

On the sizes of vertex- k -maximal r -uniform hypergraphs *

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Abstract Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of non-empty subsets of V called edges. If all edges of H have the same cardinality r , then H is a r -uniform hypergraph; if E consists of all r -subsets of V , then H is a complete r -uniform hypergraph, denoted by K_n^r , where $n = |V|$. A hypergraph $H' = (V', E')$ is called a subhypergraph of $H = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A r -uniform hypergraph $H = (V, E)$ is vertex- k -maximal if every subhypergraph of H has vertex-connectivity at most k , but for any edge $e \in E(K_n^r) \setminus E(H)$, $H + e$ contains at least one subhypergraph with vertex-connectivity at least $k+1$. In this paper, we first prove that for given integers n, k, r with $k, r \geq 2$ and $n \geq k+1$, every vertex- k -maximal r -uniform hypergraph H of order n satisfies $|E(H)| \geq \binom{n}{r} - \binom{n-k}{r}$, and this lower bound is best possible. Next, we conjecture that for sufficiently large n , every vertex- k -maximal r -uniform hypergraph H on n vertices satisfies $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + \left(\frac{n}{k} - 2\right)\binom{k}{r}$, where $k, r \geq 2$ are integers. And the conjecture is verified for the case $r > k$.

Keywords: Vertex-connectivity; Vertex- k -maximal hypergraphs; r -uniform hypergraphs

1 Introduction

In this paper, we consider finite simple graphs. For graph-theoretical terminologies and notation not defined here, we follow [4]. For a graph G , we use $\kappa(G)$ to denote the *vertex-connectivity* of G . The *complement* of a graph G is denoted by G^c . For $X \subseteq E(G^c)$, $G + X$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G + e$ for $G + \{e\}$. The *floor* of a real number x , denoted by $\lfloor x \rfloor$, is the greatest integer not larger than x ; the *ceil* of a real number x , denoted by $\lceil x \rceil$, is the least integer greater than or equal to x . For two integers n and k , we define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ when $k \leq n$ and $\binom{n}{k} = 0$ when $k > n$.

Matula [14] first explicitly studied the quantity $\bar{\kappa}(G) = \max\{\kappa(G') : G' \subseteq G\}$. For a positive integer k , the graph G is *vertex- k -maximal* if $\bar{\kappa}(G) \leq k$ but for any edge $e \in E(G^c)$, $\bar{\kappa}(G + e) > k$. Because $\kappa(K_n) = n - 1$, a vertex- k -maximal graph G with at most $k + 1$ vertices must be a complete graph.

The *union* of two graphs G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* of two graphs G_1 and G_2 , denoted

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by $G_1 \vee G_2$, is the graph obtained from the union of G_1 and G_2 by adding all the edges that connect the vertices of G_1 with G_2 . Let $G_{n,k} = ((p-1)K_k \cup K_q) \vee K_k^c$, where $n = pk + q \geq 2k$ ($1 \leq q \leq k$) and $(p-1)K_k$ is the union of $p-1$ complete graphs on k vertices. Then $G_{n,k}$ is vertex- k -maximal and $|E(G_{n,k})| \leq \frac{3}{2}(k - \frac{1}{3})(n - k)$, where the equality holds if n is a multiple of k . Mader [11] conjectured that, for large order of graphs, the graph $G_{n,k}$ would in fact present the best possible upper bound for the sizes of a vertex- k -maximal graph.

Conjecture 1. (Mader [11]) *Let $k \geq 2$ be an integer. Then for sufficiently large n , every vertex- k -maximal graph on n vertices satisfies $|E(G)| \leq \frac{3}{2}(k - \frac{1}{3})(n - k)$.*

Some progresses towards Conjecture 1 are listed in the following.

Theorem 1.1. *Let $k \geq 2$ be an integer.*

(i) (Mader [10], see also [11]) *Conjecture 1 holds for $k \leq 6$.*

(ii) (Mader [10], see also [11]) *For sufficiently large n , every vertex- k -maximal graph G on n vertices satisfies $|E(G)| \leq (1 + \frac{1}{\sqrt{2}})k(n - k)$.*

(iii) (Yuster [18]) *If $n \geq \frac{9k}{4}$, then every vertex- k -maximal graph G on n vertices satisfies $|E(G)| \leq \frac{193}{120}k(n - k)$.*

(iv) (Bernshteyn and Kostochka [3]) *If $n \geq \frac{5k}{2}$, then every vertex- k -maximal graph G on n vertices satisfies $|E(G)| \leq \frac{19}{12}k(n - k)$.*

In [17], Xu, Lai and Tian obtained the lower bound of the sizes of vertex- k -maximal graphs.

