n \in \{4x^2 - x, \dots, 4x^2 + 3x - 3\}$ then

$$\psi(n) \le g_n(x) - 1$$

2.1 Proof of Theorem 2.

In order to prove Theorem 2 we only need to show that $\psi(m_a) \ge (m_a - a)q$. We will do this by exhibiting a complete edge-colouring of K_{m_a} with $(m_a - a)q$ colours.

For the construction of such an edge-colouring, we need some definitions and remarks.

A projective plane consists of a set of n points, a set of lines, and an incidence relation between points and lines having the following properties:

- 1. Given any two distinct points there is exactly one line incident with both of them.
- 2. Given any two distinct lines there is exactly one point incident with both of them.
- 3. There are four points, such that no line is incident with more than two of them.

Such plane has $n = q^2 + q + 1$ points (for some number q) and n lines; each line contains q + 1 points and each point belongs to q + 1 lines. The number q is called the *order* of the projective plane. A projective plane of order q is called Π_q . If q is a prime power there exists Π_q , which is called the *algebraic projective plane* since it arises from finite fields.

Let \mathbb{P} be the set of points of Π_q and let $\mathbb{L} = \{l_1, \ldots, l_n\}$ be the set of lines of Π_q . Now, we will identify the points of Π_q with the set of vertices of the complete graph K_n . In a natural way, the set of points of each line of Π_q induces a subgraph isomorphic to K_{q+1} in K_n . For each line $l_i \in \mathbb{L}$, let $l_i = (V(l_i), E(l_i))$ be the subgraph of K_n induced by the set of q + 1 points of l_i . By the properties of the projective plane, for every pair $\{i, j\} \subseteq \{1, \ldots, n\}, |V(l_i) \cap V(l_j)| = 1$ and $\{E(l_1), \ldots, E(l_n)\}$ is a partition of $E(K_n)$. In this way, when we say that a graph G isomorphic to K_n is a representation of the projective plane Π_q , we will understand that V(G) is identified with the points of Π_q and that there is a family of subgraphs (lines) $\{l_1, \ldots, l_n\}$ of G such that for each line l_i of Π_q , l_i is the subgraph induced by the set of points of l_i .

Given two graphs G and H, the directed sum, $G \oplus H$, is defined as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup (V(G) \times V(H))$ (the set of edges $V(G) \times V(H)$ for short; we write it as V(G)V(H)-edges). Given $S \subseteq V(G)$, $G \setminus S$ is the subgraph of G induced by $V(G) \setminus S$.

Let *m* be a positive integer. Given an edge-colouring $\Gamma: E(K_m) \to \mathcal{C}$, we will say that a vertex $x \in V(K_m)$ is an *owner* of a set of colours $\mathcal{C}' \subseteq \mathcal{C}$ whenever for every $c \in \mathcal{C}'$ there is $y \in V(K_m)$ such that $\Gamma(xy) = c$; and given a subgraph *G* of K_m , we will say that *G* is an *owner* of a set of colours $\mathcal{C}' \subseteq \mathcal{C}$ if each vertex of *G* is an owner of \mathcal{C}' . By this, Γ is a complete edge-colouring if for every pair of colours in \mathcal{C} there is a vertex in K_m which is an owner of both colours.

Lemma 1. Let $n = q^2 + q + 1$, with q a natural number such that Π_q exists, and let t be a positive integer. Let G be a subgraph of K_{n+t} isomorphic to K_n and let G be a representation of Π_q . Let $\Gamma: E(K_{n+t}) \to C$ be an edge-colouring of K_{n+t} . Suppose that each line l_i of G is an owner of a set of colours $\mathcal{C}_i \subseteq \mathcal{C}$. Then for every pair of colours $\{c_1, c_2\} \subseteq \bigcup_{i=1}^n \mathcal{C}_i$ there is $x \in V(G)$ which is an owner of c_1 and c_2 .

