List Coloring a Cartesian Product with a Complete Bipartite Factor

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Abstract

We study the list chromatic number of the Cartesian product of any graph G and a complete bipartite graph with partite sets of size a and b, denoted $\chi_{\ell}(G \Box K_{a,b})$. We have two motivations. A classic result on the gap between list chromatic number and the chromatic number tells us $\chi_{\ell}(K_{a,b}) = 1 + a$ if and only if $b \ge a^a$. Since $\chi_{\ell}(K_{a,b}) \le 1 + a$ for any $b \in \mathbb{N}$, this result tells us the values of b for which $\chi_{\ell}(K_{a,b})$ is as large as possible and far from $\chi(K_{a,b}) = 2$. In this paper we seek to understand when $\chi_{\ell}(G \Box K_{a,b})$ is far from $\chi(G \Box K_{a,b}) = \max{\chi(G), 2}$. It is easy to show $\chi_{\ell}(G \Box K_{a,b}) \le \chi_{\ell}(G) + a$. In 2006, Borowiecki, Jendrol, Král, and Miškuf showed that this bound is attainable if bis sufficiently large; specifically, $\chi_{\ell}(G \Box K_{a,b}) = \chi_{\ell}(G) + a$ whenever $b \ge (\chi_{\ell}(G) + a - 1)^{a|V(G)|}$. Given any graph G and $a \in \mathbb{N}$, we wish to determine the smallest b such that $\chi_{\ell}(G \Box K_{a,b}) = \chi_{\ell}(G) + a$. In this paper we show that the list color function, a list analogue of the chromatic polynomial, provides the right concept and tool for making progress on this problem. Using the list color function, we prove a general improvement on Borowiecki et al.'s 2006 result, and we compute the smallest such b exactly for some large families of chromatic-choosable graphs.

Keywords. graph coloring, list coloring, Cartesian product, list color function, chromatic choosable.

Mathematics Subject Classification. 05C15.

1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [19] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, ...\}$. For $k \in \mathbb{N}$, we write [k] for the set $\{1, 2, ..., k\}$. If G is a graph and $S \subseteq V(G)$, we write G[S] for the subgraph of G induced by S. For $v \in V(G)$, we write $d_G(v)$ for the degree of vertex v in the graph G. If G and H are vertex disjoint graphs, we write $G \lor H$ for the join of G and H.

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1.1 List Coloring

List coloring is a variation on the classic vertex coloring problem. It was introduced in the 1970's independently by Vizing [17] and Erdős, Rubin, and Taylor [4]. In the classic vertex coloring problem we seek a proper k-coloring of a graph G which is a coloring of the vertices of G with colors from [k] such that adjacent vertices receive different colors. The chromatic number of a graph, denoted $\chi(G)$, is the smallest k such that G has a proper k-coloring. For list coloring, we associate a list assignment, L, with a graph G such that each vertex $v \in V(G)$ is assigned a list of colors L(v) (we say L is a list assignment for G). The graph G is L-colorable if there exists a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a proper L-coloring of G). A list assignment L is called a k-assignment for G if |L(v)| = k for each $v \in V(G)$. The list chromatic number of a graph G, denoted $\chi_{\ell}(G)$, is the smallest k such that G is L-colorable whenever L is a k-assignment for G. We say G is k-choosable if $k \geq \chi_{\ell}(G)$.

It is immediately obvious that for any graph G, $\chi(G) \leq \chi_{\ell}(G)$. Both Vizing [17] and Erdős, Rubin, and Taylor [4] observed bipartite graphs can have arbitrarily large list chromatic number. This implies the gap between $\chi(G)$ and $\chi_{\ell}(G)$ can be arbitrarily large. The following result illustrates this.

Theorem 1. For $a, b \in \mathbb{N}$, $\chi_{\ell}(K_{a,b}) = a + 1$ if and only if $b \ge a^a$.

It is worth mentioning that Theorem 1 is often attributed to Vizing [17] or Erdős, Rubin, and Taylor [4], but it is not stated in those papers. It is best described as a folklore result. In this paper we prove results similar in flavor to Theorem 1 for Cartesian products.

Graphs in which $\chi(G) = \chi_{\ell}(G)$ are known as *chromatic-choosable* graphs [12]. The notion of chromatic-choosability has received considerable attention in the literature. Many families of graphs have been conjectured to be chromatic-choosable (see for example [2], [6], and [7]), and there are several families of graphs that have been shown to be chromatic-choosable (see for example [5], [8], [11], [13], and [16]). We are studying how Cartesian products with a complete bipartite factor can be far from being chromatic-choosable.

1.2 List Coloring Cartesian Products

The Cartesian product of graphs M and H, denoted $M \Box H$, is the graph with vertex set $V(M) \times V(H)$ and edges created so that (u, v) is adjacent to (u', v') if and only if either u = u' and $vv' \in E(H)$ or v = v' and $uu' \in E(M)$. Throughout this paper, if $G = M \Box H$ and $u \in V(M)$ (resp. $u \in V(H)$), we let V_u be the subset of V(G) consisting of the vertices with first (resp. second) coordinate u. By the definition of Cartesian product of graphs, it is easy to see $G[V_u]$ is a copy of H (resp. M), and we refer to $G[V_u]$ as the copy of H (resp. M) corresponding to u.