Theorem 1.2. (Xu, Lai and Tian [17]) *Let n, k be integers with $n \geq k + 1 \geq 3$. If G is a vertex- k -maximal graph on n vertices, then $|E(G)| \geq (n - k)k + \frac{k(k-1)}{2}$. Furthermore, this bound is best possible.*

The related studies on edge- k -maximal graphs have been conducted by quite a few researchers, as seen in [7,9,12,13,15], among others. For corresponding digraph problems, see [1,8], among others.

Let $H = (V, E)$ be a hypergraph, where V is a finite set and E is a set of non-empty subsets of V , called edges. An edge of cardinality 2 is just a graph edge. For a vertex $u \in V$ and an edge $e \in E$, we say u is *incident with e* or e is *incident with u* if $u \in e$. If all edges of H have the same cardinality r , then H is a *r -uniform hypergraph*; if E consists of all r -subsets of V , then H is a *complete r -uniform hypergraph*, denoted by K_n^r , where $n = |V|$. For $n < r$, the complete r -uniform hypergraph K_n^r is just the hypergraph with n vertices and no edges. The *complement* of a r -uniform hypergraph $H = (V, E)$, denoted by H^c , is the r -uniform hypergraph with vertex set V and edge set consisting of all r -subsets of V not in E . A hypergraph $H' = (V', E')$ is called a *subhypergraph* of $H = (V, E)$, denoted by $H' \subseteq H$, if $V' \subseteq V$ and $E' \subseteq E$. For $X \subseteq E(H^c)$, $H + X$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \cup X$; for $X' \subseteq E(H)$, $H - X'$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \setminus X'$. We use $H + e$ for $H + \{e\}$ and $H - e'$ for $H - \{e'\}$ when $e \in E(H^c)$ and $e' \in E(H)$. For $Y \subseteq V(H)$, we use $H[Y]$ to denote the hypergraph *induced* by Y , where $V(H[Y]) = Y$ and $E(H[Y]) = \{e \in E(H) : e \subseteq Y\}$. $H - Y$ is the hypergraph induced by $V(H) \setminus Y$.

Let H be a hypergraph and V_1, V_2, \dots, V_l be subsets of $V(H)$. An edge $e \in E(H)$ is (V_1, V_2, \dots, V_l) -crossing if $e \cap V_i \neq \emptyset$ for $1 \leq i \leq l$. If in addition, $e \subseteq \cup_{i=1}^l V_i$, then e is *exact*- (V_1, V_2, \dots, V_l) -crossing. The set of all (V_1, V_2, \dots, V_l) -crossing edges of H is denoted by $E_H[V_1, V_2, \dots, V_l]$; the set of all exact- (V_1, V_2, \dots, V_l) -crossing edges of H is denoted by $E_{H[V_1 \cup V_2 \cup \dots \cup V_l]}[V_1, V_2, \dots, V_l]$. Let $d_H(V_1, V_2, \dots, V_l) = |E_H[V_1, V_2, \dots, V_l]|$ and $d_{H[V_1 \cup V_2 \cup \dots \cup V_l]}(V_1, V_2, \dots, V_l) = |E_{H[V_1 \cup V_2 \cup \dots \cup V_l]}[V_1, V_2, \dots, V_l]|$. For a vertex $u \in V(H)$, we call $d_H(u) := d_H(\{u\}, V(H) \setminus \{u\})$ the *degree* of u in H . The *minimum degree* $\delta(H)$ of H is defined as $\min\{d_H(u) : u \in V\}$; the *maximum degree* $\Delta(H)$ of H is defined as $\max\{d_H(u) : u \in V\}$. When $\delta(H) = \Delta(H) = k$, we call H *k-regular*.