Proof. Let $\{c_1, c_2\} \subseteq \bigcup_{i=1}^n C_i$. If there is $i \in \{1, \ldots, n\}$ such that $\{c_1, c_2\} \subseteq C_i$, then since l_i is an owner of C_i it follows that each $x \in V(l_i)$ is an owner of c_1 and c_2 . If $c_1 \in C_i$ and $c_2 \in C_j$ $(i \neq j)$, there is $x \in V(G)$ such that $x = V(l_i) \cap V(l_j)$, and then since l_i and l_j are owners of C_i and C_j , respectively, x is an owner of c_1 and c_2 .

Now, we define different edge-colourations for some special graphs that will be used later.

It is well known (see [5], [9]) that any complete graph of even order r admits a 1-factorization and that any complete graph of odd order r admits a 2-factorization by Hamiltonian cycles.

Definition 1. Let r be an even integer. An edge-colouring $\Gamma: E(K_r) \to \{1, 2, ..., r-1\}$ will be said to be of *Type 1* if for every $i \in \{1, 2, ..., r-1\}$ the set $\{xy \in E(K_r): \Gamma(xy) = i\}$ is a perfect matching of K_r .

Definition 2. Let r be an odd integer. An edge-colouring $\Gamma: E(K_r) \to \{1, \ldots, r\}$ will be said to be of *Type* 2 if we obtain Γ in the following way: Let G be the graph (isomorphic to K_{r+1}) obtained by adding to K_r a new vertex x_0 and all the $x_0V(K_r)$ -edges. Let Γ' be an edge-colouring of Type 1 of G and, for every $e \in E(K_r)$, let $\Gamma(e) := \Gamma'(e)$.

Definition 3. Let r be an odd integer and $x, y \in V(K_r)$. An edge-colouring $\Gamma: E(K_r - xy) \rightarrow \{1, \ldots, r-2\}$ will be said to be of *Type 3* if we obtain Γ in the following way: Let G be the graph (isomorphic to K_{r-1}) obtained by deleting the vertex x and all the $xV(K_{r-1})$ -edges. Let Γ' be an edge-colouring of Type 1 of G and, for every $e \in E(K_r - xy)$, let $\Gamma(e) := \Gamma'(e)$ if $e \in E(G)$, and $\Gamma(xw) := \Gamma'(yw)$ for every $w \in V(G) - y$.

Definition 4. Let r be an odd integer. An edge-colouring $\Gamma_i: E(C_r) \to \{i, i + \frac{r-1}{2}\}$ will be said to be of *Type 4* if we obtain Γ_i in the following way: Let G be the graph (isomorphic to P_r) obtained by deleting the edge $x_0 y \in E(C_r)$. Let $\Gamma'_i: E(G) \to \{i, i + \frac{r-1}{2}\}$ be a proper edge-colouring of G(remember that *proper* means that each vertex has different colours in its edges) and, for every $e \in E(C_r)$, let $\Gamma_i(e) := \Gamma'_i(e)$ be if $e \in E(G)$, and $\Gamma_i(x_0 y) := \Gamma'_i(x_0 w)$ for $w = N(x_0) - y$. Observe that x_0 is an owner of one colour.

Definition 5. Let r be an odd integer. An edge-colouring $\Gamma: E(K_r) \to \{1, \ldots, r-1\}$ will be said to be of *Type 5* in x_0 if we obtain Γ in the following way: Let $\{G_1, \ldots, G_{\frac{r-1}{2}}\}$ be a 2-factorization of K_r such that $G_i = C_r$ for each $i \in \{1, \ldots, \frac{r-1}{2}\}$ and x_0 is the same in each G_i . Let Γ_i be a edge-colouring of G_i of Type 4 and, for every $e \in E(K_r)$, let $\Gamma(e) := \Gamma_i(e)$ be if $e \in G_i$. Observe that x_0 is an owner of $\frac{r-1}{2}$ colours.