It is well known that $\chi(G \Box H) = \max{\chi(G), \chi(H)}$. On the other hand, the list chromatic number of the Cartesian product of graphs is not nearly as well understood. In 2006, Borowiecki, Jendrol, Král, and Miškuf [3] showed the following.

Theorem 2 ([3]). For any graphs G and H, $\chi_{\ell}(G \Box H) \leq \min\{\chi_{\ell}(G) + \operatorname{col}(H), \operatorname{col}(G) + \chi_{\ell}(H)\} - 1$.

Here $\operatorname{col}(G)$, the coloring number of a graph G, is the smallest integer d for which there exists an ordering, v_1, v_2, \ldots, v_n , of the elements in V(G) such that each vertex v_i has at most d-1 neighbors among $v_1, v_2, \ldots, v_{i-1}$. For any graph G, it is easy to see that Theorem 2 implies $\chi_{\ell}(G \Box K_{a,b}) \leq \chi_{\ell}(G) + a$. In proving that the bound in Theorem 2 is tight, Borowiecki, Jendrol, Král, and Miškuf also proved the following.

Theorem 3 ([3]). Suppose G is a graph with n vertices. Then, $\chi_{\ell}(G \Box K_{a,b}) = \chi_{\ell}(G) + a$ whenever $b \ge (\chi_{\ell}(G) + a - 1)^{an}$.

One motivation for this paper was: For any given graph G, can we improve upon the bound on b in Theorem 3? With this question in mind, for $a \in \mathbb{N}$, we let $f_a(G)$ be the smallest b such that $\chi_{\ell}(G \square K_{a,b}) = \chi_{\ell}(G) + a$. Along the lines of Theorem 1, computing $f_a(G)$ for some graph G and $a \in \mathbb{N}$ yields: $\chi_{\ell}(G \square K_{a,b}) = \chi_{\ell}(G) + a$ if and only if $b \geq f_a(G)$. Theorem 1 says that $f_a(K_1) = a^a$.

There are several other observations about $f_a(G)$ that are immediate. First, $\chi_\ell(G \square K_{a,0}) = \chi_\ell(G) < \chi_\ell(G) + a$ which implies that $f_a(G) \ge 1$. Second, Theorem 3 implies that $f_a(G) \le (\chi_\ell(G) + a - 1)^{a|V(G)|}$. This means $f_a(G)$ exists and is a natural number. Also, if G is a disconnected graph with components: H_1, H_2, \ldots, H_r , we have $f_a(G) = \max_{H_i, \chi_\ell(H_i) = \chi_\ell(G)} f_a(H_i)$. So, we will restrict our attention to connected graphs.

In this paper we use a list analogue of the chromatic polynomial called the list color function to find an upper bound on $f_a(G)$ that is an improvement on the result of Theorem 3. We also present some further results on computing $f_a(G)$ in the special case where G is a strongly chromatic-choosable graph.

1.3 The List Color Function and Strong Chromatic-Choosability

We let P(G, k) be the chromatic polynomial of the graph G; that is, P(G, k) is equal to the number of proper k-colorings of G. It can be shown that P(G, k) is a polynomial in k (see [1]). This notion was extended to list coloring as follows. If L is a list assignment for G, we use P(G, L) to denote the number of proper L-colorings of G. The list color function $P_{\ell}(G, k)$ is the minimum value of P(G, L) where the minimum is taken over all possible k-assignments L for G. Since a k-assignment could assign the same k colors to every vertex in a graph, it is clear that $P_{\ell}(G, k) \leq P(G, k)$ for each $k \in \mathbb{N}$. In general, the list color function can differ significantly from the chromatic polynomial for small values of k. However, Wang, Qian, and Yan [18] recently showed: If G is a connected graph with m edges, then $P_{\ell}(G, k) = P(G, k)$ whenever $k > \frac{m-1}{\ln(1+\sqrt{2})}$. Also see [10] and [15] for earlier results on the list color function.

In the case G is a complete graph or odd cycle, it is well known (see [14]) that $P(C_n, k) = (k-1)^n + (-1)^n (k-1)$ and $P(K_n, k) = \prod_{i=0}^{n-1} (k-i)$. It is easy to see that for each $n, k \in \mathbb{N}$, $P(K_n, k) = P_\ell(K_n, k)$, and it was recently shown in [9] that for each $n, k \in \mathbb{N}$, $P(C_n, k) = P_\ell(C_n, k)$.

In [8] we introduced the notion of strong chromatic-choosability, and we used the list color function to exactly compute f_1 for graphs that are strongly chromatic-choosable. Strong chromatic-choosability is a notion of criticality in the context of chromatic-choosability. A graph G is strong k-chromatic-choosable if $\chi(G) = k$ and every (k - 1)-assignment, L, for which G is not L-colorable has the property that the lists are the same on all vertices ¹. We

¹List assignments that assign the same list of colors to every vertex of a graph are called *constant*.

say G is strongly chromatic-choosable if it is strong $\chi(G)$ -chromatic-choosable. Note that if G is strong k-chromatic-choosable, then the only reason G is not (k-1)-choosable is that a proper (k-1)-coloring of G does not exist. Simple examples of strongly chromatic-choosable graphs include complete graphs, odd cycles, and the join of a complete graph and odd cycle (see [8] for many other examples). See Section 3 for a summary of their properties, etc. The following is proven in [8].

