Spanning Properties of Theta-Theta-6

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Abstract

We show that, unlike the Yao-Yao graph YY_6 , the Theta-Theta graph $\Theta\Theta_6$ defined by six cones is a spanner for sets of points in convex position. We also show that, for sets of points in non-convex position, the spanning ratio of $\Theta\Theta_6$ is unbounded.

1 Introduction

Let S be a set of n points in the plane and let G = (S, E) be a weighted geometric graph with vertex set S and a set E of (directed or undirected) edges between pairs of points, where the weight of an edge $uv \in E$ is equal to the Euclidean distance |uv| between u and v. The length of a path in G is the sum of the weights of its constituent edges. The distance $d_G(u, v)$ in G between two points $u, v \in S$ is the length of a shortest path in G between U and U. The graph U is called a U-spanner if any two points U, U in the plane are at distance U in the smallest integer U for which this property holds is called the spanning ratio of U.

The Yao graph $Y_k(S)$ and the Theta graph $\Theta_k(S)$ are defined for a fixed integer k > 0 as follows. Partition the plane into k equiangular cones by extending k equally-separated rays starting at the origin, with the first ray in the direction of the positive x-axis. Then translate the cones to each point $u \in S$, and connect u to a "nearest" neighbor in each cone. The difference between Yao and Theta graphs is in the way the "nearest" neighbor is defined. For a fixed point $u \in S$ and a cone C(u) with apex u, a Yao edge $\overrightarrow{uv} \in C(u)$ minimizes the Euclidean distance |uv| between u and v, whereas a Theta edge $\overrightarrow{uv} \in C(u)$ minimizes the projective distance ||uv|| from u to v, which is the Euclidean distance between u and the orthogonal projection of v on the bisector of C(u). Ties are arbitrarily broken.

Each of the graphs Θ_k and Y_k has out-degree k, but in-degree n-1 in the worst case (consider, for example, the case of n-1 points uniformly distributed on the circumference of a circle centered at the n^{th} point: for any $k \geq 6$, the center point has in-degree n-1). This is a significant drawback in certain wireless networking applications where a wireless node can communicate with only a limited number of neighbors. To reduce the in-degrees, a second filtering step can be applied to the set of incoming edges in each cone. This filtering step eliminates, for each each point $u \in S$ and each cone with apex u, all but a "shortest" incoming edge. The result of this filtering step applied on Θ_k (Y_k) is the Theta-Theta (Yao-Yao) graph $\Theta\Theta_k$ (Y_k). Again, the definition of "shortest" differs for Yao and Theta graphs: a shortest Yao edge $\overrightarrow{vu} \in C(u)$ minimizes |vu|, and a shortest Theta edge $\overrightarrow{vu} \in C(u)$ minimizes |vu|. Again, ties are arbitrarily broken.

Yao and Theta graphs (and their Yao-Yao and Theta-Theta sparse variants) have many important applications in wireless networking [1], motion planning [9] and walkthrough animations [15].

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We refer the readers to the books by Li [18] and Narasimhan and Smid [21] for more details on their uses, and to the comprehensive survey by Eppstein [14] for related topics on geometric spanners. Many such applications take advantage of the spanning and sparsity properties of these graphs, which have been extensively studied. Molla [20] showed that Y_2 and Y_3 may not be spanners, and her examples can be used to show that Θ_2 and Θ_3 are not spanners either. On the other hand, it has been shown that, for any $k \geq 4$, Y_k and Θ_k are spanners: Y_4 is a 54.6-spanner [13] and Θ_4 is a 17-spanner [4]; Y_5 is a 3.74-spanner [2] and Θ_5 is a 9.96-spanner [6]; Y_6 is a 5.8-spanner [2] and Θ_6 is a 2-spanner [3]; for $k \geq 7$, the spanning ratio of Y_k is $\frac{1+\sqrt{2-2\cos(2\pi/k)}}{2\cos(2\pi/k)-1}$ [5] and the spanning ratio of Θ_k is $\frac{1}{1-2\sin(\pi/k)}$ [22]; improved bounds on the spanning ratio of Y_k for odd $k \geq 5$, and for Θ_k for even $k \geq 6$, also exist [7].

In contrast with Yao and Theta graphs, our knowledge of Yao-Yao and Theta-Theta graphs is more limited. Li et al. [19] proved that YY_k is connected for k > 6 and provided substantial experimental evidence suggesting that YY_k is a spanner for large k values. This conjecture has been partly confirmed by Bauer and Damian [11] who showed that, for $k \ge 6$, YY_{6k} is a spanner with spanning ratio 11.76. This spanning ratio has been improved to 7.82 in [10] for a more general class of graphs called canonical k-cone graphs, which include both YY_{6k} and $\Theta\Theta_{6k}$, for $k \ge 6$. The same paper establishes a spanning ratio of 16.76 for YY_{30} and $\Theta\Theta_{30}$. Recent breakthroughs show that YY_{2k} , for any $k \ge 42$, is a spanner with spanning ratio $6.03 + O(k^{-1})$ [17], and YY_k for odd $k \ge 3$ is not a spanner [16]. For small values $k \le 5$, Damian et al. [12] show that YY_4 is not a spanner, and Barba et al. [2] show that YY_5 is not a spanner, and their constructions can also be used to show that $\Theta\Theta_4$ and $\Theta\Theta_5$ are not spanners. Molla [20] showed that YY_6 is also not a spanner, even for sets of points in convex position. This paper fills in one of the gaps in our knowledge of Theta-Theta graphs and shows that $\Theta\Theta_6$ is an 8-spanner for sets of points in convex position, but has unbounded spanning ratio for sets of points in non-convex position.

