BERGE'S CONJECTURE AND AHARONI-HARTMAN-HOFFMAN'S CONJECTURE FOR LOCALLY IN-SEMICOMPLETE DIGRAPHS

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Abstract. Let k be a positive integer and let D be a digraph. A *path partition* $\mathcal{P}$ of D is a set of vertex-disjoint paths which covers $V(D)$. Its *k-norm* is defined as $\sum_{P\in\mathcal{P}} \min\{|V(P)|, k\}$. A path partition is *$k$-optimal* if its k-norm is minimum among all path partitions of D. A *partial k-coloring* is a collection of k disjoint stable sets. A partial k-coloring $\mathcal{C}$ is *orthogonal* to a path partition $\mathcal{P}$ if each path $P \in \mathcal{P}$ meets $\min\{|P|, k\}$ distinct sets of $\mathcal{C}$. Berge (1982) conjectured that every k-optimal path partition of D has a partial k-coloring orthogonal to it. A *(path) k-pack* of D is a collection of at most k vertex-disjoint paths in D. Its *weight* is the number of vertices it covers. A k-pack is *optimal* if its weight is maximum among all k-packs of D. A *coloring* of D is a partition of $V(D)$ into stable sets. A k-pack $\mathcal{P}$ is *orthogonal* to a coloring $\mathcal{C}$ if each set $C \in \mathcal{C}$ meets $\min\{|C|, k\}$ paths of $\mathcal{P}$. Aharoni, Hartman and Hoffman (1985) conjectured that every optimal k-pack of D has a coloring orthogonal to it. A digraph D is *semicomplete* if every pair of distinct vertices of D is adjacent. A digraph D is *locally in-semicomplete* if, for every vertex $v \in V(D)$, the in-neighborhood of v induces a semicomplete digraph. *Locally out-semicomplete* digraphs are defined similarly. In this paper, we prove Berge's and Aharoni-Hartman-Hoffman's Conjectures for locally in/out-semicomplete digraphs.

§1. Introduction

The digraphs considered in this text do not contain loops or parallel arcs, but may contain cycles of length two. Let D be a digraph. We denote the vertex set of D by $V(D)$ and its arc set by $A(D)$. If u and v are vertices of D, then we denote the arc with tail in u and head in v by uv. Vertices u and v are *adjacent* in D if $uv \in A(D)$ or $vu \in A(D)$; otherwise they are *nonadjacent*. The *neighborhood*, *in-neighborhood*, and *out-neighborhood* of a vertex $v \in V(D)$ are the sets $\{u \in V(D)\colon uv \in A(D) \text{ or } vu \in A(D)\}$, $\{u \in V(D)\colon uv \in A(D)\}$, and $\{u \in V(D)\colon vu \in A(D)\}$, respectively.

A *path* in D is a nonempty sequence of distinct vertices $P = v_1v_2\dots v_\ell$ such that $v_iv_{i+1} \in A(D)$ for $1 \le i < \ell$. We define $V(P) = \{v_1, v_2, \dots, v_\ell\}$ and $e(P) = v_\ell$. The *order* of P, denoted by $|P|$, is equal to ℓ and a path is *trivial* if its order is one. We denote the order of a longest path in D by $\lambda(D)$. For a set $\mathcal{P}$ of vertex-disjoint paths of D, we define $V(\mathcal{P}) = \cup_{P\in\mathcal{P}} V(P)$.

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A set S of vertices of D is *stable* if all of its vertices are pairwise nonadjacent. The *stability number* of D, denoted by $\alpha(D)$, is equal to the cardinality of a maximum stable set of D. A *path partition* of D is a set of vertex-disjoint paths of D which covers $V(D)$. A path partition $\mathcal{P}$ of D is *optimal* if $|\mathcal{P}|$ is minimum among all possible path partitions of D. The cardinality of an optimal path partition of D is denoted by $\pi(D)$. In 1950, Dilworth [8] proved the following result.

Theorem 1.1 (Dilworth [8]). *For every transitive acyclic digraph D, we have $\pi(D) = \alpha(D)$.*

Note that this equality is not valid for digraphs in general; for example, if D is a directed cycle with 5 vertices, then $\pi(D) = 1$ and $\alpha(D) = 2$. However, Gallai and Milgram [11] proved that the following inequality holds for arbitrary digraphs.

Theorem 1.2 (Gallai-Milgram [11]). *For every digraph D, we have $\pi(D) \leq \alpha(D)$.*

Let k be a positive integer and let D be a digraph. The *k-norm* of a path partition $\mathcal{P}$ of D is defined as $\sum_{P \in \mathcal{P}} \min\{|P|, k\}$ and denoted by $|\mathcal{P}|_k$. A path partition $\mathcal{P}$ of D is *k-optimal* if $|\mathcal{P}|_k$ is minimum among all possible path partitions of D. The k-norm of a k-optimal path partition of D is denoted by $\pi_k(D)$. A *partial k-coloring* $\mathcal{C}$ of D is a collection of k disjoint stable sets of D called *color classes* (empty color classes are allowed). The *weight* of a partial k-coloring $\mathcal{C}$, denoted by $||\mathcal{C}||$, is defined as $\sum_{C \in \mathcal{C}} |C|$. A partial k-coloring $\mathcal{C}$ of D is *optimal* if $||\mathcal{C}||$ is maximum among all possible partial k-colorings of D. The weight of an optimal partial k-coloring of D is denoted by $\alpha_k(D)$. In 1976, Greene and Kleitman [13] proved the following result.