Theorem 4 ([8]). Let M be a strong k-chromatic-choosable graph. Then, $f_1(M) = P_{\ell}(M, k)$.

We will generalize this result in Theorem 8, as stated in the next section.

1.4 Outline of Results and an Open Questions

In this subsection we present an outline of the paper while stating our results and mentioning some open questions. Recall that $f_a(G)$ is defined to be the smallest b such that $\chi_\ell(G \Box K_{a,b}) = \chi_\ell(G) + a$.

In Section 2, we prove that for any graph G, $\chi_{\ell}(G \Box K_{a,b}) = \chi_{\ell}(G) + a$ whenever $b \geq (P_{\ell}(G, \chi_{\ell}(G) + a - 1))^a$. Thus, starting with any chromatic-choosable G, and taking its Cartesian product with a sequence of appropriate $K_{a,b}$ (with a = 0, 1, 2, ...), we can construct a sequence of graphs that at each step get one farther from being chromatic-choosable: for any $s \geq t \geq 2$ there exists a graph H with $\chi(H) = t$ and $\chi_{\ell}(H) = s$.

Theorem 5. For any graph G and $a \in \mathbb{N}$, $f_a(G) \leq (P_\ell(G, \chi_\ell(G) + a - 1))^a$.

It is easy to see that if G has at least one edge, then $P_{\ell}(G, \chi_{\ell}(G) + a - 1) < (\chi_{\ell}(G) + a - 1)^{|V(G)|}$. This implies that Theorem 5 is an improvement on Theorem 3 whenever G has an edge. We will see many examples in Section 3 that illustrate that the bound in Theorem 5 is tight (notice Theorem 4 shows the bound is tight when a = 1 and G is strongly chromatic-choosable). However, it is not the case that $f_a(G) = (P_{\ell}(G, \chi_{\ell}(G) + a - 1))^a$ for all graphs G and $a \in \mathbb{N}$ since it is easy to see that $f_1(C_{2n}) = 1$, yet $P_{\ell}(C_{2n}, 2) = 2$. This observation leads us to the following open question.

Question 6. For what graphs does $f_a(G) = (P_\ell(G, \chi_\ell(G) + a - 1))^a$ for each $a \in \mathbb{N}$?

It is possible to slightly modify the proof idea of Theorem 5 to obtain the following more general result.

Theorem 7. Suppose H is a bipartite graph with partite sets A and B where |A| = a and |B| = b. Let $\delta = \min_{v \in B} d_H(v)$. If $b \ge (P_\ell(G, \chi_\ell(G) + \delta - 1))^a$, then $\chi_\ell(G \square H) \ge \chi_\ell(G) + \delta$.

Notice that Theorem 7 gives us conditions on when $\chi_{\ell}(G \Box H)$ is guaranteed to be far from $\chi(G \Box H) = \max{\chi(G), 2}$ for any bipartite graph H. Furthermore, notice that when $\delta = a, H = K_{a,b}$. So, Theorem 7 implies Theorem 5.

In Section 3 we prove that if M is a strong k-chromatic choosable graph and $k \ge a+1$, then $\chi_{\ell}(M \square K_{a,b}) = \chi_{\ell}(G) + a$ if and only if $b \ge (P_{\ell}(M, \chi_{\ell}(M) + a - 1))^a$. This is a generalization of Theorem 4.

Theorem 8. If M is strongly chromatic-choosable and $\chi(M) \ge a + 1$, then $f_a(M) = (P_\ell(M, \chi_\ell(M) + a - 1))^a$.

So, we have that the bound in Theorem 5 is tight when our graph is strongly chromaticchoosable and its chromatic number is sufficiently large. There are infinite families of such graphs constructed in [8]. Theorem 8 and the fact that $P(K_n, k) = P_\ell(K_n, k)$ and $P(C_n, k) = P_\ell(C_n, k)$ whenever $n, k \in \mathbb{N}$ imply the following.

Corollary 9. The following statements hold.

(i) For any $l \in \mathbb{N}$, $f_2(C_{2l+1}) = (P_\ell(C_{2l+1}, 4))^2 = (3^{2l+1} - 3)^2 = 9(9^l - 1)^2$. (ii) For $n \in \mathbb{N}$ satisfying $n \ge a+1$, $f_a(K_n) = (P_\ell(K_n, n+a-1))^a = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$.

Notice that Corollary 9 (ii) shows that the bound in Theorem 5 is tight for any $a \in \mathbb{N}$. We do not know of any strongly chromatic-choosable graph M for which $f_a(M) < (P_\ell(M, \chi_\ell(M) + a - 1))^a$. This leads us to the following question.

