Graphs with conflict-free connection number two^{*}

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February 12, 2018

Abstract

An edge-colored graph G is *conflict-free connected* if any two of its vertices are connected by a path, which contains a color used on exactly one of

^{*}Supported by NSFC No.11531011.

[†]Financial support by the Free State of Saxony (Landesstipendium) is thankfully acknowledged.

 $^{^{\}ddagger}$ This work was supported by the Slovak Research and Development Agency under the contract No. APVV-15-0116 and by the Slovak VEGA Grant 1/0368/16.

[§]Part of this research was done while the author was visiting the Center for Combinatorics. Financial support is gratefully acknowledged.

its edges. The conflict-free connection number of a connected graph G, denoted by cfc(G), is the smallest number of colors needed in order to make G conflict-free connected. For a graph G, let C(G) be the subgraph of G induced by its set of cut-edges. In this paper, we first show that, if G is a connected non-complete graph G of order $n \ge 9$ with C(G) being a linear forest and with the minimum degree $\delta(G) \ge \max\{3, \frac{n-4}{5}\}$, then cfc(G) = 2. The bound on the minimum degree is best possible. Next, we prove that, if G is a connected non-complete graph of order $n \ge 33$ with C(G) being a linear forest and with $d(x) + d(y) \ge \frac{2n-9}{5}$ for each pair of two nonadjacent vertices x, y of V(G), then cfc(G) = 2. Both bounds, on the order n and the degree sum, are tight. Moreover, we prove several results concerning relations between degree conditions on G and the number of cut edges in G.

Keywords: edge-coloring; conflict-free connection number; degree condition. **AMS subject classification 2010:** 05C15, 05C40, 05C07.

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not described here. Let G be a graph. We use $V(G), E(G), n(G), m(G), and \delta(G)$ to denote the vertex-set, edge-set, number of vertices, number of edges, and minimum degree of G, respectively. For $v \in V(G)$, let N(v) denote the neighborhood of v in G, deg(x) denote the degree of v in G.

Let G be a nontrivial connected graph with an associated *edge-coloring* c: $E(G) \rightarrow \{1, 2, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of G are assigned different colors by c, then c is a proper (*edge-)coloring*. For a graph G, the minimum number of colors needed in a proper coloring of G is referred to as the *edge-chromatic number* of G and denoted by $\chi'(G)$. A path of an edge-colored graph G is said to be a rainbow path if no two edges on the path have the same color. The graph G is called rainbow connected if every pair of distinct vertices of G is connected by a rainbow path in G. An edge-coloring of a connected graph is a rainbow connection coloring if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al.[7] in 2008. For a connected graph G, the rainbow connection number rc(G) of G is defined as the smallest number of colors that are needed in order to make G rainbow connected. Readers interested in this topic are referred to [17, 18, 19] for a survey.

Inspired by the rainbow connection coloring and the proper coloring in graphs,

Andrews et al.[1] and Borozan et al.[4] independently introduced the concept of a proper connection coloring. Let G be a nontrivial connected graph with an edgecoloring. A path in G is called a *proper path* if no two adjacent edges of the path receive the same color. An edge-coloring c of a connected graph G is a *proper connection coloring* if every pair of distinct vertices of G is connected by a proper path in G. And if k colors are used, then c is called a *proper connection* k-coloring. An edge-colored graph G is *proper connected* if any two vertices of G are connected by a proper path. For a connected graph G, the minimum number of colors that are needed in order to make G proper connected is called the *proper connection number* of G, denoted by pc(G). Let G be a nontrivial connected graph of order n and size m (number of edges). Then we have that $1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m$. For more details, we refer to [2, 13, 14, 15] and a dynamic survey [16].

Our research was motivated by the following three results.

Theorem 1.1 [5] If G is a 2-connected graph of order n = n(G) and minimum degree $\delta(G) > max\{2, \frac{n+8}{20}\}$, then $pc(G) \le 2$.

Theorem 1.2 [5] For every integer $d \ge 3$, there exists a 2-connected graph of order n = 42d such that $pc(G) \ge 3$.

Theorem 1.3 [14] Let G be a connected noncomplete graph of order $n \geq 5$. If $G \notin \{G_1, G_2\}$ and $\delta(G) \geq \frac{n}{4}$, then pc(G) = 2, where G_1 and G_2 are two exceptional graphs on 7 and 8 vertices.