Theorem 1.3 (Greene-Kleitman [13]). *For every transitive acyclic digraph D and every positive integer k, we have $\pi_k(D) = \alpha_k(D)$.*

Since $\pi(D) = \pi_1(D)$ and $\alpha(D) = \alpha_1(D)$, Theorem 1.1 is the particular case of Theorem 1.3 in which $k = 1$. In 1981, Linial [17] conjectured that Theorem 1.3 can be extended to arbitrary digraphs in the same way that Theorem 1.2 extends Theorem 1.1.

Conjecture 1.1 (Linial [17]). *For every digraph D and every positive integer k, we have $\pi_k(D) \leq \alpha_k(D)$.*

In an attempt to unify Theorem 1.2 and a result proved independently by Gallai [10] and Roy [19] (Theorem 1.5), Berge proposed the following conjecture, which is a strengthening of Conjecture 1.1. A path partition $\mathcal{P}$ and a partial k-coloring $\mathcal{C}$ are *orthogonal* if each path $P \in \mathcal{P}$ meets $\min\{|P|, k\}$ distinct color classes of $\mathcal{C}$ (we also say that $\mathcal{P}$ is orthogonal to $\mathcal{C}$ and vice-versa).

Berge's Conjecture [5]. *Let D be a digraph and let k be a positive integer. If $\mathcal{P}$ is a k-optimal path partition of D, then there exists a partial k-coloring of D orthogonal to $\mathcal{P}$.*

Berge's Conjecture remains open, but we know it holds for $k = 1$ [16], $k = 2$ [6], when $\lambda(D) = 3$ [5], when the k-optimal path partition has only paths of order at most k [5] or if it has only paths of order at least k [1], acyclic digraphs [2, 7], digraphs where all directed cycles are pairwise vertex-disjoint [20], bipartite digraphs [5], digraphs containing a Hamiltonian path [5], and $k \geq \lambda(D) - 3$ [15].

Now we exchange the roles of paths and stable sets in the concepts discussed so far and present some similar results. Let D be a digraph. A *coloring* of D is a partition of $V(D)$ into stable sets called *color classes*. A coloring $\mathcal{C}$ of D is *optimal* if $|\mathcal{C}|$ is minimum among all possible colorings of D. The *chromatic number* of D, denoted by $\chi(D)$, is the cardinality of an optimal coloring of D. Mirsky [18] proved the following dual of Theorem 1.1.

Theorem 1.4 (Mirsky [18])**.** *For every transitive acyclic digraph D, we have $\chi(D) = \lambda(D)$.*

Similarly to Theorem 1.2, Gallai [10] and Roy [19], independently, proved the following result.

Theorem 1.5 (Gallai-Roy [10, 19])**.** *For every digraph D, we have $\chi(D) \leq \lambda(D)$.*

Let k be a positive integer. The *k-norm* of a coloring $\mathcal{C}$, denoted by $|\mathcal{C}|_k$, is $\sum_{C\in\mathcal{C}} \min\{|C|, k\}$. A coloring $\mathcal{C}$ of a digraph D is *k-optimal* if $|\mathcal{C}|_k$ is minimum among all possible path partitions of D. The k-norm of a k-optimal coloring of a digraph D is denoted by $\chi_k(D)$. A *(path) k-pack* $\mathcal{P}$ is a set of at most k vertex-disjoint paths of a digraph D. The *weight* of a k-pack $\mathcal{P}$, denoted by $||\mathcal{P}||$, is defined as $|V(\mathcal{P})|$ (i.e., the number of vertices $\mathcal{P}$ covers). A k-pack $\mathcal{P}$ of a digraph D is *optimal* if $||\mathcal{P}||$ is maximum among all possible k-packs of D. The weight of an optimal k-pack of a digraph D is denoted by $\lambda_k(D)$. Note that $\chi(D) = \chi_1(D)$ and $\lambda(D) = \lambda_1(D)$. Greene [12] proved the following theorem for transitive acyclic digraphs.

Theorem 1.6 (Greene [12])**.** *For every transitive acyclic digraph D and every positive integer k, we have $\chi_k(D) = \lambda_k(D)$.*

Linial [17] proposed Conjecture 1.2 for arbitrary digraphs.