Question 10. Does there exist a strongly chromatic-choosable graph M such that $f_a(M) < (P_\ell(M, \chi_\ell(M) + a - 1))^a$?

Another interesting question involves only complete graphs.

Question 11. Is it the case that $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each $n, a \in \mathbb{N}$?

Since $f_a(K_1) = a^a$, the answer to Question 11 is yes when n = 1. We have a rather tedious argument, which for the sake of brevity will not be presented in this paper, that shows $f_2(K_2) = 36$. One could ask questions analogous to Question 11 for any family of strongly chromatic-choosable graphs. We end Section 3 by proving a general lower bound on f_a for strongly chromatic-choosable graphs.

Theorem 12. Suppose M is a strong k-chromatic-choosable graph. Then,

$$\frac{(P_{\ell}(M, k+a-1))^a}{2^{k-1}} \le f_a(M).$$

Considering Theorem 8, Theorem 12 gives us something new when $\chi(M) < a + 1$.

2 General Upper Bound

In this section we will prove Theorem 5. Before we prove this theorem, we introduce some notation and terminology that will be used for the remainder of this paper. Whenever we have a graph of the form $H = G \Box K_{a,b}$ with $a, b \in \mathbb{N}$, we will assume that the vertex set of the first factor is $\{v_1, v_2, \ldots, v_n\}$. We also assume that the partite sets of the copy of $K_{a,b}$ used to form H are $\{u_1, u_2, \ldots, u_n\}$ and $\{w_1, w_2, \ldots, w_b\}$. If L is a list assignment for $H = G \Box K_{a,b}$ and f is a proper L-coloring of $H[\bigcup_{j=1}^{a} V_{u_j}]$, then we say f is a bad coloring for the copy of G corresponding to w_l if there is no proper L-coloring for $H[V_{w_l}]$ where L'is the list assignment for $H[V_{w_l}]$ given by $L'(v_i, w_l) = L(v_i, w_l) - \{f(v_i, u_j) : j \in [a]\}$ for each $i \in [n]^2$. We now present a straightforward lemma related to this notion of bad coloring.

²We use $L(v_i, w_l)$ rather than the technically correct $L((v_i, w_l))$.

Lemma 13. Suppose $H = G \Box K_{a,b}$ with $a, b \in \mathbb{N}$ and L is a list assignment for H. Suppose C is the set of all proper L-colorings of $H[\bigcup_{j=1}^{a} V_{u_j}]$. For each $f \in C$ there exists an $l \in [b]$ such that f is a bad coloring for the copy of G corresponding to w_l if and only if there is no proper L-coloring of H.

Proof. We prove the only if direction first. Suppose for the sake of contradiction that c is a proper L-coloring of H. Let c' be the proper L-coloring of $H[\bigcup_{j=1}^{a} V_{u_j}]$ obtained by restricting the domain of c to $\bigcup_{j=1}^{a} V_{u_j}$ (clearly $c' \in C$). We know that there is a $d \in [b]$ such that c' is a bad coloring for the copy of G corresponding to w_d . Let $L'(v_i, w_d) = L(v_i, w_d) - \{c'(v_i, u_j) : j \in [a]\}$ for each $i \in [n]$. Restricting the domain of c to V_{w_d} yields a proper L'-coloring of the copy of G corresponding to w_d which is a contradiction.

We now prove the contrapositive of the converse. Suppose there is a $g \in C$ such that for each $t \in [b]$, g is not a bad coloring for the copy of G corresponding to w_t . Let $L'(v_i, w_t) =$ $L(v_i, w_t) - \{c'(v_i, u_j) : j \in [a]\}$ for each $i \in [n]$ and $t \in [b]$. For each $t \in [b]$, since g is not a bad coloring for the copy of G corresponding to w_t , there is a proper L'-coloring of $H[V_{w_t}]$. Coloring each copy of G corresponding to a vertex in $\{w_1, w_2, \ldots, w_b\}$ according to a proper L'-coloring extends g to a proper L-coloring of H.

We are now ready to prove Theorem 5

Proof. Suppose $H = G \Box K_{a,b}$ and $\chi_{\ell}(G) = k$. Let $t = P_{\ell}(G, k + a - 1)$. In order to prove the desired, we must show that $\chi_{\ell}(H) = k + a$ when $b = t^a$. We already know $\chi_{\ell}(H) \leq k + a$ (by Theorem 2). So, we suppose that $b = t^a$, and we will construct a (k + a - 1)-assignment, L, for H such that there is no proper L-coloring of H.