Given a hypergraph H , we define a *walk* in H to be an alternating sequence $v_1, e_1, v_2, \dots, e_s, v_{s+1}$ of vertices and edges of H such that: $v_i \in V(H)$ for $i = 1, \dots, s+1$; $e_i \in E(H)$ for $i = 1, \dots, s$; and $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, s$. A *path* is a walk with additional restrictions that the vertices are all distinct and the edges are all distinct. A hypergraph H is *connected* if for every pair of vertices $u, v \in V(H)$, there is a path connecting u and v ; otherwise H is *disconnected*. A *component* of a hypergraph H is a maximal connected subhypergraph of H . A subset $X \subseteq V$ is called a *vertex-cut* of H if $H - X$ is disconnected. We define the *vertex-connectivity* of H , denoted by $\kappa(H)$, as follows: if H had at least one vertex-cut, then $\kappa(H)$ is the cardinality of a minimum vertex-cut of H ; otherwise $\kappa(H) = |V(H)| - 1$. We call a hypergraph H *k-vertex-connected* if $\kappa(H) \geq k$. Let $\bar{\kappa}(H) = \max\{\kappa(H') : H' \subseteq H\}$. For a positive integer k , the r -uniform hypergraph H is *vertex-k-maximal* if $\bar{\kappa}(H) \leq k$ but for any edge $e \in E(H^c)$, $\bar{\kappa}(H + e) > k$. Since $\kappa(K_n^r) = n - r + 1$, we note that H is complete if H is a vertex- k -maximal r -uniform hypergraph with $n - r + 1 \leq k$, where $n = |V(H)|$. The edge- k -maximal hypergraph can be defined similarly. For results on the connectivity of hypergraphs, see [2,5,6] for references.

In [16], we determined, for given integers n, k and r , the extremal sizes of an edge- k -maximal r -uniform hypergraph on n vertices.

Theorem 1.3. (Tian, Xu, Lai and Meng [16]) *Let k and r be integers with $k, r \geq 2$, and let $t = t(k, r)$ be the largest integer such that $\binom{t-1}{r-1} \leq k$. That is, t is the integer satisfying $\binom{t-1}{r-1} \leq k < \binom{t}{r-1}$. If H is an edge- k -maximal r -uniform hypergraph with $n = |V(H)| \geq t$, then*

- (i) $|E(H)| \leq \binom{t}{r} + (n - t)k$, and this bound is best possible;
- (ii) $|E(H)| \geq (n - 1)k - ((t - 1)k - \binom{t}{r})\lfloor \frac{n}{t} \rfloor$, and this bound is best possible.

The main goal of this research is to investigate, for given integers n, k and r , the extremal sizes of a vertex- k -maximal r -uniform hypergraph on n vertices. Section 2 below is devoted to the study of some properties of vertex- k -maximal r -uniform hypergraphs. In Section 3, we give the best possible lower bound of the sizes of vertex- k -maximal r -uniform hypergraphs. We propose a conjecture on the upper bound of the sizes of vertex- k -maximal r -uniform hypergraphs and verify the conjecture for the case $r > k$ in Section 4.

2 Properties of vertex- k -maximal r -uniform hypergraphs

Combining the definition of vertex- k -maximal r -uniform hypergraph with $\kappa(K_n^r) = n - r + 1$, we obtain that H is isomorphic to K_n^r if H is a vertex- k -maximal r -uniform hypergraph with

$$n = |V(H)| \leq k + r - 1.$$

Lemma 2.1. *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r - 1$. If H is a vertex- k -maximal r -uniform hypergraph on n vertices, then $\bar{\kappa}(H) = \kappa(H) = k$.*

Proof. Since H is vertex- k -maximal, we have $\kappa(H) \leq \bar{\kappa}(H) \leq k$. In order to complete the proof, we only need to show that $\kappa(H) \geq k$.

If $n = k + r - 1$, then H is complete and $\kappa(H) = n - r + 1 = k$. Thus, assume $n \geq k + r$, and so H is not complete. On the contrary, assume $\kappa(H) < k$. Since H is not complete, H has a vertex-cut S with $|S| = \kappa(H) < k$. Let C_1 be a component of $H - S$ and $C_2 = H - (S \cup V(C_1))$. By $|V(C_1) \cup V(C_2)| = n - |S| \geq k + r - (k - 1) = r + 1$, we can choose a r -subset $e \subseteq V(C_1) \cup V(C_2)$ such that $e \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Then $e \in E(H^c)$.

Since H is vertex- k -maximal, we have $\bar{\kappa}(H + e) \geq k + 1$. Hence $H + e$ contains a subhypergraph H' with $\kappa(H') = \bar{\kappa}(H + e) \geq k + 1$. Since $\bar{\kappa}(H) \leq k$, H' cannot be a subhypergraph of H , and so $e \in E(H')$. Since $V(H') \cap V(C_i) \neq \emptyset$ for $i = 1, 2$, it follows that $V(H') \cap S$ is a vertex-cut of $H' - e$.

