# Graphs with conflict-free connection number two* 

Hong Chang ${ }^{1}$, Trung Duy Doan ${ }^{2,3}{ }^{\dagger}$, Zhong Huang ${ }^{1}$, Stanislav Jendrol ${ }^{4} \ddagger$ Xueliang Li ${ }^{1}$, Ingo Schiermeyer ${ }^{2}$ §<br>${ }^{1}$ Center for Combinatorics and LPMC<br>Nankai University, Tianjin 300071, China<br>${ }^{2}$ Institut für Diskrete Mathematik und Algebra<br>Technische Universität Bergakademie Freiberg<br>09596 Freiberg, Germany<br>${ }^{3}$ School of applied Mathematics and Informatics<br>Hanoi University of Science and Technology, Hanoi, Vietnam<br>${ }^{4}$ Institute of Mathematics, P. J. Šafárik University<br>Jesenná 5, 04001 Košice, Slovakia<br>Email: changh@mail.nankai.edu.cn, trungdoanduy@gmail.com, stanislav.jendrol@upjs.sk 2120150001@mail.nankai.edu.cn, lxl@nankai.edu.cn, Ingo.Schiermeyer@tu-freiberg.de

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#### Abstract

An edge-colored graph $G$ is conflict-free connected if any two of its vertices are connected by a path, which contains a color used on exactly one of


[^0]its edges. The conflict-free connection number of a connected graph $G$, denoted by $c f c(G)$, is the smallest number of colors needed in order to make $G$ conflict-free connected. For a graph $G$, let $C(G)$ be the subgraph of $G$ induced by its set of cut-edges. In this paper, we first show that, if $G$ is a connected non-complete graph $G$ of order $n \geq 9$ with $C(G)$ being a linear forest and with the minimum degree $\delta(G) \geq \max \left\{3, \frac{n-4}{5}\right\}$, then $c f c(G)=2$. The bound on the minimum degree is best possible. Next, we prove that, if $G$ is a connected non-complete graph of order $n \geq 33$ with $C(G)$ being a linear forest and with $d(x)+d(y) \geq \frac{2 n-9}{5}$ for each pair of two nonadjacent vertices $x, y$ of $V(G)$, then $c f c(G)=2$. Both bounds, on the order $n$ and the degree sum, are tight. Moreover, we prove several results concerning relations between degree conditions on $G$ and the number of cut edges in $G$.

Keywords: edge-coloring; conflict-free connection number; degree condition.
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## 1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not described here. Let $G$ be a graph. We use $V(G), E(G), n(G), m(G)$, and $\delta(G)$ to denote the vertex-set, edge-set, number of vertices, number of edges, and minimum degree of $G$, respectively. For $v \in V(G)$, let $N(v)$ denote the neighborhood of $v$ in $G, \operatorname{deg}(x)$ denote the degree of $v$ in $G$.

Let $G$ be a nontrivial connected graph with an associated edge-coloring $c$ : $E(G) \rightarrow\{1,2, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is a proper (edge-)coloring. For a graph $G$, the minimum number of colors needed in a proper coloring of $G$ is referred to as the edge-chromatic number of $G$ and denoted by $\chi^{\prime}(G)$. A path of an edge-colored graph $G$ is said to be a rainbow path if no two edges on the path have the same color. The graph $G$ is called rainbow connected if every pair of distinct vertices of $G$ is connected by a rainbow path in $G$. An edge-coloring of a connected graph is a rainbow connection coloring if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [7] in 2008. For a connected graph $G$, the rainbow connection number $\operatorname{rc}(G)$ of $G$ is defined as the smallest number of colors that are needed in order to make $G$ rainbow connected. Readers interested in this topic are referred to [17, 18, 19] for a survey.

Inspired by the rainbow connection coloring and the proper coloring in graphs,

Andrews et al. [1] and Borozan et al. (4) independently introduced the concept of a proper connection coloring. Let $G$ be a nontrivial connected graph with an edgecoloring. A path in $G$ is called a proper path if no two adjacent edges of the path receive the same color. An edge-coloring $c$ of a connected graph $G$ is a proper connection coloring if every pair of distinct vertices of $G$ is connected by a proper path in $G$. And if $k$ colors are used, then $c$ is called a proper connection $k$-coloring. An edge-colored graph $G$ is proper connected if any two vertices of $G$ are connected by a proper path. For a connected graph $G$, the minimum number of colors that are needed in order to make $G$ proper connected is called the proper connection number of $G$, denoted by $p c(G)$. Let $G$ be a nontrivial connected graph of order $n$ and size $m$ (number of edges). Then we have that $1 \leq p c(G) \leq \min \left\{\chi^{\prime}(G), r c(G)\right\} \leq m$. For more details, we refer to [2, 13, 14, 15] and a dynamic survey [16].