2 Definitions

Throughout the paper, S is a fixed set of n points in the plane and k > 1 is a fixed integer. The graphs Y_k and Θ_k use a set of k equally-separated rays starting at the origin. These rays define k equiangular cones $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$, each of angle $\theta = 2\pi/k$, with the lower ray of \mathcal{C}_1 extending in the direction of the positive x-axis. Refer to Figure 1. We assume that each cone is half-open and half-closed, meaning that it includes the clockwise bounding ray, but it excludes the counterclockwise bounding ray. Let $\mathcal{C}_i(a)$ denote a copy of \mathcal{C}_i translated to a, for each $a \in S$ and each $i = 1, \ldots, k$.

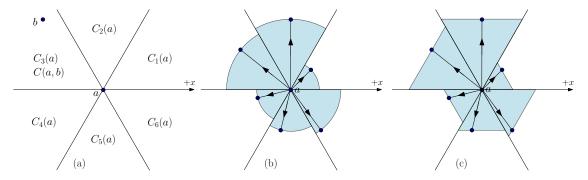


Figure 1: (a) Cones defining Y_6 and Θ_6 (b) Y_6 edges minimize Euclidean distances (c) Θ_6 edges minimize projective distances.

The directed graphs \overrightarrow{Y}_k and $\overrightarrow{\Theta}_k$ are constructed as follows. In each cone $C_i(a)$, for each $i=1,\ldots,k$

and each $a \in S$, extend a directed edge from a to a "nearest" point b that lies in $C_i(a)$. Yao and Theta graphs differ only in the way "nearest" is defined. A point b is "nearest" to a in Y_k if it minimizes the Euclidean distance |ab|, whereas b is "nearest" to a in Θ_k if it minimizes the projective distance |ab|. See Figure 2a,b for simple graph examples illustrating these definitions.

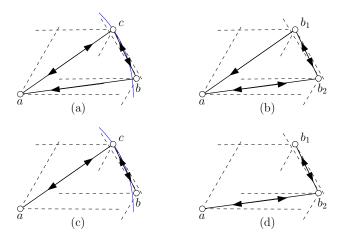


Figure 2: Graph examples (a) Y_6 (b) Θ_6 (c) YY_6 (d) $\Theta\Theta_6$.

The Yao-Yao graph $\overrightarrow{YY_k} \subseteq \overrightarrow{Y_k}$ and Theta-Theta graph $\overrightarrow{\Theta\Theta_k} \subseteq \overrightarrow{\Theta_k}$ are obtained by applying a filtering step to the set of incoming edges at each vertex in $\overrightarrow{Y_k}$ and $\overrightarrow{\Theta_k}$, respectively. Specifically, for each $a \in S$ and each i = 1, ..., k, these graphs retain a "shortest" incoming edge that lies in $C_i(a)$ and and discard the rest of incoming edges, if any. Recall that a "shortest" Yao edge $\overrightarrow{ba} \in C_i(a)$ minimizes |ba|, whereas a "shortest" Theta edge $\overrightarrow{ba} \in C_i(a)$ minimizes |ba|. Figure 2c(d) depicts the graph YY_6 ($\Theta\Theta_6$) after this filtering step has been applied to the graph Y_6 (Θ_6) from Figure 2a(b).

3 Background: YY_6 is not a Spanner

Molla [20] gave an example of a set of points in convex position for which YY_6 is not a spanner. We briefly review her construction here and show that the result does not hold for $\Theta\Theta_6$. The construction begins with a strip of equilateral triangles between two horizontal lines with vertices $\{a_1, a_2, \ldots, a_n\}$ on the lower line (which we call the a-line) and $\{b_1, b_2, \ldots, b_n\}$ on the upper line (which we call the b-line). See the left of Figure 3a. Next the a-line is rotated clockwise about a_1 and the b-line is rotated counterclockwise about b_1 by a small angle $\alpha > 0$, to guarantee that $|a_{i-1}a_i| < |b_{i-1}a_i|$ and $|b_{i-1}b_i| < |a_ib_i|$, for $i = 2, \ldots, n$. The points are also slightly perturbed to ensure that $C_2(a_i)$ and $C_5(b_i)$ are all empty, for $i = 1, \ldots, n$. The result is depicted in the right of Figure 3a.

The graphs Y_6 and YY_6 induced by the set of points $S = \{a_1, \ldots a_4\} \cup \{b_1, \ldots, b_4\}$ are depicted in Figure 3b. Note that, with the exception of a_1b_1 , YY_6 includes none of the Y_6 edges incident on both the a-line and the b-line. This is because, for i > 1, $\overrightarrow{b_{i-1}a_i}$ and $\overrightarrow{a_{i-1}a_i}$ both lie in $C_3(a_i)$ and YY_6 maintains only the shorter of the two, which is $\overrightarrow{a_{i-1}a_i}$. Similarly, $\overrightarrow{a_ib_i}$ and $\overrightarrow{b_{i-1}b_i}$ both lie in $C_4(b_i)$ and YY_6 maintains only the shorter of the two, which is $\overrightarrow{b_{i-1}b_i}$. This shows that the shortest path in YY_6 between a_n and b_n is a Hamiltonian path of length at least 2n-1, which grows arbitrarily large with n. It follows that YY_6 is not a spanner.