Let G_i be the copy of G corresponding to u_i . We inductively assign lists of size (k+a-1) to the vertices in $\bigcup_{i=1}^{a} V_{u_i}$ as follows. We begin by assigning lists, L(v), to each $v \in V(G_1)$ such that there are exactly t distinct proper L-colorings of G_1 . Then, for each $1 < i \leq a$, we assign lists, L(v), to each $v \in V(G_i)$ such that there are exactly t distinct proper L-colorings of G_i and

$$\left(\bigcup_{v\in V(G_i)} L(v)\right) \bigcap \left(\bigcup_{j=1}^{i-1} \bigcup_{v\in V(G_j)} L(v)\right) = \emptyset$$

(this can be done by taking the lists for G_1 , thinking of the colors as natural numbers, and adding a sufficiently large natural number to each color in each list). Now, for $i \in [a]$, we let $c_{i,1}, c_{i,2}, \ldots, c_{i,t}$ denote the *t* distinct proper *L*-colorings of G_i . We note that there are exactly t^a proper *L*-colorings of $H[\bigcup_{i=1}^a V_{u_i}]$ (since for each $i \in [a]$ we have *t* choices in how we color G_i). Suppose we index the t^a proper *L*-colorings of $H[\bigcup_{i=1}^a V_{u_i}]$ as: $c^{(1)}, c^{(2)}, \ldots, c^{(t^a)}$.

Now, suppose that L' is a (k-1)-assignment for G such that there is no proper L'-coloring of G and

$$\left(\bigcup_{v\in V(G)} L'(v)\right) \bigcap \left(\bigcup_{j=1}^{a} \bigcup_{v\in V(G_j)} L(v)\right) = \emptyset.$$

Let G'_d be the copy of G corresponding to w_d . For each $d \in [t^a]$ we assign a list, L(v), of size (k + a - 1) to each $v \in V(G'_d)$ as follows. Suppose that the coloring $c^{(d)}$ is formed via the colorings: $c_{1,b_1}, c_{2,b_2}, \ldots, c_{a,b_a}$ (note that b_j is between 1 and t for each $j \in [a]$).

By construction, we know that $|\{c_{j,b_j}(v_i, u_j) : j \in [a]\}| = a$ for each $i \in [n]$. So, for each $(v_i, w_d) \in V(G'_d)$, we let

$$L(v_i, w_d) = L'(v_i) \bigcup \{ c_{j, b_j}(v_i, u_j) : j \in [a] \}.$$

This completes the construction of our (k + a - 1)-assignment for H.

Finally, notice that by construction $c^{(d)}$ is a bad coloring for G'_d for each $d \in [t^a]$. Lemma 13 implies that there is no proper L-coloring of H.

It is fairly easy to modify the idea of the proof of Theorem 5 in order to obtain a proof of Theorem 7. We would simply obtain graph H' from H by deleting edges in H until all vertices in B have degree δ . Then, we could use the same construction idea to obtain a $(\chi_{\ell}(G) + \delta - 1)$ -assignment for $G \Box H'$, L, such that there is no proper L-coloring of $G \Box H'$. This would then imply $\chi_{\ell}(G) + \delta - 1 < \chi_{\ell}(G \Box H') \leq \chi_{\ell}(G \Box H)$.

3 Computing f_a for Strongly Chromatic-Choosable Graphs

In this section we will prove Theorems 8 and 12. Suppose that M is a strong k-chromaticchoosable graph. Recall that this means $\chi(M) = k$ and every (k-1)-assignment, L, for which M is not L-colorable is constant. In this section our focus is studying $f_a(M)$. By Theorem 4 we know that $f_1(M) = P_{\ell}(M, k)$. So, we assume $a \geq 2$ throughout this section. Also, since the strong 1-chromatic-choosable graphs are edgeless graphs, we only consider strongly chromatic-choosable graphs with chromatic number at least 2 throughout this section (since we know $f_a(K_1) = a^a$ for each $a \in \mathbb{N}$).

There are several properties of strongly chromatic-choosable graphs that follow immediately from the definition. We now mention some of these results (all proofs of these results can be found in [8]).

Proposition 14 ([8]). Suppose M is a strong k-chromatic-choosable graph. Then, (i) $\chi_{\ell}(M) = k$ (i.e. M is chromatic-choosable); (ii) If L is a list assignment for M with $|L(v)| \ge k - 1$ for each $v \in V(M)$ and L is not a constant (k - 1)-assignment, then there exists a proper L-coloring of M; (iii) $M \lor K_p$ is strong (k + p)-chromatic-choosable for any $p \in \mathbb{N}$; (iv) For any $v \in V(M)$, $\chi(M - \{v\}) \le \chi_{\ell}(M - \{v\}) < k$; (v) k = 2 if and only if M is a K_2 ; (vi) k = 3 if and only if M is an odd cycle.

Proposition 15 ([8]). Suppose M is a strong k-chromatic-choosable graph. Suppose L is an arbitrary m-assignment for M with $m \ge k$. Then, for any $v \in V(M)$ and any $\alpha \in L(v)$, there is a proper L-coloring, c, for M such that $c(v) = \alpha$. Consequently,

$$P_{\ell}(M,m) \ge m \max_{v \in V(M)} P_{\ell}(M - \{v\}, m - 1) \ge m.$$

We now prove two lemmas that will lead to the proof of Theorem 8.

Lemma 16. Suppose M is a strong k-chromatic-choosable graph and $H = M \Box K_{a,b}$. Suppose that L is a (k + a - 1)-assignment for H such that there exist l, x, and y with $x \neq y$ and $L(v_l, u_x) \cap L(v_l, u_y) \neq \emptyset$. Then, there is a proper L-coloring of H.