A coloring of the vertices of a hypergraph H is called *conflicted-free* if each hyperedge E of H has a vertex of unique color that is not repeated in E. The smallest number of colors required for such a coloring is called the *conflict-free chromatic number* of H. This parameter was first introduced by Even et al. [12] in a geometric setting, in connection with frequency assignment problems for cellular networks. One can find many results on the conflict-free coloring, see [9, 10, 20].

Recently, Czap et al. [8] introduced the concept of a conflict-free connection of graphs. An edge-colored graph G is called *conflict-free connected* if each pair of distinct vertices is connected by a path which contains at least one color used on exactly one of its edges. This path is called a *conflict-free path*, and this coloring is called a *conflict-free connection coloring* of G. The *conflict-free connection number* of a connected graph G, denoted by cfc(G), is the smallest number of colors needed to color the edges of G so that G is conflict-free connected. In [8], they showed that it is easy to compute the conflict-free connection number for 2-connected graphs and very difficult for other connected graphs, including trees.

This paper is organized as follows. In Section 2, we list some fundamental results on the conflict-free connection of graphs. In Sections 3 and 4, we prove our main results.

2 Preliminaries

At the very beginning, we state some fundamental results on the conflict-free connection of graphs, which will be used in the sequel.

Lemma 2.1 [8] If P_n is a path on n edges, then $cfc(P) = \lceil \log_2(n+1) \rceil$.

Let C(G) be the subgraph of G induced on the set of cut-edges of G. The following lemmas respectively provide a necessary condition and a sufficient condition for graphs G with cfc(G) = 2.

Recall that a linear forest is a forest where each of its components is a path.

Lemma 2.2 [8] If cfc(G) = 2 for a connected graph G, then C(G) is a linear forest whose each component has at most three edges.

Lemma 2.3 [8] If G is a connected graph, and C(G) is a linear forest in which each component is of order 2, then cfc(G) = 2.

The following lemma, which can be seen as a corollary of Lemma 2.3 for C(G) being empty, is of extra interest. A rigorous proof can be found in [11].

Lemma 2.4 [8, 11] If G is a 2-edge-connected non-complete graph, then cfc(G) = 2.

A *block* of a graph G is a maximal connected subgraph of G that has no cutvertex. If G is connected and has no cut-vertex, then G is a block. An edge is a block if and only if it is a cut-edge, this block is called *trivial*. Therefore, any nontrivial block is 2-connected.

Lemma 2.5 [8] Let G be a connected graph. Then from its every nontrivial block an edge can be chosen so that the set of all such chosen edges forms a matching.

Let C(G) be a linear forest consisting of k $(k \ge 0)$ components Q_1, Q_2, \ldots, Q_k with $n_i = |V(Q_i)|$ such that $2 \le n_1 \le n_2 \le \cdots \le n_k$. We now present a stronger result than Lemma 2.3, which will be important to show our main results. **Theorem 2.6** If G is a connected non-complete graph with C(G) being a linear forest with $2 = n_1 = n_2 = \cdots = n_{k-1} \le n_k \le 4$ or C(G) being edgeless, then cfc(G) = 2.

Proof. If C(G) is edgeless then the theorem is true by Lemma 2.4. If C(G) a linear forest with at least one edge, then G is a non-complete graph and therefore $cfc(G) \geq 2$. It remains to verify the converse. Note that one can choose from each nontrivial block an edge so that all the chosen edges create a matching set S by Lemma 2.5. We define an edge-coloring of G as follows. First, we color all edges from S with color 2, and the edges in $E(G) \setminus \{S \cup Q_k\}$ with color 1. Next, we only need to color the edges of Q_k . If $n_k = 2$, then color the unique edge of Q_k with color 1. If $n_k = 3$, then color two edges of Q_k with colors 1 and 2. Suppose $n_k = 4$. It follows that Q_k is a path of order 4, say $w_1w_2w_3w_4$. We color the two edges w_1w_2 and w_3w_4 with color 1, and w_2w_3 with color 2. It is easy to check that this coloring is a conflict-free connection coloring of G. Thus, we have $cfc(G) \leq 2$, and hence cfc(G) = 2.