For the same point set S, the graphs Θ_6 and $\Theta\Theta_6$ are depicted in Figure 3c. Note that, if projective distances are used, then $||a_{i-1}a_i|| > ||b_{i-1}a_i||$ and $||b_{i-1}b_i|| > ||a_ib_i||$, for i = 2, ..., n. These properties force $\Theta\Theta_6$ to maintain $\overrightarrow{b_{i-1}a_i} \in \mathcal{C}_3(a_i)$ and $\overrightarrow{a_ib_i} \in \mathcal{C}_4(b_i)$, for each i = 2, ..., n. The

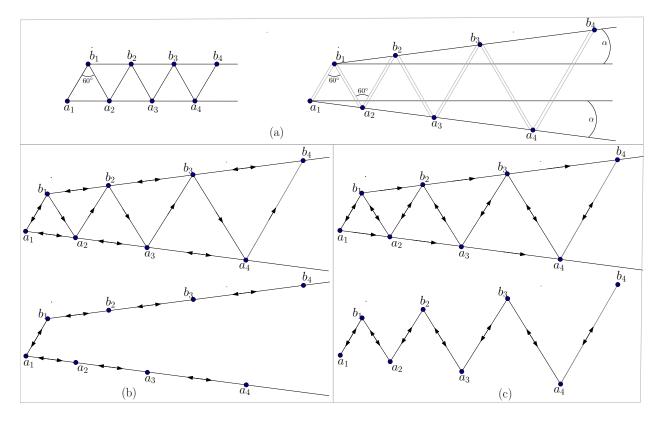


Figure 3: (a) Point set $\{a_1, \ldots, a_4\} \cup \{b_1, \ldots, b_4\}$ (b) Graphs Y_6 (top) and YY_6 (bottom) (c) Graphs Θ_6 (top) and $\Theta\Theta_6$ (bottom).

result is the zig-zag path depicted in Figure 3c which shows that, for this particular point set, $\Theta\Theta_6$ is a spanner. In the next section we show that $\Theta\Theta_6$ is a spanner for any set of points in convex position.

4 $\Theta\Theta_6$ is a Spanner for Points in Convex Position

It has been established in [20] (and revisited in Section 3 of this paper) that YY_6 is not a spanner for sets of points in convex position. In this section we show that, unlike YY_6 , the graph $\Theta\Theta_6$ is an 8-spanner for sets of points in convex position (in the next section we will show that this result does not hold for sets of points in non-convex position). This is the first result that marks a difference in the spanning properties of YY-graphs and $\Theta\Theta$ -graphs.

Throughout this section, we assume that S is a set of points in convex position. For simplicity, we also assume that the points in S are in general position, meaning that no two points lie on a line parallel to one of the rays that define the cones. This implies that there is a unique nearest point in each cone of Θ_6 and $\Theta\Theta_6$. We begin with a few definitions.

For any $a, b \in S$, let C(a, b) denote the cone with apex a that contains b. For any ordered pair of vertices a and b, let T(a, b) be the canonical triangle delimited by the rays bounding C(a, b) and the perpendicular through b on the bisector of C(a, b). See Figure 4a. For a fixed point $a \in S$ and $i \in \{1, ..., k\}$, let $p_{\Theta_6}(a, i)$ denote the path in Θ_6 that starts at a and follows the Θ_6 -edges that lie in cones C_i . See, for example, the path $p_{\Theta_6}(a, 1)$ depicted in Figure 4. Note that this path is monotone with respect to the bisector of C_i . This along with the fact that the point set S is finite implies that the path itself is finite and well defined. We say that two edges ab and cd cross if they share a point other than an endpoint (a, b, c) or d.

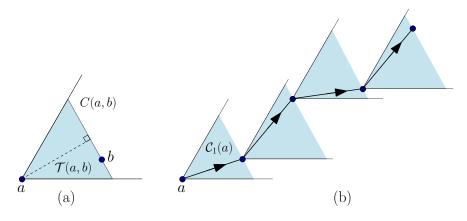


Figure 4: (a) Canonical triangle $\mathcal{T}(a,b)$ (b) Path $p_{\Theta_6}(a,1)$.

The half- Θ_6 -graph introduced in [3] takes only "half" the edges of Θ_6 , those belonging to non-consecutive cones. Thus, the Θ_6 -graph is the union of two half- Θ_6 -graphs: one that includes all Θ_6 -edges that lie in cones C_1 , C_3 , C_5 , and one that includes all Θ_6 -edges that lie in cones C_2 , C_4 , C_6 . Bonichon et al. [3] show that half- Θ_6 is a triangular-distance Delaunay triangulation, computed as the dual of the Voronoi diagram based on the triangular distance function. This, combined with Chew's proof that any triangular-distance Delaunay triangulation is a 2-spanner [8], yields the following result.

Theorem 1 [3] The half- Θ_6 -graph is a plane 2-spanner.

Next we introduce two preliminary lemma that will be useful in proving the main result of this section.

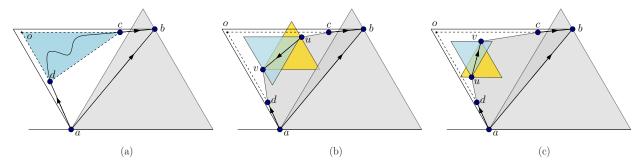


Figure 5: Lemma 2 (a) There is a path $p_{\Theta\Theta_6}(d,c)$ that lies inside $\triangle cod$ (b) If $\overrightarrow{uv} \in \Theta_6$, $\overrightarrow{uv} \in \mathcal{C}_4(u)$, then $\overrightarrow{uv} \in \Theta\Theta_6$ (c) If $\overrightarrow{uv} \in \Theta_6$, $\overrightarrow{uv} \in \mathcal{C}_2(u)$, then $\overrightarrow{uv} \in \Theta\Theta_6$.

Lemma 2 Let S be a set of points in convex position and let $a,b,c,d \in S$ be distinct points such that $b \in \mathcal{C}_1(a)$ and $\overrightarrow{ab} \in \Theta_6 \setminus \Theta\Theta_6$; $c \in \mathcal{C}_4(b)$ and $\overrightarrow{cb} \in \Theta\Theta_6$; $d \in \mathcal{C}_2(a)$ and $\overrightarrow{ad} \in \Theta_6$. Let o be the intersection point between the upper ray of $\mathcal{C}_4(c)$ and the left ray of $\mathcal{C}_2(d)$. Then there is a path in $\Theta\Theta_6$ between c to d that lies in $\triangle cod$ and is no longer than |oc| + |od|.