Proof. Suppose that $\alpha \in L(v_l, u_x) \cap L(v_l, u_y)$. We begin by finding a proper L-coloring for each of the copies of M corresponding to u_1, \ldots, u_a . When it comes to the copies of Mcorresponding to u_x and u_y , we ensure that our proper L-colorings for these copies color both (v_l, u_x) and (v_l, u_y) with α . We know this is possible by Proposition 15. For the remaining copies of M, we simply take any proper L-coloring, and we know there must be at least one such coloring since $k + a - 1 \ge k + 1 > k$. Let c_j be the proper L-coloring that we found for the copy of M corresponding to u_j for each $j \in [a]$.

Now, for $i \in [n]$ and $t \in [b]$ (we are defining a list assignment for the yet to be colored vertices), we let $L'(v_i, w_t) = L(v_i, w_t) - \{c_j(v_i, u_j) : j \in [a]\}$. It is easy to see that for each i and $t, |L'(v_i, w_t)| \ge k + a - 1 - a = k - 1$, and $|L'(v_l, w_t)| \ge k + a - 1 - (a - 1) = k$. So, for any $t \in [b]$, we see that L' restricted to the copy of M corresponding to w_t is not a constant (k-1)-assignment. Proposition 14 then implies that our proper L-coloring of $H[\bigcup_{j=1}^{a} V_{u_j}]$ is not bad for the copy of M corresponding to w_t . Lemma 13 then implies there is a proper L-coloring of H.

Lemma 17. Suppose M is a strong k-chromatic-choosable graph and $H = M \Box K_{a,1}$ with $k \ge a + 1$. Suppose that L is a (k + a - 1)-assignment for H such that the lists $L(v_i, u_1), L(v_i, u_2), \ldots, L(v_i, u_a)$ are pairwise disjoint for each $i \in [n]$. Then, there is at most one proper L-coloring of $H[\bigcup_{i=1}^{a} V_{u_i}]$ that is bad for the copy of M corresponding to w_1 .

Proof. Throughout this proof for each $j \in [a]$, we let M_j be the copy of M corresponding to u_j . Furthermore, let M^* be the copy of M corresponding to w_1 . Suppose for the sake of contradiction that there exist two distinct proper L-colorings, c and c', of $H[\bigcup_{j=1}^{a} V_{u_i}]$ that are bad for M^* . Since M is strong k-chromatic-choosable, the list assignments:

$$L'(v_i, w_1) = L(v_i, w_1) - \{c(v_i, u_j) : j \in [a]\} \text{ and } L''(v_i, w_1) = L(v_i, w_1) - \{c'(v_i, u_j) : j \in [a]\}$$

for $i \in [n]$ are both constant (k-1)-assignments for M^* . Now, for $j \in [a]$, let c_j be the proper *L*-coloring for M_j obtained by restricting the domain of c to $V(M_j)$, and let c_{a+j} be the proper *L*-coloring for M_j obtained by restricting the domain of c' to $V(M_j)$. Since c and c' are different, we may assume without loss of generality that $c_1(v_m, u_1) \neq c_{a+1}(v_m, u_1)$ for some $m \in [n]$. Suppose $c_{a+1}(v_m, u_1) = b$. Since $L(v_m, u_1), L(v_m, u_2), \ldots, L(v_m, u_a)$ are pairwise disjoint, we know that:

$$b \notin \bigcup_{j=2}^{a} L(v_m, u_j)$$
 and $b \notin \{c_j(v_m, u_j) : j \in [a]\}.$

Let A be the set of (k-1) colors that L' assigns to all the vertices in M^* , and let B be the set of (k-1) colors that L'' assigns to all the vertices in M^* . We know that for $i \in [n]$, $L(v_i, w_1) = A \bigcup \{c_j(v_i, u_j) : j \in [a]\}$. So, for $i \in [n]$,

$$B = \left(A \bigcup \{c_j(v_i, u_j) : j \in [a]\}\right) - \{c_{a+j}(v_i, u_j) : j \in [a]\}.$$
(1)

Since |B| = k - 1, $b \notin \{c_j(v_m, u_j) : j \in [a]\}$, and equation (1) holds for i = m, it must be the case that $b \in A$. It immediately follows that $b \in \{c_{a+j}(v_i, u_j) : j \in [a]\}$ for each $i \in [n]$. To see why, note that if this did not hold we would have that some of the lists obtained from L'' would contain b and some would not.

Now, for $j \in [a]$, let

$$C_j = \{v_l : c_{a+j}(v_l, u_j) = b\}.$$

Note C_1, \ldots, C_a are pairwise disjoint because if C_r and C_s both contained some vertex v_p and $r \neq s$, then $b \in L(v_p, u_r) \cap L(v_p, u_s)$ which is a contradiction. Since $b \in \{c_{a+j}(v_i, u_j) : j \in [a]\}$ for each $i \in [n]$, we have that $V(M) = \bigcup_{j=1}^{a} C_j$. Since c_{a+j} colors all the vertices in $V(M_j)$ with first coordinate in C_j with the color b, we know that C_j is an independent set of vertices in M. Thus, $\{C_1, \ldots, C_a\}$ is a partition of V(M) into a independent sets. This implies that $k = \chi(M) \leq a$ which is a contradiction. This completes the proof.