Conjecture 1.2 (Linial [17])**.** *For every digraph D and every positive integer k, we have $\chi_k(D) \leq \lambda_k(D)$.*

Note that Theorems 1.4, 1.5, 1.6, and Conjecture 1.2 can be seen as dual versions of Theorems 1.1, 1.2, 1.3, and Conjecture 1.1, respectively, where the roles of paths and stable sets are exchanged. Therefore, it is natural to ask if there exists a dual version of Berge's Conjecture. By exchanging the roles of paths and stable sets, we end up with the following definition of orthogonality. A coloring $\mathcal{C}$ and a k-pack $\mathcal{P}$ are *orthogonal* if each color class $C \in \mathcal{C}$ meets $\min\{|C|, k\}$ distinct paths of $\mathcal{P}$ (we also say that $\mathcal{C}$ is orthogonal to $\mathcal{P}$ and vice-versa). The natural dual version of Berge's Conjecture states that a k-optimal coloring of a digraph D, for some positive integer k, has a k-pack of D orthogonal to it. This statement is a strengthening of Conjecture 1.2, however it is false. For example, take D defined as $V(D) = \{v_1, v_2, v_3, v_4, v_5\}$ and $A(D) = \{v_1v_2, v_1v_5, v_3v_2, v_3v_4, v_5v_4\}$, and take $\mathcal{C} = \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}$. As an alternative strengthening of Conjecture 1.2, Aharoni, Hartman, and Hoffman [2] gave the following conjecture.

Aharoni-Hartman-Hoffman's Conjecture [2]**.** *Let D be a digraph and let k be a positive integer. If $\mathcal{P}$ is an optimal k-pack of D, then there exists a coloring of D orthogonal to $\mathcal{P}$.*

This conjecture remains open, but we know it holds for $k = 1$ [10,19], $k \geq \pi(D)$ [14], when the optimal k-pack has at least one trivial path [14], bipartite digraphs [14], and acyclic digraphs [2].

A digraph is *semicomplete* if all its vertices are pairwise adjacent. A digraph D is *(locally) in-semicomplete* (respectively, *out-semicomplete*) if, for every vertex $v \in V(D)$, the in-neighborhood (respectively, out-neighborhood) of v induces a semicomplete digraph. One important characterization of in-semicomplete digraphs which we use throughout the text is the following.

Theorem 1.7 ([3])**.** *A digraph D is in-semicomplete if, and only if, for every vertex v and every pair of internally vertex-disjoint paths P and Q such that $v = e(P) = e(Q)$, there exists a path R such that $V(R) = V(P) \cup V(Q)$ and $e(R) = v$.*

In-semicomplete digraphs generalize semicomplete digraphs, which in turn generalize *tournaments*. We refer the reader to the book by Bang-Jensen and Gutin [3] for further information on this class of digraphs. We just would like to remark that there are a few results and problems in literature considering in-semicomplete digraphs, such as Bondy's Conjecture and Laborde, Payan, and Xuong's Conjecture, presented next. The former states that for every digraph D and every choice of positive integers λ_1, λ_2 such that $\lambda(D) = \lambda_1 + \lambda_2$, there exists a partition of D into two digraphs D_1 and D_2 such that $\lambda(D_i) = \lambda_i$, for $i = 1, 2$. The latter states that in every digraph there exists a maximal stable set that intersects every longest path. These conjectures, still open for arbitrary digraphs, were proved for in-semicomplete digraphs by Bang-Jensen *et al.* [4] and Galeana-Sánchez and Gómez [9], respectively. In this paper, we prove Berge's Conjecture and Aharoni-Hartman-Hoffman's Conjecture for in-semicomplete and out-semicomplete digraphs.

§2. Results for Berge's Conjecture

Given a path P and a positive integer k, if $|P| > k$ then we say P is *k-long*, otherwise we say it is *k-short*. For a set $\mathcal{P}$ of vertex-disjoint paths of a digraph D and a positive integer k, we define $e(\mathcal{P}) = \{e(P) \colon P \in \mathcal{P}\}$, $\mathcal{P}^{>k} = \{P \in \mathcal{P} \colon |P| > k\}$, and $\mathcal{P}^{\leq k} = \{P \in \mathcal{P} \colon |P| \leq k\}$. To simplify notation, given a set S and an element x, we denote by $S + x$ the union $S \cup \{x\}$ and by $S - x$ the difference $S \backslash \{x\}$.

Next lemma shows that it is possible to convert one path partition $\mathcal{P}$ into another path partition $\mathcal{Q}$ whose k-norm is either smaller than $\mathcal{P}$'s or it is the same as $\mathcal{P}$'s but $e(\mathcal{Q})$ is a stable set.