Proof Note that, since the points in S are in convex position, the point o exists and lies outside the convex quadrilateral abcd. Refer to Figure 5a. Consider the paths $p_c = p_{\Theta_6}(c, 4)$ and $p_d = p_{\Theta_6}(d, 2)$.

¹The triangular distance from a point a to a point b is the side length of the smallest equilateral triangle centered at a that touches b and has one horizontal side.

Since p_c and p_d are in the same half- Θ_6 graph, Theorem 1 tells us that p_c and p_d do not cross. This implies that p_c and p_d meet in a point $e \in \triangle cod$. Let p(c, e) be the piece of p_c extending from c to e, and p(d, e) the piece of p_d extending from d to e. Note that $p(c, d) = p(c, e) \cup p(d, e)$ is a convex path that lies inside $\triangle cod$, which implies that |p(c, d)| < |oc| + |od|.

To complete the proof, it remains to show that p_{cd} is a path in $\Theta\Theta_6$. To do so, we consider an arbitrary edge $\overrightarrow{uv} \in p(c,d) \in \Theta_6$, and show that $\overrightarrow{uv} \in \Theta\Theta_6$. Assume first that $\overrightarrow{uv} \in p(c,e)$, meaning that $v \in \mathcal{C}_4(u)$. Refer to Figure 5b. The convexity property of S implies that no points may lie in $\mathcal{T}(v,u)$ and above u. Ignoring the piece of $\mathcal{T}(v,u)$ that extends above u, the rest of $\mathcal{T}(v,u)$ lies inside $\mathcal{T}(u,v) \cup abcuvd \cup \mathcal{T}(a,b)$. This region, however, is empty of points in S: $\mathcal{T}(u,v)$ is empty of points in S because $\overrightarrow{uv} \in \Theta_6$; abcuvd is a convex polygon empty of points in S, by the convexity property of S; and $\mathcal{T}(a,b)$ is empty of points in S, because $\overrightarrow{ab} \in \Theta_6$. It follows that $\mathcal{T}(v,u)$ is empty of points in S and therefore $\overrightarrow{uv} \in \Theta\Theta_6$.

The arguments for the case when $\overrightarrow{uv} \in p(d,e)$ are similar: in this case, $v \in \mathcal{C}_2(u)$; no points in S may lie in $\mathcal{T}(v,u)$ and left of $\mathcal{C}_2(u)$; ignoring the piece of $\mathcal{T}(v,u)$ that extends left of $\mathcal{C}_2(u)$, the rest of $\mathcal{T}(v,u)$ lies inside $\mathcal{T}(u,v) \cup abcvud \cup \mathcal{T}(a,b)$, which is empty of points in S. It follows that $\mathcal{T}(v,u)$ is empty of points in S and therefore $\overrightarrow{uv} \in \Theta\Theta_6$.

Lemma 3 For any edge \overrightarrow{ab} in the Θ_6 -graph induced by a set of points S in convex position, there is a path between a and b in $\Theta\Theta_6$ no longer than 4|ab|.

Proof Assume without loss of generality that $\overrightarrow{ab} \in \mathcal{C}_1(a)$ and let α be the angle formed by ab with the lower ray of $\mathcal{C}_1(a)$. Let i_1 (h_1) be the intersection point between the upper ray of $\mathcal{C}_1(a)$ and the horizontal (perpendicular) through b. Refer to Figure 6a. Let i_2 (h_2) , i_3 (h_3) and i_4 (h_4) be copies of i_1 (h_1) rotated counterclockwise by $\pi/3$, $2\pi/3$ and $2\pi/3 + \alpha$, respectively. Note that $|ah_1| < |ab|$ and $|bi_1| = 2|i_1h_1|$. We show that there is a convex path $p(a,b) \in \Theta\Theta_6$ between a and b that lies inside the convex region $\mathcal{R} = abi_2i_3i_4$ (shaded in Figure 6a). The length of such a path is

$$|p(a,b)| < |bi_2| + |i_2i_3| + |i_3i_4| + |i_4a|$$

$$= 2|i_1h_1| + |i_1i_2| + |i_2i_3| + |i_3i_4| + |i_4a|$$

$$< 2|i_1h_1| + 4|i_1i_2| < 4(|i_1h_1| + |i_1i_2|) = 4|ah_1|$$

$$< 4|ab|$$

It remains to prove the existence of such a path $p(a,b) \in \Theta\Theta_6$. If $\overrightarrow{ab} \in \Theta\Theta_6$, then p(a,b) = ab and the lemma trivially holds. Otherwise, there is $\overrightarrow{c_1b} \in \Theta\Theta_6$, with $c_1 \in \mathcal{C}(b,a)$. By definition $||c_1b|| < ||ab||$, which implies that c_1 lies in $\mathcal{C}_2(a)$ or $\mathcal{C}_6(a)$. Assume without loss of generality that $c_1 \in \mathcal{C}_2(a)$; the case where $c_1 \in \mathcal{C}_6(a)$ is symmetric. Because $\mathcal{C}_2(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_2} \in \mathcal{C}_2(a)$. Refer to Figure 6b. If b_2 and c_1 coincide, let $p(b_2, c_1)$ be the empty path; otherwise, $p(b_2, c_1) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies in a triangular region inside $\mathcal{T}(a, c_1)$ (shaded in Figure 6b). If $\overrightarrow{ab_2} \in \Theta\Theta_6$, then

$$p(a,b) = ab_2 \oplus p(b_2,c_1) \oplus c_1b$$

is a convex path (by the convexity property of S) from a to b in $\Theta\Theta_6$ that lies inside \mathcal{R} , so the lemma holds. Here \oplus denotes the concatenation operator. If $\overrightarrow{ab_2} \not\in \Theta\Theta_6$, then there is $\overrightarrow{c_2b_2} \in \Theta\Theta_6$, with $c_2 \in \mathcal{C}(b_2, a)$. By definition $||c_2b_2|| < ||ab_2||$, which implies that $c_2 \in \mathcal{C}_3(a)$. Because $\mathcal{C}_3(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_3} \in \mathcal{C}_3(a)$. If b_3 and c_2 coincide, let $p(b_3, c_2)$ be the empty path;