Our research was motivated by the following three results.
Theorem 1.1 5] If $G$ is a 2-connected graph of order $n=n(G)$ and minimum degree $\delta(G)>\max \left\{2, \frac{n+8}{20}\right\}$, then $p c(G) \leq 2$.

Theorem 1.2 [5] For every integer $d \geq 3$, there exists a 2-connected graph of order $n=42 d$ such that $p c(G) \geq 3$.

Theorem 1.3 [14] Let $G$ be a connected noncomplete graph of order $n \geq 5$. If $G \notin\left\{G_{1}, G_{2}\right\}$ and $\delta(G) \geq \frac{n}{4}$, then $p c(G)=2$, where $G_{1}$ and $G_{2}$ are two exceptional graphs on 7 and 8 vertices.

A coloring of the vertices of a hypergraph $H$ is called conflicted-free if each hyperedge $E$ of $H$ has a vertex of unique color that is not repeated in $E$. The smallest number of colors required for such a coloring is called the conflict-free chromatic number of $H$. This parameter was first introduced by Even et al. [12] in a geometric setting, in connection with frequency assignment problems for cellular networks. One can find many results on the conflict-free coloring, see [9, 10, 20].

Recently, Czap et al. 8] introduced the concept of a conflict-free connection of graphs. An edge-colored graph $G$ is called conflict-free connected if each pair of distinct vertices is connected by a path which contains at least one color used on exactly one of its edges. This path is called a conflict-free path, and this coloring is called a conflict-free connection coloring of $G$. The conflict-free connection number of a connected graph $G$, denoted by $c f c(G)$, is the smallest number of colors needed to color the edges of $G$ so that $G$ is conflict-free connected. In [8], they showed that it is easy to compute the conflict-free connection number for 2-connected graphs and very difficult for other connected graphs, including trees.

This paper is organized as follows. In Section 2, we list some fundamental results on the conflict-free connection of graphs. In Sections 3 and 4, we prove our main results.

## 2 Preliminaries

At the very beginning, we state some fundamental results on the conflict-free connection of graphs, which will be used in the sequel.

Lemma 2.1 [8] If $P_{n}$ is a path on $n$ edges, then $c f c(P)=\left\lceil\log _{2}(n+1)\right\rceil$.

Let $C(G)$ be the subgraph of $G$ induced on the set of cut-edges of $G$. The following lemmas respectively provide a necessary condition and a sufficient condition for graphs $G$ with $c f c(G)=2$.

Recall that a linear forest is a forest where each of its components is a path.
Lemma 2.2 [8] If $c f c(G)=2$ for a connected graph $G$, then $C(G)$ is a linear forest whose each component has at most three edges.

Lemma 2.3 [8] If $G$ is a connected graph, and $C(G)$ is a linear forest in which each component is of order 2 , then $c f c(G)=2$.

The following lemma, which can be seen as a corollary of Lemma 2.3 for $C(G)$ being empty, is of extra interest. A rigorous proof can be found in [11.

Lemma 2.4 [8, 11] If $G$ is a 2-edge-connected non-complete graph, then $c f c(G)=2$.

A block of a graph $G$ is a maximal connected subgraph of $G$ that has no cutvertex. If $G$ is connected and has no cut-vertex, then $G$ is a block. An edge is a block if and only if it is a cut-edge, this block is called trivial. Therefore, any nontrivial block is 2 -connected.

Lemma 2.5 [8] Let $G$ be a connected graph. Then from its every nontrivial block an edge can be chosen so that the set of all such chosen edges forms a matching.

Let $C(G)$ be a linear forest consisting of $k(k \geq 0)$ components $Q_{1}, Q_{2}, \ldots, Q_{k}$ with $n_{i}=\left|V\left(Q_{i}\right)\right|$ such that $2 \leq n_{1} \leq n_{2} \leq \cdots \leq n_{k}$. We now present a stronger result than Lemma 2.3, which will be important to show our main results.