Lemma 2.1. *Let $\mathcal{P}$ be a path partition of an in-semicomplete digraph D and let k be a positive integer. Then there exists a path partition $\mathcal{Q}$ of D such that one of the following conditions holds:*

(i) $|\mathcal{Q}|_k < |\mathcal{P}|_k$ *and* $e(\mathcal{Q}) \subset e(\mathcal{P})$*;*
(ii) $|\mathcal{Q}|_k = |\mathcal{P}|_k$*,* $e(\mathcal{Q}) \subseteq e(\mathcal{P})$*,* $e(\mathcal{Q})$ *is stable, and every partial k-coloring of D orthogonal to $\mathcal{Q}$ is also orthogonal to $\mathcal{P}$.*

Proof. If $e(\mathcal{P})$ is stable, then $\mathcal{Q} = \mathcal{P}$ satisfies case (ii) and the result follows. Thus, we may assume that $e(\mathcal{P})$ is not stable and, therefore, there exists a pair of vertices u and v in $e(\mathcal{P})$ such that $uv \in A(D)$. Let P_1 and P_2 be the paths in $\mathcal{P}$ such that $e(P_1) = u$ and $e(P_2) = v$. By Theorem 1.7, there exists a path Q in D such that $V(Q) = V(P_1) \cup V(P_2)$ and $e(Q) = v$. Let $\mathcal{Q}$ be the path partition of D defined as $\mathcal{P} - P_1 - P_2 + Q$. Note that $e(\mathcal{Q}) \subset e(\mathcal{P})$.

Suppose first that at least one of P_1 and P_2 is a k-long path. Hence, Q is k-long and

$$\begin{aligned}|\mathcal{Q}|_k &= \sum_{P\in\mathcal{Q}} \min\{|P|,k\} = \sum_{P\in\mathcal{Q}-Q} \min\{|P|,k\} + \min\{|Q|,k\} = \sum_{P\in\mathcal{Q}-Q} \min\{|P|,k\} + k \\ &< \sum_{P\in\mathcal{P}-P_1-P_2} \min\{|P|,k\} + \min\{|P_1|,k\} + \min\{|P_2|,k\} = \sum_{P\in\mathcal{P}} \min\{|P|,k\} = |\mathcal{P}|_k \ ,\end{aligned}$$

where the inequality follows because $\min\{|P_1|,k\} + \min\{|P_2|,k\}$ is at least $k+1$, since at least one of P_1 or P_2 is k-long. Therefore, case (i) holds and we may assume that there is no arc in $A(D)$ for which one of its endpoints is in $e(\mathcal{P}^{>k})$ and the other is in $e(\mathcal{P})$. Hence, we have that P_1 and P_2 are k-short paths.

Note that $|\mathcal{P}|_k = \sum_{P\in\mathcal{P}-P_1-P_2} \min\{|P|,k\} + \min\{|P_1|,k\} + \min\{|P_2|,k\}$ and $|\mathcal{Q}|_k = \sum_{P\in\mathcal{P}-P_1-P_2} \min\{|P|,k\} + \min\{|Q|,k\}$. If $|Q| > k$, that is, if Q is k-long, then $|\mathcal{Q}|_k < |\mathcal{P}|_k$ and so $\mathcal{Q}$ satisfies case (i) of the lemma.

Thus, we may assume Q is k-short and, therefore, $|\mathcal{Q}|_k = |\mathcal{P}|_k$. The previous argument shows that if there exists an arc of D connecting two vertices of $e(\mathcal{P})$, then both of them must be ends of k-short paths of $\mathcal{P}$. From $\mathcal{Q}$ we show how to find a new path partition of D which satisfies either (i) or (ii).

First suppose that there exists a partial k-coloring $\mathcal{C}$ orthogonal to $\mathcal{Q}$ and let P be a path in $\mathcal{Q} - Q = \mathcal{P} - P_1 - P_2$. Hence, P meets $\min\{|P|,k\}$ color classes of $\mathcal{C}$, since $\mathcal{C}$ is orthogonal to $\mathcal{Q}$. In order to prove that $\mathcal{C}$ is also orthogonal to $\mathcal{P}$, it remains to show that P_i meets $\min\{|P_i|,k\} = |P_i|$ color classes for $i \in \{1,2\}$. Since Q is k-short, we know that each of its vertices meets a different color class of $\mathcal{C}$. Therefore, every vertex in P_i, for $i \in \{1,2\}$, also meets a distinct color class of $\mathcal{C}$, after all $V(Q) = V(P_1) \cup V(P_2)$, and so $\mathcal{C}$ is indeed orthogonal to $\mathcal{P}$. So, we assume there exists no partial k-coloring orthogonal to $\mathcal{Q}$.

If $e(\mathcal{Q}^{\leq k})$ is stable, then $e(\mathcal{Q})$ is stable and case (ii) holds. Thus, we may assume that $e(\mathcal{Q}^{\leq k})$ is not stable and the remaining proof is by induction on the number ℓ of k-short paths in $\mathcal{P}$.