Since $|V(C_1) \cup V(C_2)| = n - |S| \geq k + r - (k - 1) = r + 1 \geq 3$, one of C_i , say C_1 , contains at least two vertices. Let $u_1 \in e \cap V(C_1)$. Then $S' = (V(H') \cap S) \cup \{u_1\}$ is a vertex-cut of H' , and so we obtain

$$k + 1 > |S| + 1 \geq |V(H') \cap S| + 1 = |S'| \geq \kappa(H') \geq k + 1,$$

a contradiction. \square

Let H be a vertex- k -maximal r -uniform hypergraph with $|V(H)| \geq k + r$. By Lemma 2.1, $\bar{\kappa}(H) = \kappa(H) = k$. By $|V(H)| \geq k + r$, H is not complete, thus H contains vertex-cuts. Let S be a minimum vertex-cut of H , C_1 be a component of $H - S$ and $C_2 = H - (S \cup V(C_1))$. We call (S, H_1, H_2) a *separation triple* of H , where $H_1 = H[S \cup V(C_1)]$ and $H_2 = H[S \cup V(C_2)]$.

Lemma 2.2. *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r$, and H be a vertex- k -maximal r -uniform hypergraph on n vertices. Assume (S, H_1, H_2) is a separation triple of H . If $e \in E(H_1^c) \cup E(H_2^c)$, then any subhypergraph H' of $H + e$ with $\kappa(H') \geq k + 1$ is either a subhypergraph of $H_1 + e$ or a subhypergraph of $H_2 + e$. Furthermore, if $e \subseteq E(H_i^c) \setminus E((H[S])^c)$, then H' is a subhypergraph of $H_i + e$ for $i = 1, 2$.*

Proof. Let $e \in E(H_1^c) \cup E(H_2^c)$. Since H is vertex- k -maximal, we have $\bar{\kappa}(H + e) \geq k + 1$. Let H' be a subhypergraph of $H + e$ with $\kappa(H') = \bar{\kappa}(H + e) \geq k + 1$. We assume, on the contrary, that $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) \neq \emptyset$. This, together with $e \in E(H_1^c) \cup E(H_2^c)$, implies that $S \cap V(H')$ is a vertex-cut of H' . Hence $k = |S| \geq |S \cap V(H')| \geq \kappa(H') \geq k + 1$, a contradiction. Therefore, we cannot have both $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) \neq \emptyset$. If $V(H') \cap (V(H_1) - S) = \emptyset$, then H' is a subhypergraph of $H_2 + e$; if $V(H') \cap (V(H_2) - S) = \emptyset$, then H' is a subhypergraph of $H_1 + e$.

If $e \subseteq E(H_1^c) \setminus E((H[S])^c)$, then $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) = \emptyset$, thus H' is a subhypergraph of $H_1 + e$. Similarly, if $e \subseteq E(H_2^c) \setminus E((H[S])^c)$, then H' is a subhypergraph of $H_2 + e$. \square

Lemma 2.3. *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r$, and H be a vertex- k -maximal r -uniform hypergraph on n vertices. Assume (S, H_1, H_2) is a separation triple of H and $n_i = |V(H_i)|$ for $i = 1, 2$. Then*

$$(i) E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset, \text{ and}$$

$$(ii) d_H(V(H_1) - S, S, V(H_2) - S) = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r}.$$

Proof. (i) By contradiction, assume $E_{H^c}[V(H_1) - S, S, V(H_2) - S] \neq \emptyset$. Let $e \in E_{H^c}[V(H_1) - S, S, V(H_2) - S]$. Since H is vertex- k -maximal, there is a subhypergraph H' of $H + e$ such that $\kappa(H') = \bar{\kappa}(H + e) \geq k + 1$. By $\bar{\kappa}(H) \leq k$, $e \in E(H')$. This, together with $e \in E_{H^c}[V(H_1) - S, S, V(H_2) - S]$, implies $V(H') \cap S \neq \emptyset$ and $V(H') \cap (V(H_i) - S) \neq \emptyset$ for $i = 1, 2$. Hence $S \cap V(H')$ is a vertex-cut of H' . But then we obtain $k = |S| \geq |S \cap V(H')| \geq \kappa(H') \geq k + 1$, a contradiction. It follows $E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset$.