2.2 The edge-colouring

Proof of Theorem 2. To prove this theorem we will exhibit a complete edge-colouring of K_{m_a} with $(m_a - a)q$ colours. The cases for $a \in \{-1, 0, 1, 2\}$ are given in [1, 2]. Let $a \in \{3, 4, \ldots, \frac{q}{2} + 1\}$. Let C be a set of $(m_a - a)q$ colours and let $\{C_1, C_2, \ldots, C_n\}$ be a partition of C in the following way: C_i is a set of q colours, for $1 \le i \le q - 2a + 3$; C_i is a set of q - 1 colours, for $q - 2a + 4 \le i \le a(q - 1) + q + 1$; C_i is a set of q + 1 colours, for $a(q - 1) + q + 2 \le i \le q^2 + q$ and C_n is a set of q - 1 colours.

Let G be a subgraph of K_{m_a} isomorphic to K_n and let $H = K_{m_a} \setminus V(G)$. Clearly H is isomorphic to K_{q-a} and $K_{m_a} = G \oplus H$. Let G be a representation of Π_q and let $L = \{l_1, \ldots, l_n\}$ be the set of lines of G.

Let $V(H) = \{h_1, \ldots, h_{q-a}\}$, let $v_0 \in V(G)$ and let ℓ be a line l of G such that $v_0 \notin V(\ell)$.

Let W, U and V be a partition of $V(\ell)$ such that $W = \{w_1, \ldots, w_{q-2a+3}\}$, $U = \{u_1, \ldots, u_{a-2}\}$ and $V = \{v_1, \ldots, v_a\}$, then $\ell = \langle W \rangle \oplus \langle U \rangle \oplus \langle V \rangle$. Let $L_0 = \{\ell_x : x \in \ell \text{ and } v_0 \in \ell_x\} \subseteq L$. Let L_W , L_U and L_V be a partition of L_0 such that $L_W = \{\ell_x : x \in W\}$, $L_U = \{\ell_x : x \in U\}$ and $L_V = \{\ell_x : x \in V\}$.

For $i \in \{1, ..., a\}$, let $L_i = \{\ell_j^{v_i} : v_i \in \ell_j^{v_i}, j \in \{1, ..., q-1\}\}$ be the set of lines $l \neq \ell$ such that $l \notin L_0$. For $i \in \{1, ..., a\}$ and $j \in \{1, ..., q-1\}$, let $Z = \{z_j^{v_i} = \ell_j^{v_i} \cap \ell_{v_{i+1}} : i \neq a\} \cup \{z_j^{v_a} = \ell_j^{v_a} \cap \ell_{v_1}\} \subseteq V(G)$ and let $Y = V(G) - Z \cup V(\ell) \cup \{v_0\}$. Without loss of generality, let $l_i = \ell_{w_i}$ for $i \in \{1, \ldots, q-2a+3\}$, $l_{i+q-2a+3} = \ell_{u_i}$ for $i \in \{1, \ldots, a-2\}$, $l_{i+q-a+1} = \ell_{v_i}$ for $i \in \{1, \ldots, a\}$ and let $l_{i(q-1)+2+j} = \ell_j^{v_i}$ for $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, q-1\}$, $L' = \{l_i : a(q-1) + q + 2 \le i \le q^2 + q\}$ and $\ell = l_n$. In Fig 2 we give a description of K_{m_a} .



Figure 2: K_{m_a} .

With the aim of defining some subset of edges of K_{m_a} we first define a special function h:

For $i \in \{1, \dots, q - 2a + 3\}$ and $j \in \{1, \dots, \frac{q}{2} - a + 1\}$, let

$$h(i+j) = \begin{cases} i+j & \text{if } i+j \le q-2a+3\\ i+j-(q-2a+1) & \text{if } i+j > q-2a+3 \end{cases}$$

and let

$$E'_{w_i} = \{w_i w_{h(i+1)}, \dots, w_i w_{h(i+\frac{q}{2}-a+1)}, w_i u_1, \dots, w_i u_{a-2}, w_i v_a\}$$

be a set of $\frac{q}{2}$ edges.