Since $e(\mathcal{P}^{\leq k})$ is not stable, we have $\ell \geq 2$. If $\ell = 2$, then $\mathcal{P}^{\leq k} = \{P_1, P_2\}$. Therefore, $e(\mathcal{Q}^{\leq k}) = e(Q)$ and so $\mathcal{Q}$ satisfies case (ii). Now suppose $\ell > 2$. By the induction hypothesis applied to $\mathcal{Q}$, there exists a path partition $\mathcal{Q}'$ of D such that case (i) or (ii) holds. If (i) holds, then $e(\mathcal{Q}') \subset e(\mathcal{Q})$ and $|\mathcal{Q}'|_k < |\mathcal{Q}|_k$. By construction of $\mathcal{Q}$, this also means that $e(\mathcal{Q}') \subset e(\mathcal{P})$ and $|\mathcal{Q}'|_k < |\mathcal{P}|_k$, and so case (i) holds for $\mathcal{P}$. Otherwise (ii) holds for $\mathcal{Q}'$, that is, $|\mathcal{Q}'|_k = |\mathcal{Q}|_k = |\mathcal{P}|_k$, $e(\mathcal{Q}') \subseteq e(\mathcal{Q}) \subset e(\mathcal{P})$, $e(\mathcal{Q}')$ is stable, and every partial k-coloring orthogonal to $\mathcal{Q}'$ is also orthogonal to $\mathcal{Q}$ and, therefore, to $\mathcal{P}$. Thus, (ii) holds for $\mathcal{P}$. ■

Given a path $P = v_1v_2\dots v_\ell$, we write $v_iP = v_iv_{i+1}\dots v_\ell$, $Pv_j = v_1v_2\dots v_j$, and $v_iPv_j = v_iv_{i+1}\dots v_j$ to denote the appropriate subpath of P. Also, using the definitions of $\mathcal{P}^{>k}$ and $\mathcal{P}^{\leq k}$ given above, note that the k-norm of a path partition $\mathcal{P}$ can equivalently be defined as $k|\mathcal{P}^{>k}| + |V(\mathcal{P}^{\leq k})|$.

Next theorem is the first main result of this paper. It shows that any path partition of an in-semicomplete digraph either has a partial k-coloring orthogonal to it or can be turned into a new path partition with smaller k-norm. Corollaries 2.1 and 2.2 state the meaning of such result for Berge's Conjecture.

Theorem 2.1. *Let D be an in-semicomplete digraph, let k be a positive integer, and let $\mathcal{P}$ be a path partition of D. Then there exists*

(i) *a partial k-coloring of D orthogonal to $\mathcal{P}$; or*
(ii) *a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}|_k < |\mathcal{P}|_k$ and $e(\mathcal{Q}) \subset e(\mathcal{P})$.*

Proof. By Lemma 2.1, there exists a path partition $\mathcal{Q}$ of D such that either (a) $|\mathcal{Q}|_k < |\mathcal{P}|_k$ and $e(\mathcal{Q}) \subset e(\mathcal{P})$ or (b) $|\mathcal{Q}|_k = |\mathcal{P}|_k$, $e(\mathcal{Q}) \subseteq e(\mathcal{P})$, $e(\mathcal{Q})$ is stable, and every partial k-coloring orthogonal to $\mathcal{Q}$ is also orthogonal to $\mathcal{P}$. If (a) holds, then case (ii) holds directly. Therefore, we may assume that (b) holds. Note that this reduces the problem of proving the result for $\mathcal{P}$ to the problem of proving it for $\mathcal{Q}$ and, thus, from now on we can only consider $\mathcal{Q}$.

The remaining proof follows by induction on k. If $k = 1$, then $e(\mathcal{Q})$ is a partial 1-coloring orthogonal to $\mathcal{Q}$ and (i) holds. Otherwise, $k > 1$ and let $D' = D[V(D)\backslash e(\mathcal{Q})]$ and $\mathcal{Q}' = \{Qu_{\ell-1} \colon Q = u_1u_2\dots u_\ell \in \mathcal{Q}\}$. Note that $\mathcal{Q}'$ is a path partition for D'. By the induction hypothesis applied to D', $\mathcal{Q}'$, and $k-1$, we have that there exists (a) a partial $(k-1)$-coloring $\mathcal{C}$ of D' orthogonal to $\mathcal{Q}'$ or (b) a path partition $\mathcal{R}'$ or D' such that $|\mathcal{R}'|_{k-1} < |\mathcal{Q}'|_{k-1}$ and $e(\mathcal{R}') \subset e(\mathcal{Q}')$. If (a) holds, then $\mathcal{C} + e(\mathcal{Q})$ is a partial k-coloring orthogonal to $\mathcal{Q}$ and case (i) holds.