(ii) By (i), $E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset$. This implies that if e is a r -subset such that $e \cap S \neq \emptyset$ and $e \cap (V(H_i) - S) \neq \emptyset$ for $i = 1, 2$, then $e \in E(H)$. Since the number of r -subsets contained in $V(H_1)$ or $V(H_2)$ is $\binom{n_1}{r} + \binom{n_2}{r} - \binom{k}{r}$, and the number of r -subsets exactly intersecting $V(H_1) - S$ and $V(H_2) - S$ is $\binom{n-k}{r} - \binom{n_1-k}{r} - \binom{n_2-k}{r}$, we have

$$\begin{aligned} d_H(V(H_1) - S, S, V(H_2) - S) &= |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &= \binom{n}{r} - ((\binom{n_1}{r} + \binom{n_2}{r}) - \binom{k}{r}) - ((\binom{n-k}{r} - \binom{n_1-k}{r} - \binom{n_2-k}{r})) \\ &= \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r}. \end{aligned}$$

This completes the proof. \square

3 The lower bound of the sizes of vertex- k -maximal r -uniform hypergraphs

The *union* of two hypergraphs H_1 and H_2 , denoted by $H_1 \cup H_2$, is the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. The *r -join* of two hypergraphs H_1 and H_2 , denoted by $H_1 \vee_r H_2$, is the hypergraph obtained from the union of H_1 and H_2 by adding all the edges with cardinality r that connect the vertices of H_1 with H_2 .

Definition 1. Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k + 1$. We define $H_L(n; k, r)$ to be $K_k^r \vee_r (K_{n-k}^r)^c$.

Lemma 3.1. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k + 1$. If $H = H_L(n; k, r)$, then*

$$(i) H \text{ is vertex-}k\text{-maximal, and}$$

$$(ii) |E(H)| = \binom{n}{r} - \binom{n-k}{r}.$$

Proof. (i) By Definition 1, H is obtained from the union of K_k^r and $(K_{n-k}^r)^c$ by adding all edges with cardinality r connecting $V(K_k^r)$ with $V((K_{n-k}^r)^c)$.

Since $V(K_k^r)$ is a vertex-cut of H and $H - V(K_k^r) = (K_{n-k}^r)^c$, there is no subhypergraph with vertex-connectivity at least $k + 1$, and so $\bar{\kappa}(H) \leq k$. If $E(H^c) = \emptyset$, then H is vertex- k -maximal

by the definition of vertex- k -maximal hypergraph. If $E(H^c) \neq \emptyset$, then for any $e \in E(H^c)$, e must be contained in $V((K_{n-k}^r)^c)$, and so $(H+e)[V(K_k^r) \cup e]$ is isomorphic to K_{k+r}^r and $\kappa((H+e)[V(K_k^r) \cup e]) = k+1$. That is $\bar{\kappa}(H+e) \geq k+1$. Thus H is vertex- k -maximal.

(ii) holds by a direct calculation. \square

Theorem 3.2. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k+1$. If H is vertex- k -maximal, then $|E(H)| \geq \binom{n}{r} - \binom{n-k}{r}$.*

Proof. We will prove the theorem by induction on n . If $n \leq k+r-1$, then by H is vertex- k -maximal, we have $H \cong K_n^r$. Thus $|E(H)| = \binom{n}{r} = \binom{n}{r} - \binom{n-k}{r}$ by $n-k \leq r-1$.

Now we assume that $n \geq k+r$, and that the theorem holds for smaller value of n . Since H is vertex- k -maximal and $n \geq k+r$, we have H is not complete. By Lemma 2.1, $\bar{\kappa}(H) = \kappa(H) = k$, and so H has a separation triple (S, H_1, H_2) with $|S| = k$. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n_1, n_2 \geq k+1$ and $n = n_1 + n_2 - k$.

Since H is vertex- k -maximal, for any $e \in E((H[S])^c)$, there is a $(k+1)$ -vertex-connected subhypergraph H' of $H+e$. By Lemma 2.2, H' is either a subhypergraph of H_1+e or a subhypergraph H_2+e . Define

$$E_1 = \{e : e \in E((H[S])^c) \text{ and } \bar{\kappa}(H_1+e) = k\}$$

$$E_2 = \{e : e \in E((H[S])^c) \text{ and } \bar{\kappa}(H_2+e) = k\}$$

Claim. Each of the following holds.

(i) $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \subseteq E((H[S])^c)$.

(ii) There is a subset $E'_1 \subseteq E_1$ such that $H_1 + E'_1$ is vertex- k -maximal.

(iii) There is a subset $E'_2 \subseteq E_2$ such that $H_2 + E'_2$ is vertex- k -maximal.

By the definition, $E_1 \cup E_2 \subseteq E((H[S])^c)$. Since H is vertex- k -maximal, we have $E_1 \cap E_2 = \emptyset$, and so Claim (i) holds.