Theorem 2.6 If $G$ is a connected non-complete graph with $C(G)$ being a linear forest with $2=n_{1}=n_{2}=\cdots=n_{k-1} \leq n_{k} \leq 4$ or $C(G)$ being edgeless, then $c f c(G)=2$.

Proof. If $C(G)$ is edgeless then the theorem is true by Lemma 2.4. If $C(G)$ a linear forest with at least one edge, then $G$ is a non-complete graph and therefore $\operatorname{cfc}(G) \geq 2$. It remains to verify the converse. Note that one can choose from each nontrivial block an edge so that all the chosen edges create a matching set $S$ by Lemma 2.5. We define an edge-coloring of $G$ as follows. First, we color all edges from $S$ with color 2, and the edges in $E(G) \backslash\left\{S \cup Q_{k}\right\}$ with color 1. Next, we only need to color the edges of $Q_{k}$. If $n_{k}=2$, then color the unique edge of $Q_{k}$ with color 1. If $n_{k}=3$, then color two edges of $Q_{k}$ with colors 1 and 2 . Suppose $n_{k}=4$. It follows that $Q_{k}$ is a path of order 4 , say $w_{1} w_{2} w_{3} w_{4}$. We color the two edges $w_{1} w_{2}$ and $w_{3} w_{4}$ with color 1 , and $w_{2} w_{3}$ with color 2 . It is easy to check that this coloring is a conflict-free connection coloring of $G$. Thus, we have $c f c(G) \leq 2$, and hence $c f c(G)=2$.

Remark 1: The following example points out that Theorem 2.6 is optimal in sense of the number of components with more than two vertices of the linear forest $C(G)$ of a graph $G$.

For $t \geq 3$, let $S_{n}$ be the graph with $n=5 t$ vertices, consisting of the path $P_{6}=v_{0} v_{1} v_{2} v_{3} v_{4} v_{5}$ with complete graphs $K_{t}$ attached to the vertices $v_{i}, i \in\{0,1,4,5\}$ and one more $K_{t}$ sharing the edge $v_{2} v_{3}$ with $P_{6}$. Observe that $\delta\left(S_{n}\right)=t-1=\frac{n-5}{5}$, and $C\left(S_{n}\right)$ is a linear forest with two components of order 3 , paths $v_{0} v_{1} v_{2}$ and $v_{3} v_{4} v_{5}$. In any conflict-free connection coloring of $S_{n}$ with two colors the edges $v_{0} v_{1}$ and $v_{1} v_{2}$ (resp. $v_{3} v_{4}$ and $v_{4} v_{5}$ ) receive different colors. But then any $v_{0}-v_{5}$ path has a conflict. This means that $\operatorname{cfc}\left(S_{n}\right) \geq 3$.

## 3 Degree conditions and the number of cut-edges

Theorem 3.1 Let $G$ be a connected graph of order $n \geq k^{2}, k \geq 3$. If $\delta(G) \geq \frac{n-k+1}{k}$, then $G$ has at most $k-2$ cut edges.

Proof. Assume or the sake of contradiction that $G$ has at least $k-1$ cut edges. Let $B$ be a set of $k-1$ cut edges of $G$. Then the graph $G \backslash B$ has exactly $k$ components $G_{1}, \ldots, G_{k}$. Consider the following two cases.

Case 1. For every $j \in[k]$ there is a vertex $v_{j} \in V\left(G_{j}\right)$ such that $N\left(v_{j}\right) \subseteq V\left(G_{j}\right)$.

Then every component $G_{j}$ has at least $\frac{n-k+1}{k}+1$ vertices and we have

$$
n=|V(G)|=\sum_{j=1}^{k}\left|V\left(G_{j}\right)\right| \geq k \cdot\left(\frac{n-k+1}{k}+1\right)=n+1,
$$

a contradiction.
Case 2. There exists some $i \in[k]$ such that $N(v) \nsubseteq V\left(G_{i}\right)$ for every vertex $v \in V\left(G_{i}\right)$. Then $a=\left|V\left(G_{i}\right)\right| \leq k-1$ and every vertex $v \in V\left(G_{i}\right)$ is incident with a cut edge from $B$. Let $m_{i}$ denote the degree sum of all the vertices of $V\left(G_{i}\right)$ within $G\left[V\left(G_{i}\right) \cup B\right]$. Then we have

$$
\frac{n-k+1}{k} \cdot a \leq m_{i} \leq a \cdot(a-1)+k-1 .
$$

This, together with the bounds on $a$, provides

$$
0 \leq a \cdot\left(a-1-\frac{n-k+1}{k}\right)+k-1 \leq(k-1) \cdot\left(k-2-\frac{n-k+1}{k}\right)+k-1 .
$$

This leads to $n \leq k^{2}-1$, a contradiction.
The next theorem shows that the bound on the minimum degree in Theorem 3.1 cannot be lowered.