So we assume that case (b) holds. Let $e(\mathcal{Q}') = \{v_1, v_2, \dots, v_\ell\}$ and $u_i \in e(\mathcal{Q})$ be the sucessor of v_i in its path in $\mathcal{Q}$. Note that each path of $\mathcal{R}'$ ends at some v_i, so name such path as R'_i. Let

$\mathcal{R} = \{R'_i v_i u_i \colon R'_i \in \mathcal{R}'^{>k-1}\} \cup \mathcal{R}'^{\leq k-1} \cup (e(\mathcal{Q}) \backslash \{u_i \colon R'_i \in \mathcal{R}'^{>k-1}\})$. In other words, $\mathcal{R}$ is built by the extensions of all $(k-1)$-long paths of $\mathcal{R}'$, plus all $(k-1)$-short paths of $\mathcal{R}'$, plus all single vertices of $e(\mathcal{Q})$ which were not used to extend the $(k-1)$-long paths of $\mathcal{R}'$. It is easy to see that $\mathcal{R}$ is a path partition for D.

We know that $|\mathcal{R}'|_{k-1} = (k-1)|\mathcal{R}'^{>k-1}| + |V(\mathcal{R}'^{\leq k-1})| = k|\mathcal{R}'^{>k-1}| + |V(\mathcal{R}'^{\leq k-1})| - |\mathcal{R}'^{>k-1}|$, by definition. By construction of $\mathcal{R}$, $|\mathcal{R}|_k = k|\mathcal{R}'^{>k-1}| + |V(\mathcal{R}'^{\leq k-1})| + |e(\mathcal{Q})| - |\mathcal{R}'^{>k-1}| = |\mathcal{R}'|_{k-1} + |\mathcal{Q}|$. Also, by construction of $\mathcal{Q}'$, $|\mathcal{Q}'|_{k-1} = (k-1)|\mathcal{Q}^{>k}| + |V(\mathcal{Q}^{\leq k})| - |\mathcal{Q}^{\leq k}| = k|\mathcal{Q}^{>k}| + |V(\mathcal{Q}^{\leq k})| - |\mathcal{Q}^{>k}| - |\mathcal{Q}^{\leq k}| = |\mathcal{Q}|_k - |\mathcal{Q}|$. Putting everything together, we have $|\mathcal{R}|_k = |\mathcal{R}'|_{k-1} + |\mathcal{Q}| < |\mathcal{Q}'|_{k-1} + |\mathcal{Q}| = |\mathcal{Q}|_k - |\mathcal{Q}| + |\mathcal{Q}| = |\mathcal{Q}|_k$ and, therefore, case (ii) holds for $\mathcal{Q}$. ■

Corollary 2.1. *If $\mathcal{P}$ is a k-optimal path partition of an in-semicomplete digraph D, then there exists a partial k-coloring of D orthogonal to $\mathcal{P}$.*

Corollary 2.2. *If $\mathcal{P}$ is a k-optimal path partition of an out-semicomplete digraph D, then there exists a partial k-coloring of D orthogonal to $\mathcal{P}$.*

Proof. The *inverse* of a digraph (path partition) B, denoted by B^-, is the digraph (path partition) built from B by inverting its arcs, that is, if $uv \in A(B)$, then $vu \in A(B^-)$. If $\mathcal{Q}$ is a path partition of D, then $\mathcal{Q}^-$ is a path partition of D^- with $|\mathcal{Q}|_k = |\mathcal{Q}^-|_k$, and vice-versa. Thus, we have that $\mathcal{P}^-$ is k-optimal in D^-, and since D^- is an in-semicomplete digraph, by Corollary 2.1 there exists a partial k-coloring $\mathcal{C}$ orthogonal to $\mathcal{P}^-$. Clearly, $\mathcal{C}$ is also orthogonal to $\mathcal{P}$. ■

§3. Results for Aharoni-Hartman-Hoffman's Conjecture

Recall that, given a k-pack $\mathcal{P}$, we define $e(\mathcal{P}) = \{e(P) \colon P \in \mathcal{P}\}$. Similarly to the result of Lemma 2.1, the next lemma shows that it is possible to convert one k-pack $\mathcal{P}$ into another k-pack $\mathcal{Q}$ with the same weight such that $e(\mathcal{Q})$ is a stable set.

Lemma 3.1. *Let $\mathcal{P}$ be a k-pack of an in-semicomplete digraph D. Then there exists a k-pack $\mathcal{Q}$ of D such that $||\mathcal{Q}|| = ||\mathcal{P}||$, $e(\mathcal{Q}) \subseteq e(\mathcal{P})$, and $e(\mathcal{Q})$ is stable.*

Proof. The proof is by induction on $\ell = |\mathcal{P}|$. If $e(\mathcal{P})$ is stable, then $\mathcal{Q} = \mathcal{P}$ satisfies the lemma's conclusion and the result follows. Thus, we may assume $e(\mathcal{P})$ is not stable. Let u and v in $e(\mathcal{P})$ such that $uv \in A(D)$. Let P_1 and P_2 be the paths in $\mathcal{P}$ which end in u and v, respectively. By Theorem 1.7, there exists a path Q in D such that $V(Q) = V(P_1) \cup V(P_2)$ and $e(Q) = e(P_2)$. Let $\mathcal{Q}$ be the k-pack of D defined as $\mathcal{P} - P_1 - P_2 + Q$. By the induction hypothesis, there exists a k-pack $\mathcal{R}$ such that $||\mathcal{R}|| = ||\mathcal{Q}||$, $e(\mathcal{R}) \subseteq e(\mathcal{Q})$, and $e(\mathcal{R})$ is a stable set. By construction, $||\mathcal{Q}|| = ||\mathcal{P}||$ and $e(\mathcal{Q}) \subset e(\mathcal{P})$, so the result follows directly. ■