We now restate and prove Theorem 8.

Theorem 8. If M is strongly chromatic-choosable and $\chi(M) \ge a + 1$, then $f_a(M) = (P_\ell(M, \chi_\ell(M) + a - 1))^a$.

Proof. We know by Theorem 5, $f_a(M) \leq (P_\ell(M, \chi_\ell(M) + a - 1))^a$. Suppose that M is strong k-chromatic-choosable with $k \geq a + 1$ and $H = M \Box K_{a,b}$. To prove the desired, we must show that if $b < (P_\ell(M, k + a - 1))^a$, then $\chi_\ell(H) < k + a$.

Let $t = P_{\ell}(M, k + a - 1)$, M_i be the copy of M corresponding to u_i , and M'_j be the copy of M corresponding to w_j . We assume that $b < t^a$, and we let L be an arbitrary (k + a - 1)assignment for H. To prove the desired, we will show that there is a proper L-coloring of H. By Lemma 16, we may assume the lists $L(v_i, u_1), L(v_i, u_2), \ldots, L(v_i, u_a)$ are pairwise disjoint for each $i \in [n]$. For each $j \in [a]$, there are clearly at least t distinct proper L-colorings of M_j . This implies that there are at least t^a proper L-colorings of $H[\bigcup_{j=1}^a V_{u_j}]$. Let \mathcal{C} be the set of distinct proper L-colorings of $H[\bigcup_{j=1}^a V_{u_j}]$ (we know $|\mathcal{C}| \geq t^a$).

By Lemma 17 we know that for each $d \in [b]$, there is at most one coloring in \mathcal{C} that is bad for M'_d . Since $b < t^a \leq |\mathcal{C}|$, there must be some $f \in \mathcal{C}$ that is not a bad coloring for any of: M'_1, \ldots, M'_b . Lemma 13 then implies there is a proper *L*-coloring of *H*.

Theorem 8 along with the fact that the list color function of an odd cycle (resp. complete graph) is equal to the chromatic polynomial of the odd cycle (resp. complete graph) for each natural number immediately yields the following corollary.

Corollary 9. The following statements hold.

(i) For any
$$l \in \mathbb{N}$$
, $f_2(C_{2l+1}) = (P_\ell(C_{2l+1}, 4))^2 = (3^{2l+1} - 3)^2 = 9(9^l - 1)^2$.
(ii) For $n \in \mathbb{N}$ satisfying $n \ge a+1$, $f_a(K_n) = (P_\ell(K_n, n+a-1))^a = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$.

Notice that Corollary 9 Statement (ii) shows that the bound in Theorem 5 is tight for any $a \in \mathbb{N}$. Suppose M is a strongly chromatic-choosable graph. At this stage of the paper, we have $f_a(M)$ exactly in terms of the list color function of M when $\chi(M) \ge a + 1$. When $\chi(M) < a + 1$ we only have an upper bound on $f_a(M)$ in terms of the list color function of M (by Theorem 5). We will now turn our attention to proving Theorem 12 which will give us a lower bound on $f_a(M)$ in terms of the list color function of M when $\chi(M) < a + 1$. We begin with a lemma. **Lemma 18.** Suppose M is strong k-chromatic-choosable and $H = M \Box K_{a,1}$ where k < a+1. Let L be a (k + a - 1)-assignment for H such that the lists $L(v_i, u_1), L(v_i, u_2), \ldots, L(v_i, u_a)$ are pairwise disjoint for each $i \in [n]$. Let \mathcal{B} be the set of proper L-colorings of $H[\bigcup_{j=1}^{a} V_{u_j}]$ that are bad for the copy of M corresponding to w_1 . Then, $|\mathcal{B}| \leq 2^{k-1}$.

Proof. We are done if \mathcal{B} is empty. So we suppose \mathcal{B} has at least one element. For $j \in [a]$, let $X_j = \{c : c \in L(v_1, w_1) \cap L(v_1, u_j)\}$ and $x_j = |X_j|$. Note that X_1, X_2, \ldots, X_a must be pairwise disjoint. We claim that

$$|\mathcal{B}| \le \left|\prod_{j=1}^{a} X_j\right| = \prod_{j=1}^{a} x_j$$

where $\prod_{j=1}^{a} X_j$ is the Cartesian product of the sets: X_1, X_2, \ldots, X_a . Now, suppose that $f \in \mathcal{B}$. Then, for $i \in [n]$, we know that the list assignment L' given by

$$L'(v_i, w_1) = L(v_i, w_1) - \{f(v_i, u_j) : j \in [a]\}$$

is a constant (k-1)-assignment for the copy of M corresponding to w_1 . This implies that $\{f(v_1, u_1), f(v_1, u_2), \ldots, f(v_1, u_a)\}$ is a set of size a that is completely contained in $L(v_1, w_1)$. Thus,

$$(f(v_1, u_1), f(v_1, u_2), \dots, f(v_1, u_a)) \in \prod_{j=1}^a X_j.$$

So, if for each $f \in \mathcal{B}$, we let $T(f) = (f(v_1, u_1), f(v_1, u_2), \dots, f(v_1, u_a))$, we see that T is a function from \mathcal{B} to $\prod_{j=1}^{a} X_j$. In order to prove the desired, we will show that T is injective. For the sake of contradiction, suppose f and g are distinct colorings in \mathcal{B} such that T(f) = T(g). For each $i \in [n]$ let L' and L'' be the list assignments for the copy of M corresponding to w_1 given by

$$L'(v_i, w_1) = L(v_i, w_1) - \{f(v_i, u_j) : j \in [a]\} \text{ and} L''(v_i, w_1) = L(v_i, w_1) - \{g(v_i, u_j) : j \in [a]\}.$$