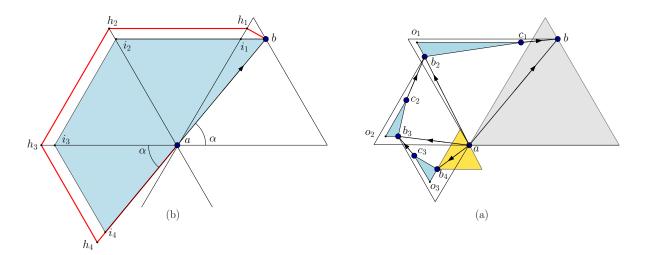


Figure 6: Lemma 3 (a) Any convex path from a to b that lies in the shaded area is no longer than 4|ab| (b) Path in $\Theta\Theta_6$ from a to b: $\overrightarrow{ab} \in \Theta_6 \setminus \Theta\Theta_6$, $\overrightarrow{c_1b}$, $\overrightarrow{c_2b_2}$, $\overrightarrow{c_3b_3}$, $\overrightarrow{ab_4} \in \Theta\Theta_6$.

otherwise, $p(b_3, c_2) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies inside $\mathcal{T}(a, c_2)$ (shaded in Figure 6b). If $\overrightarrow{ab_3} \in \Theta\Theta_6$, then

$$p(a,b) = ab_3 \oplus p(b_3, c_2) \oplus c_2b_2 \oplus p(b_2, c_1) \oplus c_1b$$

is a convex path from a to b in $\Theta\Theta_6$ that lies inside \mathcal{R} , so the lemma holds. If $\overrightarrow{ab_3} \notin \Theta\Theta_6$, then there is $\overrightarrow{c_3b_3} \in \Theta\Theta_6$, with $c_3 \in \mathcal{C}(b_3, a)$. By definition $\|c_3b_3\| < \|ab_3\|$, which implies that $c_3 \in \mathcal{C}_4(a)$. Because $\mathcal{C}_4(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_4} \in \mathcal{C}_4(a)$. If b_4 and c_3 coincide, let $p(b_4, c_3)$ be the empty path; otherwise, $p(b_4, c_3) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies inside $\mathcal{T}(a, c_3)$. The convexity property of S implies that the region of $\mathcal{T}(b_4, a)$ that extends right of the line supporting ab is empty of points in S. Ignoring this region, the rest of $\mathcal{T}(b_4, a)$ lies in $\mathcal{T}(a, b_4) \cup \mathcal{T}(a, b_3)$, which is also empty of points in S. It follows that $\mathcal{T}(b_4, a)$ is empty of points in S, therefore $\overrightarrow{ab_4} \in \Theta\Theta_6$. These together imply that

$$p(a,b) = ab_4 \oplus p(b_4,c_3) \oplus c_3b_3 \oplus p(b_3,c_2) \oplus c_2b_2 \oplus p(b_2,c_1) \oplus c_1b$$

is a convex path from a to b in $\Theta\Theta_6$ that lies inside \mathcal{R} . This completes the proof.

Lemmas 1 and 3 together yield the main result of this section.

Theorem 4 The $\Theta\Theta_6$ -graph induced by a set of points in convex position is an 8-spanner.

The following lemma establishes a lower bound of 4 on the spanning ratio of $\Theta\Theta_6$ for convex point sets. In addition, it shows that the bound 4 of Lemma 3 on the spanning ratio of $\Theta\Theta_6$ -paths spanning Θ_6 -edges is tight.

Lemma 5 The spanning ratio of the $\Theta\Theta_6$ -graph induced by a set of points in convex position is at least 4.

Proof We construct a set of points S that satisfies the claim of this lemma. Let a be an arbitrary point in the plane and let b_i be the point at unit distance from a that lies on the counterclockwise ray of $C_i(a)$, for i = 1, ..., 4. Refer to Figure 7a. Perturb the points infinitesimally so that b_1 lies strictly inside $C_1(a)$ and b_i lies strictly inside $C_i(a) \cap T(a, b_{i-1})$, for i = 2, 3, 4. We ignore

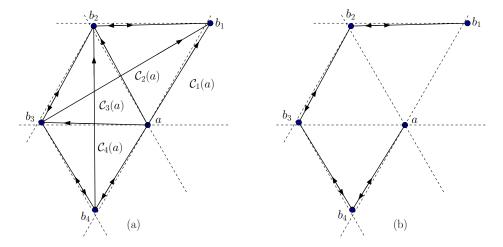


Figure 7: Set $S = \{a, b_1, b_2, b_3, b_4\}$ of points in convex position (a) Θ_6 -graph (b) $\Theta\Theta_6$ -graph.

this infinitesimal quantity from our calculations and assume that $|ab_i| = 1$, for i = 1, ..., 4 and $|b_i b_{i+1}| = 1$, for i = 1, 2, 3.

Let $S = \{a, b_1, b_2, b_3, b_4\}$. The Θ_6 -graph and $\Theta\Theta_6$ -graph induced by S are depicted in Figure 7a and Figure 7b, respectively. Note that $\overrightarrow{ab_1} \in \Theta_6$, however $\overrightarrow{ab_1} \notin \Theta\Theta_6$ and $p_{\Theta\Theta_6}(a, b_1) = ab_4 \oplus b_4b_3 \oplus b_3b_2 \oplus b_2b_1$ is a shortest path in $\Theta\Theta_6$ between a and b of length 4. This proves the claim of this lemma. It also shows that the bound of Lemma 3 is tight.