Remark 1: The following example points out that Theorem 2.6 is optimal in sense of the number of components with more than two vertices of the linear forest C(G) of a graph G.

For $t \geq 3$, let S_n be the graph with n = 5t vertices, consisting of the path $P_6 = v_0 v_1 v_2 v_3 v_4 v_5$ with complete graphs K_t attached to the vertices $v_i, i \in \{0, 1, 4, 5\}$ and one more K_t sharing the edge $v_2 v_3$ with P_6 . Observe that $\delta(S_n) = t - 1 = \frac{n-5}{5}$, and $C(S_n)$ is a linear forest with two components of order 3, paths $v_0 v_1 v_2$ and $v_3 v_4 v_5$. In any conflict-free connection coloring of S_n with two colors the edges $v_0 v_1$ and $v_1 v_2$ (resp. $v_3 v_4$ and $v_4 v_5$) receive different colors. But then any v_0 - v_5 path has a conflict. This means that $cfc(S_n) \geq 3$.

3 Degree conditions and the number of cut-edges

Theorem 3.1 Let G be a connected graph of order $n \ge k^2, k \ge 3$. If $\delta(G) \ge \frac{n-k+1}{k}$, then G has at most k-2 cut edges.

Proof. Assume or the sake of contradiction that G has at least k-1 cut edges. Let B be a set of k-1 cut edges of G. Then the graph $G \setminus B$ has exactly k components G_1, \ldots, G_k . Consider the following two cases.

Case 1. For every $j \in [k]$ there is a vertex $v_j \in V(G_j)$ such that $N(v_j) \subseteq V(G_j)$.

Then every component G_j has at least $\frac{n-k+1}{k} + 1$ vertices and we have

$$n = |V(G)| = \sum_{j=1}^{k} |V(G_j)| \ge k \cdot \left(\frac{n-k+1}{k} + 1\right) = n+1,$$

a contradiction.

Case 2. There exists some $i \in [k]$ such that $N(v) \not\subseteq V(G_i)$ for every vertex $v \in V(G_i)$. Then $a = |V(G_i)| \leq k - 1$ and every vertex $v \in V(G_i)$ is incident with a cut edge from B. Let m_i denote the degree sum of all the vertices of $V(G_i)$ within $G[V(G_i) \cup B]$. Then we have

$$\frac{n-k+1}{k} \cdot a \le m_i \le a \cdot (a-1) + k - 1.$$

This, together with the bounds on a, provides

$$0 \le a \cdot (a - 1 - \frac{n - k + 1}{k}) + k - 1 \le (k - 1) \cdot (k - 2 - \frac{n - k + 1}{k}) + k - 1.$$

This leads to $n \leq k^2 - 1$, a contradiction.

The next theorem shows that the bound on the minimum degree in Theorem 3.1 cannot be lowered.

Theorem 3.2 For every $k \ge 3$ and $t \ge 3$ there exists a connected n-vertex graph H_n with $n = k \cdot t$, $\delta(H_n) = \frac{n-k}{k}$, and k-1 cut edges.

Proof. The graph H_n consists of a path P_k on k vertices to every vertex of it a complete graph K_t is attached.

The following theorem shows that the bound k^2 on the number n of vertices in Theorem 3.1 is best possible.

Theorem 3.3 For every $k \ge 3$ there exists a graph R_n on $n = k^2 - 1$ vertices with $\delta(R_n) = \frac{n-k+1}{k}$ and k-1 cut edges.

Proof. The graph R_n is a connected graph consisting of a central block B_0 , isomorphic to the complete graph K_{k-1} , k-1 blocks B_1, \ldots, B_{k-1} , that are complete graphs on k vertices, and a matching M of k-1 cut edges. This matches the vertices of B_0 with the remaining blocks.

Theorem 3.4 Let G be a connected graph of order

$$n \ge \max\{k^2 + k, \frac{\lfloor \frac{k}{2} \rfloor \cdot k(k-2) + k^2 - 5k + 3}{k-4}\}, k \ge 5.$$

If $\deg(x) + \deg(y) \ge \frac{2n-2k+1}{k}$ for any two non-adjacent vertices x and y of G, then G has at most k-2 cut edges.