Theorem 3.2 For every $k \geq 3$ and $t \geq 3$ there exists a connected $n$-vertex graph $H_{n}$ with $n=k \cdot t, \delta\left(H_{n}\right)=\frac{n-k}{k}$, and $k-1$ cut edges.

Proof. The graph $H_{n}$ consists of a path $P_{k}$ on k vertices to every vertex of it a complete graph $K_{t}$ is attached.

The following theorem shows that the bound $k^{2}$ on the number $n$ of vertices in Theorem 3.1 is best possible.

Theorem 3.3 For every $k \geq 3$ there exists a graph $R_{n}$ on $n=k^{2}-1$ vertices with $\delta\left(R_{n}\right)=\frac{n-k+1}{k}$ and $k-1$ cut edges.

Proof. The graph $R_{n}$ is a connected graph consisting of a central block $B_{0}$, isomorphic to the complete graph $K_{k-1}, k-1$ blocks $B_{1}, \ldots, B_{k-1}$, that are complete graphs on $k$ vertices, and a matching $M$ of $k-1$ cut edges. This matches the vertices of $B_{0}$ with the remaining blocks.

Theorem 3.4 Let $G$ be a connected graph of order

$$
n \geq \max \left\{k^{2}+k, \frac{\left\lfloor\frac{k}{2}\right\rfloor \cdot k(k-2)+k^{2}-5 k+3}{k-4}\right\}, k \geq 5 .
$$

If $\operatorname{deg}(x)+\operatorname{deg}(y) \geq \frac{2 n-2 k+1}{k}$ for any two non-adjacent vertices $x$ and $y$ of $G$, then $G$ has at most $k-2$ cut edges.

Proof. Assume for the sake of contradiction that $G$ has at least $k-1$ cut edges. Let $B$ be a set of $k-1$ cut edges of $G$. Then the graph $G \backslash B$ has exactly $k$ components $G_{1}, \ldots, G_{k}$. Consider the following two cases.

Case 1. For every $j \in[k]$ there is a vertex $v_{j} \in V\left(G_{j}\right)$ such that $N\left(v_{j}\right) \subseteq V\left(G_{j}\right)$.
Case 1.1. Let $k$ be even. Then

$$
n=|V(G)|=\sum_{j=1}^{\frac{k}{2}}\left|V\left(G_{j}\right) \cup V\left(G_{k-j+1}\right)\right| \geq \frac{k}{2} \cdot\left(\frac{2 n-2 k+1}{k}+2\right)=n+\frac{1}{2}
$$

a contradiction.
Case 1.2. Let $k$ be odd. Then, w.l.o.g., we can suppose that $\left|V\left(G_{k}\right)\right| \geq \frac{n-k+1}{k}+1$. Therefore,

$$
\begin{aligned}
n=\left|V\left(G_{k}\right)\right|+\sum_{j=1}^{\frac{k-1}{2}}\left|V\left(G_{j}\right) \cup V\left(G_{k-j}\right)\right| & \geq \frac{n-k+1}{k}+1+\frac{k-1}{2} \cdot\left(\frac{2 n-2 k+1}{k}+2\right) \\
& =n+\frac{k+1}{2 k},
\end{aligned}
$$

a contradiction.
Case 2. There exists some $i \in[k]$ such that $N(v) \nsubseteq V\left(G_{i}\right)$ for every vertex $v \in V\left(G_{i}\right)$.