5 $\Theta\Theta_6$ is not a Spanner for Points in Non-Convex Position

In this section we show that there exist sets of points in non-convex position for which $\Theta\Theta_6$ has unbounded spanning ratio and therefore it is not a spanner. We show how to construct a set $S = \{a_i, b_i, c_i, d_i : i = 1, 2, ..., n\}$ of 4n points with this property.

Let a_1 and b_1 be points in the plane such that a_1b_1 forms a $\pi/3$ -angle with the horizontal through a_1 . Let r_a (r_b) be the ray with origin a_1 (b_1) pointing in the direction of the positive x-axis. Fix a small positive real value $0 < \alpha < 2$, and rotate r_a (r_b) clockwise (counterclockwise) about a_1 (b_1) by angle α . Let $a_2, a_3, \ldots a_n$ be points along r_a , and $b_2, b_3, \ldots b_n$ points along r_b , such that $\angle b_{i-1}a_ib_i = \pi/3$ for each $i = 2, \ldots, n$, and $\angle a_ib_ia_{i+1} = \pi/3$ for each $i = 1, 2, \ldots, n-1$. Refer to Figure 8a (which shows α enlarged for clarity). Note that at this point $\mathcal{C}_2(a_i)$ and $\mathcal{C}_5(b_i)$ share the line segment a_ib_i , for each $i = 1, 2, \ldots, n$. Fix an arbitrary real value

$$\delta < \frac{|a_1 a_2| \sin \alpha}{2}.\tag{1}$$

Keep a_1 in place and shift the remaining points rightward alongside their supporting rays r_a and r_b such that the horizontal distance between the right boundary ray of $C_2(a_i)$ and the left boundary ray of $C_5(b_i)$ is δ , for each i. Refer to Figure 8b. Finally, let c_i (d_i) be a copy of b_i (a_i) shifted upward (downward) by 2δ , for i = 1, 2, ..., n. Thus $b_i c_i$ and $a_i d_i$ are vertical line segments of length $|b_i c_i| = |a_i d_i| = 2\delta$.

The following property is key to establishing an unbounded spanning ratio for $\Theta\Theta_6(S)$.

Property 6 For each i = 1, 2, ..., n - 1, the point c_i lies in a small triangular region at the intersection between $C_2(a_i)$, $C_2(a_{i+1})$ and $C_4(b_{i+1})$.

To establish this property, fix an arbitrary $i \in \{1, 2, ..., n-1\}$. Let y be the intersection point between the right ray of $C_2(a_i)$ and the left ray of $C_2(a_{i+1})$. See Figure 8b, which depicts the instance

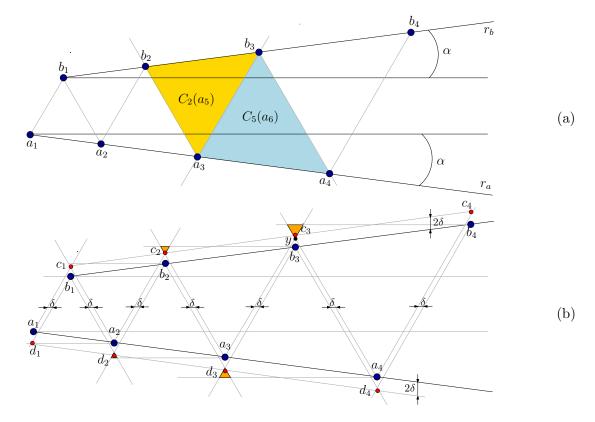


Figure 8: (a) Initial point configuration (b) Shifted point positions.

i=3. Note that $|b_iy|$ is equal to the height of an equilateral triangle of side length 2δ , which is $\delta\sqrt{3} < 2\delta = |b_ic_i|$. This means that c_i lies vertically above y, therefore $c_i \in \mathcal{C}_2(a_i) \cap \mathcal{C}_2(a_{i+1})$. To establish that $c_i \in \mathcal{C}_4(b_{i+1})$, it suffices to show that c_i lies below the horizontal line through b_{i+1} . The distance from b_i to this line is $|b_ib_{i+1}|\sin\alpha > |a_1a_2|\sin\alpha > 2\delta$ (cf. Equation 1). This along with the fact that $|b_ic_i| = 2\delta$ implies that c_i lies below the horizontal line through b_{i+1} . This settles Property 6.

Symmetric arguments establish the following property.

Property 7 For each i = 2, ..., n-1, the point d_i lies in a small triangular region at the intersection between $C_5(b_{i-1})$, $C_5(b_i)$ and $C_3(a_{i+1})$. If i = 1, $d_i \in C_5(b_i) \cap C_3(a_{i+1})$.

We use Properties 6 and 7 in identifying the set of edges in $\Theta_6(S)$ and $\Theta\Theta_6(S)$. Fix an arbitrary $i \in \{1, \ldots, n\}$. The edges in $\Theta_6(S)$ outgoing from a_i are: $\overrightarrow{a_i b_i} \in \mathcal{C}_1(a_i)$; $\overrightarrow{a_i c_{i-1}} \in \mathcal{C}_2(a_i)$, if i > 1; $\overrightarrow{a_i d_{i-1}} \in \mathcal{C}_3(a_i)$, if i > 1 (note that Property 7 implies that $||a_i d_{i-1}|| < ||a_i b_{i-1}||$); $\overrightarrow{a_i d_i} \in \mathcal{C}_5(a_i)$; and $\overrightarrow{a_i a_{i+1}} \in \mathcal{C}_6(a_i)$, if i < n. Refer to Figure 9a, which depicts the instance i = 3. (Note that the cone $\mathcal{C}_4(a_i)$ is empty of points in S.) The edges in $\Theta_6(S)$ outgoing from $a_i = 1$ are: $\overrightarrow{a_i b_{i+1}} \in \mathcal{C}_1(d_i)$, if i < n; and $\overrightarrow{a_i a_i} \in \mathcal{C}_2(d_i)$; $\overrightarrow{a_i d_{i-1}} \in \mathcal{C}_3(d_i)$, if i > 1; and $\overrightarrow{a_i a_{i+1}} \in \mathcal{C}_6(d_i)$, if i < n. (Note that the cones $\mathcal{C}_4(a_i)$ and $\mathcal{C}_5(a_i)$ are empty of points in S.) The edges in $\Theta_6(S)$ outgoing from $a_i = 1$ are: $a_i = 1$ and $a_i = 1$ are: $a_i =$