Assume first that $H_1 + E_1$ is complete. If $n_1 \leq k+r-1$, then $\bar{\kappa}(H_1 + E_1) \leq k$, and so $H_1 + E_1$ is vertex- k -maximal by the definition of vertex- k -maximal hypergraphs. If $n_1 \geq k+r$, then by $\bar{\kappa}(H_1) \leq \bar{\kappa}(H) \leq k$ and $\bar{\kappa}(H_1 + E_1) \geq k+1$, we can choose a maximum subset $E'_1 \subseteq E_1$ such that $\bar{\kappa}(H_1 + E'_1) \leq k$. It follows by the maximality of E'_1 and by the definition of vertex- k -maximal hypergraphs that $H_1 + E'_1$ is vertex- k -maximal. Next, we assume $H_1 + E_1$ is not complete. Take an arbitrary edge $e \in E((H_1 + E_1)^c)$. Then $e \in E(H^c)$, and so as H is vertex- k -maximal, $H+e$ contains a $(k+1)$ -vertex-connected subhypergraph H' with $e \in E(H')$. If $e \cap (V(H_1) - S) \neq \emptyset$, then by Lemma 2.2, H' is a subhypergraph of H_1+e . If $e \subseteq S$, then as $e \notin E_1$, we can choose H' such that H' is a subhypergraph of H_1+e . That is, $\bar{\kappa}(H_1 + E_1 + e) \geq k+1$. If $\bar{\kappa}(H_1 + E_1) \leq k$, then $H_1 + E_1$ is vertex- k -maximal. If $\bar{\kappa}(H_1 + E_1) \geq k+1$, then by $\bar{\kappa}(H_1) \leq \bar{\kappa}(H) \leq k$, we can choose a maximum subset $E'_1 \subseteq E_1$ such that $\bar{\kappa}(H_1 + E'_1) \leq k$. It also follows by the maximality of E'_1 and by the definition of vertex- k -maximal hypergraphs that $H_1 + E'_1$ is vertex- k -maximal. This verifies Claim (ii). By symmetry, Claim (iii) holds. Thus the proof of the Claim is complete.

By Claim (ii) and Claim (iii), there are $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2$ such that $H_1 + E'_1$ and $H_2 + E'_2$ are vertex- k -maximal. Since $n_1, n_2 \geq k+1$, by induction assumption, we have $|E(H_1 + E'_1)| \geq$

$\binom{n_1}{r} - \binom{n_1-k}{r}$ and $|E(H_2 + E'_2)| \geq \binom{n_2}{r} - \binom{n_2-k}{r}$. By Claim (i) and the definition of $(H[S])^c$, we have $|E'_1| + |E'_2| + |E(H[S])| \leq |E_1| + |E_2| + |E(H[S])| \leq |E((H[S])^c)| + |E(H[S])| = \binom{k}{r}$. Thus

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &= |E(H_1 + E'_1)| - |E'_1| + |E(H_2 + E'_2)| - |E'_2| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &\geq \binom{n_1}{r} - \binom{n_1-k}{r} + \binom{n_2}{r} - \binom{n_2-k}{r} - \binom{k}{r} \\ &\quad + \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r} \text{ (By Lemma 2.3)} \\ &= \binom{n}{r} - \binom{n-k}{r}. \end{aligned}$$

This proves Theorem 3.2. \square

By Lemma 3.1, the lower bound of the sizes of vertex- k -maximal hypergraphs given in Theorem 3.2 is best possible. If $r = 2$, then a r -uniform hypergraph H is just a graph. Thus Theorem 1.2 is a corollary of Theorem 3.2.

Corollary 3.3. (*Xu, Lai and Tian [17]*) *Let n, k be integers with $n \geq k+1 \geq 3$. If G is a vertex- k -maximal graph on n vertices, then $|E(G)| \geq \binom{n}{2} - \binom{n-k}{2} = (n-k)k + \frac{k(k-1)}{2}$. Furthermore, this bound is best possible.*

4 The upper bound of the sizes of vertex- k -maximal r -uniform hypergraphs

Definition 2. Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. Assume $n = pk + q$, where p, q are integers and $1 \leq q \leq k$. We define $H_U(n; k, r)$ to be $((p-1)K_k^r \cup K_q^r) \vee_r (K_k^r)^c$, where $(p-1)K_k^r$ is the union of $p-1$ complete r -uniform hypergraphs on k vertices.

Lemma 4.1. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. If $H = H_U(n; k, r)$, then*

- (i) H is vertex- k -maximal, and
- (ii) $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + \left(\frac{n}{k} - 2\right)\binom{k}{r}$, where the equality holds if n is a multiple of k .