Proof. Assume for the sake of contradiction that G has at least k-1 cut edges. Let B be a set of k-1 cut edges of G. Then the graph $G \setminus B$ has exactly k components G_1, \ldots, G_k . Consider the following two cases.

Case 1. For every $j \in [k]$ there is a vertex $v_j \in V(G_j)$ such that $N(v_j) \subseteq V(G_j)$. Case 1.1. Let k be even. Then

$$n = |V(G)| = \sum_{j=1}^{\frac{k}{2}} |V(G_j) \cup V(G_{k-j+1})| \ge \frac{k}{2} \cdot (\frac{2n-2k+1}{k}+2) = n + \frac{1}{2},$$

a contradiction.

Case 1.2. Let k be odd. Then, w.l.o.g., we can suppose that $|V(G_k)| \ge \frac{n-k+1}{k} + 1$. Therefore,

$$n = |V(G_k)| + \sum_{j=1}^{\frac{k-1}{2}} |V(G_j) \cup V(G_{k-j})| \ge \frac{n-k+1}{k} + 1 + \frac{k-1}{2} \cdot (\frac{2n-2k+1}{k} + 2)$$
$$= n + \frac{k+1}{2k},$$

a contradiction.

Case 2. There exists some $i \in [k]$ such that $N(v) \not\subseteq V(G_i)$ for every vertex $v \in V(G_i)$.

Case 2.1. There exists only one $i \in [k]$ such that all vertices $v \in V(G_i)$ have $N(v) \not\subseteq V(G_i)$. Observe that $|V(G_i)| = a \leq k - 1$. Notice that every vertex $v \in V(G_i)$ is incident with an edge from B, and there is a vertex $y \in V(G_i)$ with $\deg(y) \leq a - 1 + \frac{k-1}{a}$. For any component $G_j, j \neq i \in [k]$, there is

$$|V(G_j)| \ge \lceil \frac{2n - 2k + 1}{k} \rceil - \deg(y) + 1 \ge \lceil \frac{2n - 2k + 1}{k} \rceil - a + 1 - \frac{k - 1}{a} + 1.$$

This means that the number of vertices in G is

$$n = |V(G)| \ge (k-1) \cdot \left(\left\lceil \frac{2n-2k+1}{k} \right\rceil - a + 1 - \frac{k-1}{a} + 1\right) + a$$
$$\ge (k-1) \cdot \left(\frac{2n-2k+1}{k} - a + 1 - \frac{k-1}{a} + 1\right) + a.$$

After some manipulations we get

$$n \le \frac{k(k-1)}{k-2} \left(a \cdot \frac{k-2}{k-1} + \frac{k-1}{a} - \frac{1}{k}\right).$$

This, together with the bounds on a, provides

$$n \le \frac{k(k-1)}{k-2} \left(1 \cdot \frac{k-2}{k-1} + \frac{k-1}{1} - \frac{1}{k}\right).$$

The inequality yields

$$n \le k^2 + k + \frac{1}{k-2}.$$

Next we check whether $n = k^2 + k$ satisfies the original inequality

$$n = |V(G)| \ge (k-1) \cdot \left(\lceil \frac{2n-2k+1}{k} \rceil - a + 1 - \frac{k-1}{a} + 1 \right) + a.$$

After some manipulations we get

$$k^2 + k \ge k^2 + 2k - 2,$$

which is impossible. Then we have

$$n \le k^2 + k - 1,$$

a contradiction.

Case 2.2 There exists more than one $i \in [k]$ such that all vertices $v \in V(G_i)$ have $N(v) \not\subseteq V(G_i)$. Assume that there exists a pair of non-adjacent vertices u, w with $u \in V(G_{i_1})$ and $w \in V(G_{i_2})$. It is possible that $i_1 = i_2$. Notice that every vertex in such a component is incident with an edge from B, and the two vertices u and w are incident with at most one edge from B in common, then $deg(u) + deg(w) - 1 \leq k - 1$. It implies $n \leq \frac{k^2 + 2k - 1}{2}$, a contradiction. Now we get that every vertex in such components is adjacent to the remaining vertices of such components. Hence all possible configurations have been excluded except for two adjacent singletons $\{u\}, \{w\}$ as the only such two components V_{i_1}, V_{i_2} . As $deg(u) + deg(w) - 1 \leq k - 1$, w.l.o.g., we assume that $deg(u) \leq \lfloor \frac{k}{2} \rfloor$. For any component $G_j, j \neq i_1$ or i_2 , then

$$|V(G_j)| \ge \frac{2n - 2k + 1}{k} - \deg(u) + 1 \ge \frac{2n - 2k + 1}{k} - \lfloor \frac{k}{2} \rfloor + 1.$$