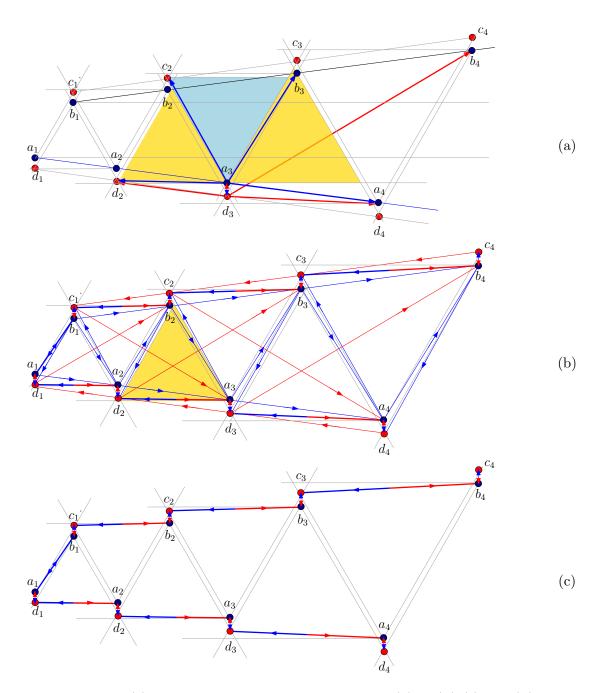


Figure 9: (a) Edges in Θ_6 outgoing from a_3 and d_3 (b) $\Theta_6(S)$ (c) $\Theta\Theta_6(S)$.

We now turn our attention to the set of incoming edges at each vertex in $\Theta_6(S)$. From among the four (three) edges directed into a_i and lying in $\mathcal{C}_3(a_i)$ for i > 2 (i = 2), the edge $\overrightarrow{d_{i-1}a_i}$ has the shortest projective distance: $\|\overrightarrow{d_{i-1}a_i}\| < \|\overrightarrow{b_{i-1}a_i}\| < \|\overrightarrow{a_{i-1}a_i}\|$, and this latter quantity is in turn smaller than $\|\overrightarrow{c_{i-2}a_i}\|$, for i > 2. This implies that $\Theta\Theta_6(S)$ keeps $\overrightarrow{d_{i-1}a_i}$ and eliminates the other three (two) edges, for i > 2 (i = 2). Note that any cone with apex a_i other than $\mathcal{C}_3(a_i)$ contains at most one edge directed into a_i , which continues to exist in $\Theta\Theta_6$.

For each i, the two edges directed into d_i that lie in $\mathcal{C}_2(d_i)$ satisfy $\|\overrightarrow{a_id_i}\| < \|\overrightarrow{b_id_i}\|$, therefore $\overrightarrow{b_id_i}$ gets eliminated from $\Theta\Theta_6(S)$ in favor of $\overrightarrow{a_id_i}$. Similarly, for i < n, $\overrightarrow{d_{i+1}d_i} \in \mathcal{C}_6(d_i)$ gets eliminated from $\Theta\Theta_6(S)$ in favor of $\overrightarrow{a_{i+1}d_i} \in \mathcal{C}_6(d_i)$. There are no edges in $\Theta_6(S)$ directed into d_i that lie in any of the cones $\mathcal{C}_1(d_i)$, $\mathcal{C}_3(d_i)$, $\mathcal{C}_4(d_i)$ and $\mathcal{C}_5(d_i)$.

For i > 1, the four edges directed into b_i that lie in $C_4(b_i)$ satisfy $\|\overrightarrow{c_{i-1}b_i}\| < \|\overrightarrow{a_ib_i}\| < \|\overrightarrow{b_{i-1}b_i}\| < \|\overrightarrow{d_{i-1}b_i}\| < \|\overrightarrow{d_{i$

The resulting $\Theta\Theta_6$ -graph is the path depicted in Figure 9c. The edge set of $\Theta\Theta_6(S)$ is $\{a_1b_1\} \cup \{a_id_i, b_ic_i : i = 1, 2, \dots, n\} \cup \{d_ia_{i+1}, c_ib_{i+1} : i = 1, 2, \dots, n-1\}.$

For an arbitrarily small α value, we have $|a_nb_n| \approx |a_1b_1|$. A shortest path $\xi_{\Theta\Theta_6}(a_n, b_n)$ in this graph between a_n and b_n has length

$$|\xi_{\Theta\Theta_{6}}(a_{n},b_{n})| > |a_{1}b_{1}| + \sum_{i=1}^{n-1} (|a_{i}d_{i}| + |d_{i}a_{i+1}|) + \sum_{i=1}^{n-1} (|b_{i}c_{i}| + |c_{i}b_{i+1}|)$$

$$> |a_{1}b_{1}| + \sum_{i=1}^{n-1} |a_{i}a_{i+1}| + \sum_{i=1}^{n-1} |b_{i}b_{i+1}| \quad \text{(by triangle inequality)}$$

$$> (2n-1) \cdot |a_{1}b_{1}|$$

This shows that the spanning ratio of $\Theta\Theta_6(S)$ is $\Omega(n)$, therefore we have the following result.