Next theorem is another main result of this paper. It shows that any k-pack of an in-semicomplete digraph either has a coloring orthogonal to it or can be turned into a k-pack with larger weight. Corollaries 3.1 and 3.2 state the meaning of such result for Aharoni-Hartman-Hoffman's Conjecture. To simplify the notation, given a k-pack $\mathcal{P}$ of D, we denote by $\overline{\mathcal{P}}$ the vertex set $V(D)\backslash V(\mathcal{P})$. Recall that, given a path $P = v_1v_2 \ldots v_\ell$, we write $v_iP = v_iv_{i+1} \ldots v_\ell$, $Pv_j = v_1v_2 \ldots v_j$, and $v_iPv_j = v_iv_{i+1} \ldots v_j$ to denote the appropriate subpaths.

Theorem 3.1. *Let D be an in-semicomplete digraph, let k be a positive integer, and let $\mathcal{P}$ be a k-pack of D. Then there exists*

(i) *a coloring of D orthogonal to $\mathcal{P}$; or*
(ii) *a k-pack $\mathcal{Q}$ of D such that $||\mathcal{Q}|| = ||\mathcal{P}|| + 1$ and $e(\mathcal{Q}) \subseteq e(\mathcal{P}) \cup \overline{\mathcal{P}}$.*

Proof. The proof is by induction on $|\overline{\mathcal{P}}|$. If $\overline{\mathcal{P}} = \varnothing$, then the coloring $\{\{v\}\colon v \in V(D)\}$ is orthogonal to $\mathcal{P}$ and case (i) holds. Thus, we may assume $\overline{\mathcal{P}} \neq \varnothing$. Let v be a vertex in $\overline{\mathcal{P}}$ and let $\mathcal{Q} = \mathcal{P} + v$. If $|\mathcal{Q}| \leq k$, then $\mathcal{Q}$ satisfies case (ii) and the result follows. Thus, we have $|\mathcal{Q}| = k + 1$, since $|\mathcal{P}| \leq k$. If $e(\mathcal{Q})$ is not stable, then there exist two paths P_1 and P_2 in $\mathcal{Q}$ such that $e(P_1)$ and $e(P_2)$ are adjacent and, by Theorem 1.7, there exists a path Q such that $V(Q) = V(P_1) \cup V(P_2)$ and $e(Q) \in \{e(P_1), e(P_2)\}$. Therefore, $\mathcal{Q} - P_1 - P_2 + Q$ is a k-pack which satisfies case (ii). Since v was chosen arbitrarily from $\overline{\mathcal{P}}$, we have that for any $v \in \overline{\mathcal{P}}$, $e(\mathcal{P}) + v$ is stable. In particular, $e(\mathcal{P})$ is stable.

Let $S \subseteq \overline{\mathcal{P}}$ be a maximum stable set in $D[\overline{\mathcal{P}}]$, let D' be the digraph $D[V(D)\backslash(e(\mathcal{P}) \cup S)]$, and let $\mathcal{R}$ be the k-pack of D' defined as $\{Pu_{\ell-1}\colon P = u_1u_2 \ldots u_\ell \in \mathcal{P}\}$. Note that $||\mathcal{R}|| = ||\mathcal{P}|| - k$ and $\overline{\mathcal{R}} \subset \overline{\mathcal{P}}$. By the induction hypothesis applied to D' and $\mathcal{R}$, we have that there exists (a) a coloring $\mathcal{C}$ of D' orthogonal to $\mathcal{R}$ or (b) a k-pack $\mathcal{B}$ of D' such that $||\mathcal{B}|| = ||\mathcal{R}|| + 1$ and $e(\mathcal{B}) \subseteq e(\mathcal{R}) \cup \overline{\mathcal{R}}$. If (a) holds, then $\mathcal{C} + (e(\mathcal{P}) \cup S)$ is a coloring orthogonal to $\mathcal{P}$ and case (i) holds. So we may assume that (b) holds. We will show that (ii) holds. By Lemma 3.1, we may assume that $e(\mathcal{B})$ is stable. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $e(\mathcal{B}_1) \subseteq e(\mathcal{R})$ and $e(\mathcal{B}_2) \subseteq \overline{\mathcal{R}}$.