Now, we will define other subsets of edges of K_{m_a} .

For $x \in U \cup W$, let

$$E_x = \{xv_1, \dots, xv_{a-1}, xh_1, \dots, xh_{q-a}\}$$

be a set of q-1 edges. For each $z_j^{v_i}$, let

$$E_{z_j^{v_i}} = \{z_j^{v_i} v_i, z_j^{v_i} h_1, \dots, z_j^{v_i} h_{q-a}\}$$

be a set of q - a + 1 edges. Let

$$E' = \{v_0 v_a, v_0 u_1, \dots, v_0 u_{a-2}, v_a u_1, \dots, v_a u_{a-2}\} \cup E(\langle U \rangle)$$

be a set of $\binom{a}{2}$ edges.

We begin by colouring the edges of K_{m_a} in the following way:

1. For $l_i \in L_W$, let $\Gamma_i : E(l_i) \to C_i$ be an edge-colouring of Type 5 in w_i and let $\mathcal{C}(w_i)$ be the subset of $\frac{q}{2}$ colours of \mathcal{C}_i which w_i is not an owner then we assign exactly the colours of $\mathcal{C}(w_i)$ to the set E'_{w_i} .

In this way, each line $l_i \in L_W$ is an owner of C_i , and we have assigned a colour to each edge of $\bigcup_{i=1}^{q-2a+3} (E(l_i) \cup E'_i)$.

2. For each $l_i \in L_U \cup L_V$ let l'_i be the subgraph of G obtained by deleting the edge $v_0 u_i$ from l_i if $i \in \{q - 2a + 4, \dots, q - a + 1\}$ and the edge $v_0 v_i$ from l_i if $i \in \{q - a + 2, \dots, q + 1\}$. Let $\Gamma_i \colon E(l'_i) \to \mathcal{C}_i$ be an edge-colouring of Type 3.

For each $l_j \in L_i$, let $l'_j = l_j \setminus E_{z_j^{v_i}}$ be and let $\Gamma_j \colon E(l'_j) \to \mathcal{C}_j$ be an edge-colouring of Type 3. Now, each line l_j in L_i is an owner of \mathcal{C}_j , and at this point we have assigned a colour to each edge of $\bigcup_{i=1}^{q-2a+3} (E(l_i) \cup E'_i) \cup \bigcup_{i=q-2a+4}^{a(q-1)+q+1} (E(l'_i))$.

3. For each $l_i \in L'$ let $\Gamma_i \colon E(l_i) \to C_i$ be an edge-colouring of Type 2. For each $l_i \in L'$, and for each $x \in V(l_i)$, let $c(x, l_i)$ be the only colour $c \in C_i$ such that for every $y \in V(l_i - x)$, $\Gamma_i(xy) \neq c$. Observe that $\bigcup_{x \in V(l_i)} c(x, l_i) = C_i$. For each x in L' let $c(x) = \{c(x, l_i) \colon x \in V(l_i) \in U\}$

$$V(l_i)$$
 and $l_i \in L'$.

For each y in Y there are a + 1 lines $l \notin L'$ such that $y \in V(l)$, then c(y) is a set of q - a colours. Colour the set of q - a edges $\{yh_1, \ldots, yh_{q-a}\}$ with the set of colours c(y).

For each z in Z there are a lines $l \notin L'$ such that $z \in V(l)$, then c(z) is a set of q - a + 1 colours. Colour the set of q - a + 1 edges E_z with the set of colours c(z).

For each $x \in U \cup V$ there are 2 lines $l \notin L'$ such that $x \in V(l)$, then c(x) is a set of q-1 colours. Colour the set of q-1 edges E_x with the set of colours c(x).

