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Rectilinear Crossings in Complete Balanced *d*-Partite *d*-Uniform Hypergraphs

Rahul Gangopadhyay · Saswata Shannigrahi

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Abstract In this paper, we study the embedding of a complete balanced d-partite d-uniform hypergraph with its nd vertices represented as points in general position in \mathbb{R}^d and each hyperedge drawn as the convex hull of d corresponding vertices. We assume that the set of vertices is partitioned into d disjoint sets, each of size n, such that each vertex in a hyperedge is from a different set. Two hyperedges are said to be crossing if they are vertex disjoint and contain a common point in their relative interiors. Using Colored Tverberg theorem with restricted dimensions, we observe that such an embedding of a complete balanced d-partite d-uniform hypergraph with nd vertices contains $\Omega\left((8/3)^{d/2}\right)(n/2)^d((n-1)/2)^d$ crossing pairs of hyperedges for $n \geq 3$ and sufficiently large d. Using Gale transform and Ham-Sandwich theorem, we improve this lower bound to $\Omega\left(2^d\right)(n/2)^d((n-1)/2)^d$ for $n \geq 3$ and sufficiently large d.

Keywords d-Partite Hypergraph · Crossing Hyperedges · Gale Transform · Colored Tverberg Theorem · Ham-Sandwich Theorem

1 Introduction

The rectilinear drawing of a graph is defined as an embedding of it in \mathbb{R}^2 such that its vertices are represented as points in general position (i.e., no three vertices are collinear) and edges are drawn as straight line segments connecting the corresponding vertices. The rectilinear crossing number of a graph G, denoted by $\overline{cr}(G)$, is defined as the minimum number of crossing pairs of

Rahul Gangopadhyay IIIT Delhi, India E-mail: rahulg@iiitd.ac.in

Saswata Shannigrahi Saint Petersburg State University, Russia E-mail: saswata.shannigrahi@gmail.com edges among all rectilinear drawings of G. Determining the rectilinear crossing number of a graph is one of the most important problems in graph theory [1,8,9,12]. In particular, finding the rectilinear crossing numbers of complete bipartite graphs is an active area of research [9,13]. Let $K_{n,n}$ denote the complete bipartite graph having n vertices in each part. For any $n \geq 5$, the best-known lower and upper bounds on $\overline{cr}(K_{n,n})$ are $(n(n-1)/5) \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$ and $\lfloor n/2 \rfloor^2 \lfloor (n-1)/2 \rfloor^2$, respectively [9,16]. For sufficiently large n, the result of Nahas [12] improved the lower bound on $\overline{cr}(K_{n,n})$ to $(n(n-1)/5) \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor + 9.9 \times 10^{-6} n^4$.

Hypergraphs are natural generalizations of graphs. A hypergraph H is a pair (V, E), where V is a set of vertices and E is a set of distinct subsets of V called hyperedges. A hypergraph H is called d-uniform if each hyperedge contains d vertices. A d-uniform hypergraph H = (V, E) is said to be d-partite if there exist sets X_1, X_2, \ldots, X_d such that $V = \bigcup_{i=1}^d X_i, \ X_i \cap X_j = \emptyset$ for any $i \neq j$, and each vertex in a hyperedge belonging to E is from a different X_i . We call X_i to be the i^{th} part of V. Moreover, such a d-partite d-uniform hypergraph is called balanced if $|X_1| = |X_2| = \ldots = |X_d|$ and complete if $|E| = |X_1 \times X_2 \times \ldots \times X_d|$. The complete balanced d-partite d-uniform hypergraph with n vertices in each part is denoted by $K_{d \times n}^d$. For $t \geq 2$, let us denote by $K_{k_1 \times n_1 + k_2 \times n_2 + \ldots + k_t \times n_t}^d$ the complete d-partite d-uniform hypergraph if $\sum_{i=1}^t k_i = d, \ n_i \neq n_{i+1}$ for all i in the range $1 \leq i \leq t-1$, and each of the first $k_1 > 0$ parts contains n_1 vertices, each of the next $k_2 > 0$ parts contains n_2 vertices, \ldots , each of the final $k_t > 0$ parts contains n_t vertices.

A d-dimensional rectilinear drawing [3] of a d-uniform hypergraph H=(V,E) is an embedding of it in \mathbb{R}^d such that its vertices are represented as points in general position (i.e., no d+1 vertices lie on a hyperplane) and its hyperedges are drawn as (d-1)-simplices formed by the corresponding vertices. In a d-dimensional rectilinear drawing of H, two hyperedges are said to be crossing if they are vertex disjoint and contain a common point in their relative interiors [4]. The d-dimensional rectilinear crossing number of H, denoted by $\overline{cr}_d(H)$, is defined as the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of H. Let us mention a few existing results on the d-dimensional rectilinear crossing number of uniform hypergraphs. Dey and Pach [4] proved that $\overline{cr}_d(H)=0$ implies the total number of hyperedges in H to be $O(|V|^{d-1})$. Gangopadhyay et al. [6] showed that $\overline{cr}_d(H)=\Omega\left(2^d\sqrt{d}\right)\binom{|V|}{2d}$ when H is a complete d-uniform hypergraph.