Let $e(\mathcal{R}) = \{v_1, v_2, \ldots, v_\ell\}$ and let $u_i \in e(\mathcal{P})$ be the successor of v_i in a path of $\mathcal{P}$. Note that every path of $\mathcal{B}_1$ ends at some v_i, so name such path as B_i. Let $\mathcal{Q}_1 = \{B_iv_iu_i\colon B_i \in \mathcal{B}_1\}$, that is, we built $\mathcal{Q}_1$ by extending all paths of $\mathcal{B}_1$. Note that $||\mathcal{Q}_1|| = ||\mathcal{B}_1|| + |\mathcal{B}_1|$, $|\mathcal{Q}_1| = |\mathcal{B}_1|$, and that $e(\mathcal{Q}_1) \subseteq e(\mathcal{P})$.

Now we will show that there exists a collection of paths in $\overline{\mathcal{P}}$ with weight $||\mathcal{B}_2||+|\mathcal{B}_2|$. Let G be the bipartite graph with vertex-set $V(G) = e(\mathcal{B}_2) \cup S$ and edge-set $E(G) = \{uv\colon u \in e(\mathcal{B}_2), v \in S$, and u and v are adjacent in $D\}$. We claim that there exists a matching in G which covers $e(\mathcal{B}_2)$. Suppose by contradiction that such matching does not exist. By Hall's Theorem, there exists $W \subseteq e(\mathcal{B}_2)$ such that $|W| > N(W)$, where $N(W) = \{u \in V(G)\colon uv \in E(G)$ and $v \in W\}$. Note that W is stable in $D[\overline{\mathcal{P}}]$, since $W \subseteq e(\mathcal{B})$, and no vertex in W is adjacent to a vertex in $S\backslash N(W)$ in D. Therefore, we have that $(S\backslash N(W)) \cup W$ is stable in $D[\overline{\mathcal{P}}]$ greater than S, which contradicts the choice of S. Hence, there exists a matching M in G which covers $e(\mathcal{B}_2)$. For each $u \in e(\mathcal{B}_2)$, let $M(u)$ be the vertex of S matched to u by M. Let $\mathcal{B}_2 = \{B_1, B_2, \ldots, B_p\}$

and let $B_i = w_1 w_2 \dots w_q$ be a path in $\mathcal{B}_2$. By Theorem 1.7, there exists a path Q_i such that $V(Q_i) = V(B_i) + M(w_q)$ and $e(Q_i) \in \{w_q, M(w_q)\}$. Let $\mathcal{Q}_2 = \{Q_i \colon B_i \in \mathcal{B}_2\}$. Note that $||\mathcal{Q}_2|| = ||\mathcal{B}_2|| + |\mathcal{B}_2|$, $|\mathcal{Q}_2| = |\mathcal{B}_2|$, and that $e(\mathcal{Q}_2) \subseteq \overline{\mathcal{P}}$.

Let $T = (e(\mathcal{P}) \cup S) \setminus (e(\mathcal{Q}_1) \cup e(\mathcal{Q}_2))$, that is, the set of vertices in $e(\mathcal{P}) \cup \overline{\mathcal{P}}$ that are not ends of a path in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Thus, $|T| = |e(\mathcal{P})| + |S| - |\mathcal{Q}_1| - |\mathcal{Q}_2| = k + |S| - |\mathcal{Q}_1| - |\mathcal{Q}_2| \geq k + 1 - |\mathcal{Q}_1| - |\mathcal{Q}_2|$.

Let U be a set of $k - |\mathcal{Q}_1| - |\mathcal{Q}_2|$ vertices in T. We can see U as a set of trivial paths. Finally, let $\mathcal{Q}$ be the k-pack of D defined as $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup U$. By the previous remarks it is easy to see that $e(\mathcal{Q}) \subseteq e(\mathcal{P}) \cup \overline{\mathcal{P}}$. At last, we have

$$\begin{aligned} ||\mathcal{Q}|| &= ||\mathcal{Q}_1|| + ||\mathcal{Q}_2|| + ||U|| = ||\mathcal{B}_1|| + |\mathcal{B}_1| + ||\mathcal{B}_2|| + |\mathcal{B}_2| + ||U|| = ||\mathcal{B}|| + |\mathcal{B}| + ||U|| \\ &= ||\mathcal{B}|| + |\mathcal{B}| + k - |\mathcal{Q}_1| - |\mathcal{Q}_2| = ||\mathcal{B}|| + |\mathcal{B}| + k - |\mathcal{B}_1| - |\mathcal{B}_2| = ||\mathcal{B}|| + |\mathcal{B}| + k - |\mathcal{B}| \\ &= ||\mathcal{R}|| + 1 + k = ||\mathcal{P}|| - k + 1 + k = ||\mathcal{P}|| + 1. \end{aligned}$$

Hence, we conclude that (ii) holds and the result follows. ■

Corollary 3.1. *If $\mathcal{P}$ is an optimal k-pack of an in-semicomplete digraph D, then there exists a coloring of D orthogonal to $\mathcal{P}$.*

Corollary 3.2. *If $\mathcal{P}$ is an optimal k-pack of an out-semicomplete digraph D, then there exists a coloring of D orthogonal to $\mathcal{P}$.*

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