This means that the number of vertices in G is

$$n = |V(G)| \ge (k-2) \cdot \left(\frac{2n-2k+1}{k} - \lfloor \frac{k}{2} \rfloor + 1\right) + 2.$$

After some manipulations we get

$$n \le \frac{\lfloor \frac{k}{2} \rfloor \cdot k(k-2) + k^2 - 5k + 2}{k-4},$$

a contradiction.

Remark 2: Observe that the graph H_n of Theorem 3.2 is a good example showing that the bound on the sum of degrees in Theorem 3.4 is tight.

The next theorem shows that the bound on n cannot be lower than $k^2 + k$.

Theorem 3.5 For every $k \ge 5$ there exists a graph D_n on $n = k^2 + k - 1$ vertices with $\deg(x) + \deg(y) \ge \frac{2n-2k+1}{k}$ for any two non-adjacent vertices x and y and having k-1 cut edges.

Proof. Let D_n be a graph consisting of a vertex v_0 , k-1 blocks B_1, \ldots, B_{k-1} , that are complete graphs on k+2 vertices, and a set M of k-1 cut edges joining the vertex of v_0 with the k-1 blocks B_1, \ldots, B_{k-1} . Observe that D_n is a connected graph on $k^2 + k - 1$ vertices such that $\deg(x) + \deg(y) \ge 2k \ge \frac{2n-2k+1}{k}$ for any two non-adjacent vertices x and y.

4 Degree conditions for cfc(G) = 2

Theorem 4.1 Let G be a connected non-complete graph of order $n \ge 25$. If C(G) induces a linear forest and $\delta(G) \ge \frac{n-4}{5}$, then cfc(G) = 2.

Proof. Observe that, by Theorem 3.1, the subgraph C(G) of any connected graph G with $\delta(G) \geq \frac{n-4}{5}$ contains at most three cut edges. As C(G) is a linear forest, we conclude that cfc(G) = 2 by Theorem 2.6.

Remark 3: The graph S_n defined in the end of Section 2 provides a good example showing the tightness of the minimum degree in Theorem 4.1.

Next, we discuss the minimum degree condition for small graphs to have conflictfree connection number 2.

Theorem 4.2 Let G be a connected non-complete graph of order $n, 9 \le n \le 24$. If C(G) induces a linear forest and $\delta(G) \ge \max\{3, \frac{n-4}{5}\}$, then cfc(G) = 2.

Proof. We may assume that $C(G) \neq \emptyset$ by Lemma 2.4. Let C(G) consist of k components Q_1, Q_2, \ldots, Q_k with $n_i = |V(Q_i)|$ such that $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. We may also assume that $3 \leq n_{k-1} \leq n_k \leq 4$ by Lemma 2.2 and Theorem 2.6. Then $G \setminus (E(Q_{k-1}) \cup E(Q_k))$ has at least five components C_1, C_2, C_3, C_4, C_5 . Since $\delta(G) \geq 3$,

it follows that $|V(C_i)| > 3$ for $1 \le i \le 5$. Notice that at most two vertices in C_i can be contained in $Q_{k-1} \cup Q_k$, then for each C_i there exists a vertex u_i such that $N(u_i) \subseteq V(C_i)$ for $1 \le i \le 5$. Thus, $|V(G)| \ge \sum_{i=1}^5 |V(C_i)| \ge \sum_{i=1}^5 (d(u_i) + 1) \ge 5(\frac{n-4}{5} + 1) = n + 1 > n$, a contradiction, which completes the proof.