Case 2.1. There exists only one $i \in[k]$ such that all vertices $v \in V\left(G_{i}\right)$ have $N(v) \nsubseteq V\left(G_{i}\right)$. Observe that $\left|V\left(G_{i}\right)\right|=a \leq k-1$. Notice that every vertex $v \in V\left(G_{i}\right)$ is incident with an edge from $B$, and there is a vertex $y \in V\left(G_{i}\right)$ with $\operatorname{deg}(y) \leq a-1+\frac{k-1}{a}$. For any component $G_{j}, j \neq i \in[k]$, there is

$$
\left|V\left(G_{j}\right)\right| \geq\left\lceil\frac{2 n-2 k+1}{k}\right\rceil-\operatorname{deg}(y)+1 \geq\left\lceil\frac{2 n-2 k+1}{k}\right\rceil-a+1-\frac{k-1}{a}+1 .
$$

This means that the number of vertices in $G$ is

$$
\begin{aligned}
n & =|V(G)| \geq(k-1) \cdot\left(\left\lceil\frac{2 n-2 k+1}{k}\right\rceil-a+1-\frac{k-1}{a}+1\right)+a \\
& \geq(k-1) \cdot\left(\frac{2 n-2 k+1}{k}-a+1-\frac{k-1}{a}+1\right)+a .
\end{aligned}
$$

After some manipulations we get

$$
n \leq \frac{k(k-1)}{k-2}\left(a \cdot \frac{k-2}{k-1}+\frac{k-1}{a}-\frac{1}{k}\right) .
$$

This, together with the bounds on $a$, provides

$$
n \leq \frac{k(k-1)}{k-2}\left(1 \cdot \frac{k-2}{k-1}+\frac{k-1}{1}-\frac{1}{k}\right) .
$$

The inequality yields

$$
n \leq k^{2}+k+\frac{1}{k-2}
$$

Next we check whether $n=k^{2}+k$ satisfies the original inequality

$$
n=|V(G)| \geq(k-1) \cdot\left(\left\lceil\frac{2 n-2 k+1}{k}\right\rceil-a+1-\frac{k-1}{a}+1\right)+a .
$$

After some manipulations we get

$$
k^{2}+k \geq k^{2}+2 k-2
$$

which is impossible. Then we have

$$
n \leq k^{2}+k-1,
$$

a contradiction.
Case 2.2 There exists more than one $i \in[k]$ such that all vertices $v \in V\left(G_{i}\right)$ have $N(v) \nsubseteq V\left(G_{i}\right)$. Assume that there exists a pair of non-adjacent vertices $u, w$ with $u \in V\left(G_{i_{1}}\right)$ and $w \in V\left(G_{i_{2}}\right)$. It is possible that $i_{1}=i_{2}$. Notice that every vertex in such a component is incident with an edge from $B$, and the two vertices $u$ and $w$ are incident with at most one edge from $B$ in common, then $\operatorname{deg}(u)+\operatorname{deg}(w)-1 \leq k-1$. It implies $n \leq \frac{k^{2}+2 k-1}{2}$, a contradiction. Now we get that every vertex in such components is adjacent to the remaining vertices of such components. Hence all possible configurations have been excluded except for two adjacent singletons $\{u\},\{w\}$ as the only such two components $V_{i_{1}}, V_{i_{2}}$. As $\operatorname{deg}(u)+\operatorname{deg}(w)-1 \leq k-1$, w.l.o.g., we assume that $\operatorname{deg}(u) \leq\left\lfloor\frac{k}{2}\right\rfloor$. For any component $G_{j}, j \neq i_{1}$ or $i_{2}$, then

$$
\left|V\left(G_{j}\right)\right| \geq \frac{2 n-2 k+1}{k}-\operatorname{deg}(u)+1 \geq \frac{2 n-2 k+1}{k}-\left\lfloor\frac{k}{2}\right\rfloor+1 .
$$

This means that the number of vertices in $G$ is

$$
n=|V(G)| \geq(k-2) \cdot\left(\frac{2 n-2 k+1}{k}-\left\lfloor\frac{k}{2}\right\rfloor+1\right)+2 .
$$

After some manipulations we get

$$
n \leq \frac{\left\lfloor\frac{k}{2}\right\rfloor \cdot k(k-2)+k^{2}-5 k+2}{k-4}
$$

a contradiction.
Remark 2: Observe that the graph $H_{n}$ of Theorem 3.2 is a good example showing that the bound on the sum of degrees in Theorem 3.4 is tight.

The next theorem shows that the bound on $n$ cannot be lower than $k^{2}+k$.