Proof. (i) By Definition 2, $H = ((p-1)K_k^r \cup K_q^r) \vee_r (K_k^r)^c$. Denote the $p-1$ complete r -uniform hypergraphs on k vertices by $K_k^r(1), \dots, K_k^r(p-1)$. Let $H_0 = H[V((K_k^r)^c)]$, $H_p = H[V(K_q^r)]$ and $H_i = H[V(K_k^r(i))]$ for $1 \leq i \leq p-1$. Then $H = H_0 \vee_r (H_1 \cup \dots \cup H_p)$.

Since $V(H_0)$ is a vertex-cut of size k and every component of $H - V(H_0)$ has at most k vertices. It follows that H contains no $(k+1)$ -vertex-connected subhypergraphs, and so $\bar{\kappa}(H) \leq k$. If $E(H^c) = \emptyset$, then H is vertex- k -maximal by the definition of vertex- k -maximal hypergraphs. Thus we assume $E(H^c) \neq \emptyset$ in the following. Let $e \in E(H^c)$. If $e \subseteq V(H_0)$, then $H' = H[V(H_1) \cup e]$ is isomorphic to K_{k+r}^r , and so $\kappa(H') = k+1$. If $e \subseteq V(H_1) \cup \dots \cup V(H_p)$, let e be exact- $(V(H_{i_1}), \dots, V(H_{i_s}))$ -crossing. We will prove that $H'' = H[V(H_0) \cup V(H_{i_1}) \cup \dots \cup V(H_{i_s})] + e$ is $(k+1)$ -vertex-connected. It suffices to prove that $H'' - S$ is connected for any $S \subseteq V(H'')$ with $|S| = k$. If $S = V(H_0)$, then, by e is exact- $(V(H_{i_1}), \dots, V(H_{i_s}))$ -crossing, $H'' - S$ is connected. So assume $V'_0 = V(H_0) \setminus S \neq \emptyset$. Let $V'_1 = (V(H_{i_1}) \cup \dots \cup V(H_{i_s})) \setminus S$. Then $H'' - S$ is isomorphic to $H[V'_0] \vee_r H[V'_1]$ if $S \cap e \neq \emptyset$; and $H'' - S$ is isomorphic to $H[V'_0] \vee_r H[V'_1] + e$ if $S \cap e = \emptyset$. Since $V'_0, V'_1 \neq \emptyset$ and $|V'_0 \cup V'_1| \geq r$, we obtain that $H'' - S$ is connected. Thus $\bar{\kappa}(H + e) \geq k+1$ for any $e \in E(H^c)$, and so H is vertex- k -maximal.

(ii) By a direct calculation, we have $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + \left(\frac{n}{k} - 2\right)\binom{k}{r}$, where the equality holds if n is a multiple of k . \square

Motivated by Conjecture 1, we propose the following conjecture for vertex- k -maximal r -uniform hypergraphs.

Conjecture 2. *Let k, r be integers with $k, r \geq 2$. Then for sufficiently large n , every vertex- k -maximal r -uniform hypergraph H on n vertices satisfies $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + \left(\frac{n}{k} - 2\right)\binom{k}{r}$.*

The following theorem confirms Conjecture 2 for the case $k < r$.

Theorem 4.2. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. If $k < r$, then every vertex- k -maximal r -uniform hypergraph H on n vertices satisfies $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + \left(\frac{n}{k} - 2\right)\binom{k}{r} = \binom{n}{r} - \binom{n-k}{r}$.*

Proof. We will prove the theorem by induction on n . If $n \leq k + r - 1$, then by H is vertex- k -maximal, we have $H \cong K_n^r$. Thus $|E(H)| = \binom{n}{r} = \binom{n}{r} - \binom{n-k}{r}$ by $n - k \leq r - 1$.

Now we assume that $n \geq k + r$, and that the theorem holds for smaller value of n . Since H is vertex- k -maximal and $n \geq k + r$, we have H is not complete. Let S be a minimum vertex-cut of H . By Lemma 2.1, $|S| = k$. Let C_1 be a minimum component of $H - S$ and $C_2 = H - (V(C_1) \cup S)$. Assume $H_1 = H[V(C_1) \cup S]$ and $H_2 = H[V(C_2) \cup S]$. Since $k < r$, we have $E((H[S])^c) = \emptyset$, and so H_1 and H_2 are both vertex- k -maximal by Lemma 2.2. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n = n_1 + n_2 - k$ and $k + 1 \leq n_1 \leq n_2$. We consider two cases in the following.

Case 1. $|V(C_1)| = 1$.