Remark 4: The following examples show that the minimum degree condition in Theorem 4.2 is best possible. Let H_i be a complete graph of order three for $1 \le i \le 2$, and take a vertex v_i of H_i for $1 \le i \le 2$. Let H be a graph obtained from H_1, H_2 by connecting v_1 and v_2 with a path of order t for $t \ge 5$. Note that $\delta(H) = 2$, but $cfc(H) \ge 3$. Another graph class is given as follows. Let G_i be a complete graph of order $\frac{n}{5}$, and take a vertex w_i of G_i for $1 \le i \le 5$. Let G be a graph obtained from G_1, G_2, G_3, G_4, G_5 by joining w_i and w_{i+1} with an edge for $1 \le i \le 4$. Notice that $\delta(G) = \frac{n-5}{5}$, but $cfc(G) \ge 3$.

Theorem 4.3 Let G be a connected noncomplete graph of order n with $4 \le n \le 8$. If C(G) induces a linear forest and $\delta(G) \ge 2$, then cfc(G) = 2.

Proof. If $|E(C(G))| \leq 3$, then the proof follows from Theorem 2.6. Otherwise the subgraph $G \setminus E(C(G))$ has at least five components. Since $\delta(G) \geq 2$, at least two components of it have at least three vertices. Thus $|V(G)| \geq 3 \times 2 + 3 = 9 > 8$, a contradiction.

Remark 5. The following example shows that the minimum degree condition in Theorem 4.3 is best possible. Let G be a path of order t with $t \ge 5$. It is easy to see that $\delta(G) = 1$, but $cfc(G) = \lceil \log_2 t \rceil \ge 3$ by Lemma 2.1.

If we do not require that C(G) is a linear forest in above theorems, then we can get the following theorem.

Theorem 4.4 Let G be a connected non-complete graph of order $n \ge 16$. If $\delta(G) \ge \frac{n-3}{4}$, then cfc(G) = 2.

Proof. Observe that Theorem 3.1 shows that C(G) of any connected graph G with $\delta(G) \geq \frac{n-3}{4}$ has at most two edges. This, when applying Theorem 2.6, immediately gives our theorem.

Remark 6: The following example shows that the minimum degree condition in Theorem 4.4 is best possible. Let H_i be a complete graph of order $\frac{n}{4}$ for $1 \le i \le 4$, and take a vertex v_i of H_i for $1 \le i \le 4$. Let H be a graph obtained from H_1, H_2, H_3, H_4 by adding the edges v_1v_2, v_1v_3, v_1v_4 . Note that $\delta(H) = \frac{n-4}{4}$, but $cfc(H) \ge 3$. On the other hand, the condition $n \ge 16$ in Theorem 4.4 is also best possible. Let G_1, G_2, G_3, G_4 be complete graphs of order 1, 4, 5, 5, respectively, and take a vertex w_i of G_i for $1 \le i \le 4$. Let G be a graph obtained from G_1, G_2, G_3, G_4 by adding the edges w_1w_2, w_1w_3, w_1w_4 . Note that $\delta(G) \ge \frac{n-3}{4}$, but $cfc(G) \ge 3$. Also the graph R_4 from Theorem 3.3 shows the sharpness of the bound of n.

Theorem 4.5 Let G be a connected non-complete graph of order $n \ge 33$. If C(G) is a linear forest, and $\deg(x) + \deg(y) \ge \frac{2n-9}{5}$ for each pair of two non-adjacent vertices x and y of V(G), then cfc(G) = 2.

Proof. From Theorem 3.4 we deduce that the subgraph C(G) of G has at most three edges. Now the proof follows from Theorem 2.6.

Remark 7: An example of the graph S_n , introduced in Remark 1, shows that the degree sum condition in Theorem 4.5 is best possible. On the other hand, the condition $n \ge 33$ in Theorem 4.5 is also best possible. Let G_i be a complete graph of order $\frac{n-2}{3}$ for $1 \le i \le 3$ and $n \le 32$, and $G_4 = v_1 u_1 u_2 v_2 v_3$ be a path of order 5. Let Gbe a graph obtained from G_1, G_2, G_3, G_4 by identifying a vertex of G_i to the vertex v_i for $1 \le i \le 3$. Note that the resulting graph G satisfies that $\deg(x) + \deg(y) \ge \frac{2n-9}{5}$ for each pair of two non-adjacent vertices x and y of V(G) and $cfc(G) \ge 3$.

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