We know that L' and L'' are both constant (k-1) assignments. Since T(f) = T(g), $L'(v_1, w_1) = L''(v_1, w_1)$ which immediately implies L' and L'' assign some list, A, of (k-1)colors to every vertex in the copy of M corresponding to w_1 . Since $f \neq g$ there are constants, r and t, such that $f(v_r, u_t) \neq g(v_r, u_t)$. Since L' and L'' are constant (k-1)assignments assigning A to every vertex in the copy of M corresponding to w_1 , we know that $\{f(v_r, u_j) : j \in [a]\} = \{g(v_r, u_j) : j \in [a]\}$. Since these two sets must have size a, $f(v_r, u_t) = g(v_r, u_p)$ for some $p \neq t$. This however contradicts the fact that $L(v_r, u_t)$ and $L(v_r, u_p)$ are disjoint. Thus, T is injective, and we have $|\mathcal{B}| \leq \left|\prod_{j=1}^{a} X_j\right|$.

To finish the proof, we must show that $\prod_{j=1}^{a} x_j \leq 2^{k-1}$. Notice that each x_j is a nonnegative integer such that $\sum_{j=1}^{a} x_j \leq k+a-1$. Under these conditions, the maximum possible value of $\prod_{j=1}^{a} x_j$ is achieved when $\sum_{j=1}^{a} x_j = k+a-1$ and each x_j equals $\lfloor (k+a-1)/a \rfloor = 1$ or $\lceil (k+a-1)/a \rceil = 2$; that is, when (k-1) of x_1, x_2, \ldots, x_a are 2 and the rest are 1.

It is fairly easy to show that the bound in Lemma 18 is tight for each $k \geq 2$. For example, suppose $G = K_3$, $V(G) = \{v_1, v_2, v_3\}$, and $H = G \Box K_{3,1}$. Suppose L is the 5-assignment for H that assigns: $\{1, 2, 3, 4, 5\}$ to (v_i, w_1) for each $i \in [3]$, $\{1, 2, 6, 7, 8\}$ to $L(v_1, u_1)$, $L(v_3, u_2)$, and $L(v_2, u_3)$, $\{3, 4, 9, 10, 11\}$ to $L(v_2, u_1)$, $L(v_1, u_2)$, and $L(v_3, u_3)$, and $\{5, 12, 13, 14, 15\}$ to $L(v_3, u_1)$, $L(v_2, u_2)$, and $L(v_1, u_3)$. It is easy to see that L satisfies the hypotheses of Lemma 18, and there are exactly 4 proper L-colorings of $H[\bigcup_{j=1}^3 V_{u_j}]$ that are bad for the copy of G corresponding to w_1 . We are now ready to restate and prove Theorem 12.

Theorem 12. Suppose M is a strong k-chromatic-choosable graph. Then,

$$\frac{(P_{\ell}(M, k+a-1))^a}{2^{k-1}} \le f_a(M)$$

Proof. Suppose that M is strong k-chromatic-choosable and $H = M \Box K_{a,b}$. By Theorem 8, the result is obvious when $k \ge a+1$. So, we assume k < a+1. Let $t = P_{\ell}(M, k+a-1)$, and let M_i be the copy of M in H corresponding to w_i . We assume $b < t^a/2^{k-1}$, and we let L be an arbitrary (k + a - 1)-assignment for H. To prove the desired, we will show there is proper L-coloring for H. By Lemma 16, we may assume that the lists $L(v_i, u_1), L(v_i, u_2), \ldots, L(v_i, u_a)$ are pairwise disjoint for each $i \in [n]$. Let C be the set of proper L-colorings of $H[\bigcup_{j=1}^a V_{u_j}]$. Clearly, $t^a \le |\mathcal{C}|$. For $d \in [b]$ let C_d be the subset of \mathcal{C} that contains all the proper L-colorings of $H[\bigcup_{j=1}^a V_{u_j}]$ that are bad for M_d . By Lemma 18 we have that $|C_d| \le 2^{k-1}$. Since $b < t^a/2^{k-1}$,

$$\sum_{d=1}^{b} |C_d| \le b(2^{k-1}) < t^a \le |\mathcal{C}|.$$

Thus, $\bigcup_{d=1}^{b} C_d$ must be a proper subset of C, and we can find an $f \in C - \bigcup_{d=1}^{b} C_d$. Lemma 13 then implies there is a proper *L*-coloring of *H*.

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