Theorem 3.5 For every $k \geq 5$ there exists a graph $D_{n}$ on $n=k^{2}+k-1$ vertices with $\operatorname{deg}(x)+\operatorname{deg}(y) \geq \frac{2 n-2 k+1}{k}$ for any two non-adjacent vertices $x$ and $y$ and having $k-1$ cut edges.

Proof. Let $D_{n}$ be a graph consisting of a vertex $v_{0}, k-1$ blocks $B_{1}, \ldots, B_{k-1}$, that are complete graphs on $k+2$ vertices, and a set $M$ of $k-1$ cut edges joining the vertex of $v_{0}$ with the $k-1$ blocks $B_{1}, \ldots, B_{k-1}$. Observe that $D_{n}$ is a connected graph on $k^{2}+k-1$ vertices such that $\operatorname{deg}(x)+\operatorname{deg}(y) \geq 2 k \geq \frac{2 n-2 k+1}{k}$ for any two non-adjacent vertices $x$ and $y$.

## 4 Degree conditions for $c f c(G)=2$

Theorem 4.1 Let $G$ be a connected non-complete graph of order $n \geq 25$. If $C(G)$ induces a linear forest and $\delta(G) \geq \frac{n-4}{5}$, then $\operatorname{cfc}(G)=2$.

Proof. Observe that, by Theorem 3.1, the subgraph $C(G)$ of any connected graph $G$ with $\delta(G) \geq \frac{n-4}{5}$ contains at most three cut edges. As $C(G)$ is a linear forest, we conclude that $c f c(G)=2$ by Theorem 2.6.

Remark 3: The graph $S_{n}$ defined in the end of Section 2 provides a good example showing the tightness of the minimum degree in Theorem 4.1.

Next, we discuss the minimum degree condition for small graphs to have conflictfree connection number 2 .

Theorem 4.2 Let $G$ be a connected non-complete graph of order $n, 9 \leq n \leq 24$. If $C(G)$ induces a linear forest and $\delta(G) \geq \max \left\{3, \frac{n-4}{5}\right\}$, then $c f c(G)=2$.

Proof. We may assume that $C(G) \neq \emptyset$ by Lemma 2.4. Let $C(G)$ consist of $k$ components $Q_{1}, Q_{2}, \ldots, Q_{k}$ with $n_{i}=\left|V\left(Q_{i}\right)\right|$ such that $2 \leq n_{1} \leq n_{2} \leq \cdots \leq n_{k}$. We may also assume that $3 \leq n_{k-1} \leq n_{k} \leq 4$ by Lemma 2.2 and Theorem 2.6. Then $G \backslash\left(E\left(Q_{k-1}\right) \cup E\left(Q_{k}\right)\right)$ has at least five components $C_{1}, C_{2}, C_{3}, C_{4}, C_{5}$. Since $\delta(G) \geq 3$,
it follows that $\left|V\left(C_{i}\right)\right|>3$ for $1 \leq i \leq 5$. Notice that at most two vertices in $C_{i}$ can be contained in $Q_{k-1} \cup Q_{k}$, then for each $C_{i}$ there exists a vertex $u_{i}$ such that $N\left(u_{i}\right) \subseteq V\left(C_{i}\right)$ for $1 \leq i \leq 5$. Thus, $|V(G)| \geq \sum_{i=1}^{5}\left|V\left(C_{i}\right)\right| \geq \sum_{i=1}^{5}\left(d\left(u_{i}\right)+1\right) \geq$ $5\left(\frac{n-4}{5}+1\right)=n+1>n$, a contradiction, which completes the proof.

Remark 4: The following examples show that the minimum degree condition in Theorem 4.2 is best possible. Let $H_{i}$ be a complete graph of order three for $1 \leq i \leq 2$, and take a vertex $v_{i}$ of $H_{i}$ for $1 \leq i \leq 2$. Let $H$ be a graph obtained from $H_{1}, H_{2}$ by connecting $v_{1}$ and $v_{2}$ with a path of order $t$ for $t \geq 5$. Note that $\delta(H)=2$, but $c f c(H) \geq 3$. Another graph class is given as follows. Let $G_{i}$ be a complete graph of order $\frac{n}{5}$, and take a vertex $w_{i}$ of $G_{i}$ for $1 \leq i \leq 5$. Let $G$ be a graph obtained from $G_{1}, G_{2}, G_{3}, G_{4}, G_{5}$ by joining $w_{i}$ and $w_{i+1}$ with an edge for $1 \leq i \leq 4$. Notice that $\delta(G)=\frac{n-5}{5}$, but $c f c(G) \geq 3$.