Now it just remains to assign colours to the edges $H \oplus \langle V \rangle \oplus \{v_0\}$ and E'.

- 4. Let $H' = H \oplus \langle V \rangle \oplus v_0$ be and let $\Gamma_n \colon E(H' v_0 v_a) \to \mathcal{C}_n$ be an edge-colouring of Type 3. In this way, H' is an owner of \mathcal{C}_n .
- 5. Let $\Gamma: E' \to \{c\}$ be an edge-colouring where $c \in \mathcal{C}$.

We have already assigned a colour to each edge in K_{m_a} . If $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{n-1} C_i$, then since every line l_i in G is an owner of C_i , by Lemma 1 it follows that there is $x \in V(G)$ which is an owner of both colours. Analogously, if $\{c_1, c_2\} \subseteq C_n$, since H' is an owner of C_n , there is $x \in V(H')$ which is an owner of both colours. Let us suppose $c_1 \in \bigcup_{i=1}^{n-1} C_i$ and $c_2 \in C_n$. If $c_1 \in C_j$ with $1 \leq j \leq a(q-1) + q + 1$, there is a vertex $x \in V \cap V(l_j)$ and x is an owner of c_1 , and since $x \in V(H')$, x is also an owner of c_2 . If $c_1 \in C_j$ with $a(q-1) + q + 2 \leq j \leq q^2 + q$, there is a vertex $x \in V(l_j)$ such that $c(x, l_j) = c_1$ and, by construction, there is $y \in V(H')$ such that $\Gamma_j(xy) = c_1$. Hence y is an owner of c_1 and since $y \in V(H')$, y is an owner of c_2 . Therefore, Γ is a complete edge-colouring of K_{m_a} and the theorem follows.

3 Proof of Theorem 3.

In order to prove Theorem 3 we need to show that $\psi(m_a) \leq (m_a - a)q$ for $a \in \{3, \ldots, \left\lceil \frac{1 + \sqrt{4q+9}}{2} \right\rceil - 1\}$, newly the cases for $a \in \{-1, 0, 1, 2\}$ are given in [1, 2].

To begin with, we prove the following lemma.

Lemma 2. Let q be an even natural number, let $a \in \{3, 4, \ldots, \frac{q}{2}+1\}$ and $m_a = (q+1)^2 - a$, then

$$\psi(m_a) \le f_{m_a}(\frac{q}{2}).$$

Proof. If $x = \frac{q}{2}$ then $m_{\frac{q}{2}+1} = 4x^2 + 3x$ and $m_3 + \frac{3}{2}q + 4 = 4(x+1)^2 - (x+1) - 1$. By Theorem 1 the lemma follows.

Finally, we prove the following.

Proof of Theorem 3. By lemma 2 we know that $\psi(m_a) \leq f_{m_a}(\frac{q}{2})$, and

$$f_{m_a}(\frac{q}{2}) = \left\lfloor \frac{q^4 + 4q^3 + (5-2a)q^2 + 2(1-a)q + a^2 - a}{q+2} \right\rfloor = (m_a - a)q + \left\lfloor \frac{\binom{a}{2}}{\frac{q}{2} + 1} \right\rfloor$$

On the other hand,

$$\left\lfloor \frac{\binom{a}{2}}{\frac{q}{2}+1} \right\rfloor = 0 \Leftrightarrow \frac{a^2-a}{q+2} < 1 \Leftrightarrow a^2 - a - (q+2) < 0 \Leftrightarrow \left(a - \frac{1-\sqrt{4q+9}}{2}\right) \left(a - \frac{1+\sqrt{4q+9}}{2}\right) < 0$$

so, $\frac{1-\sqrt{4q+9}}{2} < a < \frac{1+\sqrt{4q+9}}{2}$ and then $a \in \{3, \dots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}.$

By Theorem 2, it follows that $(m_a - a)q \leq \psi(m_a) \leq (m_a - a)q$ and the result follows.

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