By $|V(C_1)| = 1$, we obtain that $n_2 = n - 1 \geq k + r - 1 \geq 2k$. Since H_2 is vertex- k -maximal, by induction assumption, we have $|E(H_2)| \leq \binom{n-1}{r} - \binom{n-k-1}{r}$. Thus

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &\leq \binom{k}{r-1} + \binom{n-1}{r} - \binom{n-k-1}{r} + \binom{n-1}{r-1} - \binom{k}{r-1} - \binom{n-k-1}{r-1} \\ &= \binom{n}{r} - \binom{n-k}{r}. \end{aligned}$$

Case 2. $|V(C_1)| \geq 2$.

By $|V(C_1)| \geq 2$, we obtain that C_1 contains edges, and so $|V(C_1)| \geq r$. Thus $n_2 \geq n_1 \geq k + r \geq 2k + 1$. Since both H_1 and H_2 are vertex- k -maximal, by induction assumption, we have $|E(H_i)| \leq \binom{n_i}{r} - \binom{n_i-k}{r}$ for $i = 1, 2$. Thus

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &\leq \binom{n_1}{r} - \binom{n_1-k}{r} + \binom{n_2}{r} - \binom{n_2-k}{r} \\ &\quad + \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r} \text{ (By Lemma 2.3)} \\ &= \binom{n}{r} - \binom{n-k}{r}. \end{aligned}$$

This completes the proof. \square

Combining Theorem 3.2 with Theorem 4.2, we have the following corollary.

Corollary 4.3. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. If $k < r$, then every vertex- k -maximal r -uniform hypergraph H on n vertices satisfies $|E(H)| = \binom{n}{r} - \binom{n-k}{r}$.*

References

- [1] J. Anderson, H.-J. Lai, X. Lin, M. Xu, On k -maximal strength digraphs, *J. Graph Theory* 84 (2017) 17-25.
- [2] M. A. Bahmanian, M. Šajna, Connection and separation in hypergraphs, *Theory and Applications of Graphs* 2(2) (2015) 0-24.
- [3] A. Bernshteyn, A. Kostochka, On the number of edges in a graph with no $(k + 1)$ -connected subgraphs, *Discrete Mathematics* 339 (2016) 682-688.
- [4] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics 244, Springer, Berlin, 2008.
- [5] C. Chekuri, C. Xu, Computing minimum cuts in hypergraphs, In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (Barcelona, 2017)*, SIAM (2017) 1085-1100.
- [6] M. Dewar, D. Pike, J. Proos, Connectivity in Hypergraphs, arXiv:1611.07087v3.
- [7] H.-J. Lai, The size of strength-maximal graphs, *J. Graph Theory* 14 (1990) 187-197.
- [8] X. Lin, S. Fan, H.-J. Lai, M. Xu, On the lower bound of k -maximal digraphs, *Discrete Math.* 339 (2016) 2500-2510.
- [9] W. Mader, Minimale n -fach kantenzusammenhängende graphen, *Math. Ann.* 191 (1971) 21-28.
- [10] W. Mader, Existenz n -fach zusammenhängender Teilgraphen in Graphen genügend großen Kantendichte, *Abh. Math. Sem. Univ. Hamburg* 37 (1972) 86-97.
- [11] W. Mader, Connectivity and edge-connectivity in finite graphs, in: B. Bollobás (Ed.), *Surveys in Combinatorics*, Cambridge University Press, London, 1979, pp. 66-95.
- [12] D. W. Matula, The cohesive strength of graphs, *The Many Facets of Graph Theory*, Lecture Notes in Mathematics, No. 110, G. Chartrand and S. F. Kapoor, eds., Springer-Verlag, Berlin, 1969, pp. 215-221.
- [13] D. Matula, K -components, clusters, and slicings in graphs, *SIAM J. Appl. Math.* 22 (1972) 459-480.
- [14] D. W. Matula, Subgraph connectivity numbers of a graph, *Theory and Applications of Graphs*, pp. 371-383 (Springer-Verlag, Berlin 1978).
- [15] D. W. Matula, Ramsey theory for graph connectivity, *J. Graph Theory* 7 (1983) 95-105.
- [16] Y. Z. Tian, L. Q. Xu, H.-J. Lai, J. X. Meng, On the sizes of k -edge-maximal r -uniform hypergraphs, arXiv:1802.08843v3.
- [17] L. Q. Xu, H.-J. Lai, Y. Z. Tian, On the sizes of vertex- k -maximal graphs, submitted.
- [18] R. Yuster, A note on graphs without k -connected subgraphs, *Ars Combin.* 67 (2003) 231-235.