Theorem 4.3 Let $G$ be a connected noncomplete graph of order $n$ with $4 \leq n \leq 8$. If $C(G)$ induces a linear forest and $\delta(G) \geq 2$, then $c f c(G)=2$.

Proof. If $|E(C(G))| \leq 3$, then the proof follows from Theorem 2.6. Otherwise the subgraph $G \backslash E(C(G))$ has at least five components. Since $\delta(G) \geq 2$, at least two components of it have at least three vertices. Thus $|V(G)| \geq 3 \times 2+3=9>8$, a contradiction.

Remark 5. The following example shows that the minimum degree condition in Theorem 4.3 is best possible. Let $G$ be a path of order $t$ with $t \geq 5$. It is easy to see that $\delta(G)=1$, but $c f c(G)=\left\lceil\log _{2} t\right\rceil \geq 3$ by Lemma 2.1.

If we do not require that $C(G)$ is a linear forest in above theorems, then we can get the following theorem.

Theorem 4.4 Let $G$ be a connected non-complete graph of order $n \geq 16$. If $\delta(G) \geq$ $\frac{n-3}{4}$, then $\operatorname{cfc}(G)=2$.

Proof. Observe that Theorem 3.1 shows that $C(G)$ of any connected graph $G$ with $\delta(G) \geq \frac{n-3}{4}$ has at most two edges. This, when applying Theorem 2.6, immediately gives our theorem.

Remark 6: The following example shows that the minimum degree condition in Theorem 4.4 is best possible. Let $H_{i}$ be a complete graph of order $\frac{n}{4}$ for $1 \leq i \leq 4$, and take a vertex $v_{i}$ of $H_{i}$ for $1 \leq i \leq 4$. Let $H$ be a graph obtained from $H_{1}, H_{2}, H_{3}, H_{4}$ by adding the edges $v_{1} v_{2}, v_{1} v_{3}, v_{1} v_{4}$. Note that $\delta(H)=\frac{n-4}{4}$, but $c f c(H) \geq 3$. On
the other hand, the condition $n \geq 16$ in Theorem 4.4 is also best possible. Let $G_{1}, G_{2}, G_{3}, G_{4}$ be complete graphs of order $1,4,5,5$, respectively, and take a vertex $w_{i}$ of $G_{i}$ for $1 \leq i \leq 4$. Let $G$ be a graph obtained from $G_{1}, G_{2}, G_{3}, G_{4}$ by adding the edges $w_{1} w_{2}, w_{1} w_{3}, w_{1} w_{4}$. Note that $\delta(G) \geq \frac{n-3}{4}$, but $c f c(G) \geq 3$. Also the graph $R_{4}$ from Theorem 3.3 shows the sharpness of the bound of $n$.

Theorem 4.5 Let $G$ be a connected non-complete graph of order $n \geq 33$. If $C(G)$ is a linear forest, and $\operatorname{deg}(x)+\operatorname{deg}(y) \geq \frac{2 n-9}{5}$ for each pair of two non-adjacent vertices $x$ and $y$ of $V(G)$, then $\operatorname{cfc}(G)=2$.

Proof. From Theorem 3.4 we deduce that the subgraph $C(G)$ of $G$ has at most three edges. Now the proof follows from Theorem [2.6.

Remark 7: An example of the graph $S_{n}$, introduced in Remark 1, shows that the degree sum condition in Theorem 4.5 is best possible. On the other hand, the condition $n \geq 33$ in Theorem 4.5 is also best possible. Let $G_{i}$ be a complete graph of order $\frac{n-2}{3}$ for $1 \leq i \leq 3$ and $n \leq 32$, and $G_{4}=v_{1} u_{1} u_{2} v_{2} v_{3}$ be a path of order 5 . Let $G$ be a graph obtained from $G_{1}, G_{2}, G_{3}, G_{4}$ by identifying a vertex of $G_{i}$ to the vertex $v_{i}$ for $1 \leq i \leq 3$. Note that the resulting graph $G$ satisfies that $\operatorname{deg}(x)+\operatorname{deg}(y) \geq \frac{2 n-9}{5}$ for each pair of two non-adjacent vertices $x$ and $y$ of $V(G)$ and $c f c(G) \geq 3$.

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