Theorem 8 The $\Theta\Theta_6$ -graph is not a spanner.

6 Conclusions

This paper establishes the first result showing a difference in the spanning properties of two related classes of sparse graphs, namely Yao-Yao and Theta-Theta. Previous results show YY_k and $\Theta\Theta_k$ are not spanners for $k \leq 5$, and are spanners for some values of k > 6. In this paper we show that, unlike YY_6 , the graph $\Theta\Theta_6$ is a spanner for sets of points in convex position. We also show that, for sets of points in non-convex position, $\Theta\Theta_6$ is not a spanner. The spanning ratios of YY_k and $\Theta\Theta_k$, for all even k in the range [8, 28] and for some even values of k (those that are not multiples of 6) in the range [32, 82], remain unknown.

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7 Bibliography

References

- [1] Khaled Alzoubi, Xiang-Yang Li, Yu Wang, Peng-Jun Wan, and Ophir Frieder. Geometric spanners for wireless ad hoc networks. *IEEE Transactions on Parallel and Distributed Systems*, 14(4):408–421, April 2003.
- [2] Luis Barba, Prosenjit Bose, Mirela Damian, Rolf Fagerberg, Wah Loon Keng, Joseph O'Rourke, André van Renssen, Perouz Taslakian, Sander Verdonschot, and Ge Xia. New and improved spanning ratios for Yao graphs. *Journal of Computational Geometry*, 6(2):19–53, 2015.
- [3] Nicolas Bonichon, Cyril Gavoille, Nicolas Hanusse, and David Ilcinkas. Connections between Theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *Proceedings of the 36th International Conference on Graph-theoretic Concepts in Computer Science*, WG'10, pages 266–278, Berlin, Heidelberg, 2010. Springer-Verlag.
- [4] Prosenjit Bose, Jean-Lou De Carufel, Darryl Hill, and Michiel Smid. On the spanning and routing ratio of Theta-four. http://arxiv.org/abs/1808.01298v1.
- [5] Prosenjit Bose, Mirela Damian, Karim Douïeb, Joseph O'Rourke, Ben Seamone, Michiel H. M. Smid, and Stefanie Wuhrer. Pi/2-angle Yao graphs are spanners. *CoRR*, abs/1001.2913, 2010.
- [6] Prosenjit Bose, Pat Morin, André van Renssen, and Sander Verdonschot. The Theta-5 graph is a spanner. Computational Geometry Theory and Applications, 48(2):108–119, 2015. A preliminary version appeared in Proceedings of the 39th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'13).
- [7] Prosenjit Bose, André van Renssen, and Sander Verdonschot. On the spanning ratio of Thetagraphs. In *Proceedings of the 13th International Symposium on Algorithms and Data Structures*, WADS'13, pages 182–194, August 2013.
- [8] L. Paul Chew. There are planar graphs almost as good as the complete graph. *Journal of Computer and System Sciences*, 39(2):205–219, October 1989.
- [9] Kenneth L. Clarkson. Approximation algorithms for shortest path motion planning. In Proceedings of the 19th Annual ACM Conference on Theory of Computing, STOC'87, pages 56–65, 1987.
- [10] Mirela Damian. Cone-based spanners of constant degree. Computational Geometry Theory and Applications, 68:48 61, 2018. Special issue in memory of Ferran Hurtado.
- [11] Mirela Damian and Matthew Bauer. An infinite class of sparse-Yao spanners. In *Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms*, SODA'13, pages 184–196, January 6-8 2013.
- [12] Mirela Damian, Nawar Molla, and Val Pinciu. Spanner properties of $\pi/2$ -angle Yao graphs. In Proceedings of the 25th European Workshop on Computational Geometry, pages 21–24, March 2009.
- [13] Mirela Damian and Naresh Nelavalli. Improved bounds on the stretch factor of Y_4 . Computational Geometry Theory and Applications, 62(C):14–24, April 2017.

- [14] David Eppstein. Spanning trees and spanners. In J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 425–461, Amsterdam, 2000. Elsevier Science.
- [15] Matthias Fischer, Tamás Lukovszki, and Martin Ziegler. Geometric searching in walkthrough animations with weak spanners in real time. In ESA '98: Proceedings of the 6th Annual European Symposium on Algorithms, pages 163–174, 1998.
- [16] Yifei Jin, Jian Li, and Wei Zhan. Odd Yao-Yao graphs are not spanners. In 34th International Symposium on Computational Geometry (SoCG'18), June 11-14, 2018, Budapest, Hungary, volume 49, pages 1–15, 2018.
- [17] Jian Li and Wei Zhan. Almost all even Yao-Yao graphs are spanners. In 24th Annual European Symposium on Algorithms ESA'16, volume 62, pages 1–13, 2016.
- [18] Xiang-Yang Li. Wireless Ad Hoc and Sensor Networks: Theory and Applications. Cambridge University Press, New York, NY, USA, 2008.
- [19] Xiang-Yang Li, Peng-Jun Wan, Yu Wang, and Ophir Frieder. Sparse power efficient topology for wireless networks. In *HICSS'02: Proceedings of the 35th Annual Hawaii Int. Conference on System Sciences*, volume 9, page 296.2, 2002.
- [20] Nawar Molla. Yao spanners for wireless ad hoc networks. Technical report, M.S. Thesis, Department of Computer Science, Villanova University, December 2009.
- [21] Giri Narasimhan and Michiel Smid. Geometric Spanner Networks. Cambridge University Press, New York, NY, USA, 2007.
- [22] Jim Ruppert and Raimund Seidel. Approximating the d-dimensional complete Euclidean graph. In *Proceedings of the 3rd Canadian Conference on Computational Geometry*, CCCG'91, pages 207–210, 1991.