In this paper, we first use Colored Tverberg theorem with restricted dimensions and Lemma 1 to observe a lower bound on $\overline{cr}_d\left(K_{d\times n}^d\right)$ for $n\geq 3$ and sufficiently large d. We mention this lower bound in Observation 1. Let us introduce a few more definitions and notations used in its proof. Two d-uniform hypergraphs $H_1=(V_1,E_1)$ and $H_2=(V_2,E_2)$ are isomorphic if there is a bijection $f:V_1\to V_2$ such that any set of d vertices $\{u_1,u_2,\ldots,u_d\}$ is a

hyperedge in E_1 if and only if $\{f(u_1), f(u_2), \ldots, f(u_d)\}$ is a hyperedge in E_2 . A hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$ is called an *induced sub-hypergraph* of H = (V, E) if $\tilde{V} \subseteq V$ and \tilde{E} is the set of all hyperedges in E that are formed only by the vertices in \tilde{V} . The convex hull of a finite point set S is denoted by Conv(S). The convex hulls Conv(S) and Conv(S') of two finite point sets S and S' intersect if they contain a common point in their relative interiors. For u and w in the range $1 \le u$, $1 \le u$, and $1 \le u$ in the range $1 \le u$, and $1 \le u$ in the range $1 \le u$, and $1 \le u$ in the range $1 \le u$, and $1 \le u$ in the range $1 \le u$, and $1 \le u$ in the range $1 \le u$ in

Colored Tverberg Theorem with restricted dimensions. [10,15] Let $\{C_1, C_2, \ldots, C_{k+1}\}$ be a collection of k+1 disjoint finite point sets in \mathbb{R}^d . Each of these sets is of cardinality at least 2r-1, where r is a prime number satisfying the inequality $r(d-k) \leq d$. Then, there exist r disjoint sets S_1, S_2, \ldots, S_r such that $S_i \subseteq \bigcup_{j=1}^{k+1} C_j, \bigcap_{i=1}^r Conv(S_i) \neq \emptyset$ and $|S_i \cap C_j| = 1$ for all i and j satisfying $1 \leq i \leq r$ and $1 \leq j \leq k+1$.

Lemma 1 [2] Consider two disjoint point sets U and W, each a subset of a set A containing 2d points in general position in \mathbb{R}^d , such that |U| = u, |W| = w, $2 \le u$, $w \le d$ and $u + w \ge d + 1$. If the (u - 1)-simplex formed by U crosses the (w - 1)-simplex formed by W, then the (d - 1)-simplices formed by any two disjoint point sets $U' \supseteq U$ and $W' \supseteq W$ satisfying |U'| = |W'| = d and $U', W' \subset A$ also cross.

Observation 1 $\overline{cr}_d\left(K_{d\times n}^d\right) = \Omega\left((8/3)^{d/2}\right)\left(n/2\right)^d\left((n-1)/2\right)^d$ for $n \geq 3$ and sufficiently large d.

Proof Let us consider the hypergraph $H=K_{d\times n}^d$ such that its vertices are in general position in \mathbb{R}^d . Let $H'=K_{(\lceil d/2\rceil+1)\times 3+(\lfloor d/2\rfloor-1)\times 2}^d$ be an induced sub-hypergraph of it containing 3 vertices from each of the first $\lceil d/2\rceil+1$ parts and 2 vertices from each of the remaining $\lfloor d/2\rfloor-1$ parts. Let C_i denote the i^{th} part of the vertex set of H' for each i in the range $1\leq i\leq \lceil d/2\rceil+1$. Note that $C_1,C_2,\ldots,C_{\lceil d/2\rceil+1}$ are disjoint sets in \mathbb{R}^d and each of them contains 3 vertices. Clearly, these sets satisfy the condition of Colored Tverberg theorem with restricted dimensions for $k=\lceil d/2\rceil$ and r=2. Since the vertices of H' are in general position in \mathbb{R}^d , Colored Tverberg theorem with restricted dimensions implies that there exists a crossing pair of $\lceil d/2\rceil$ -simplices spanned by $U\subseteq\bigcup_{j=1}^{\lceil d/2\rceil+1}C_j$ and $W\subseteq\bigcup_{j=1}^{\lceil d/2\rceil+1}C_j$ such that $U\cap W=\emptyset$ and $|U\cap C_j|=1$, $|W\cap C_j|=1$ for each j in the range $1\leq j\leq \lceil d/2\rceil+1$. Lemma 1 implies that U and W can be extended to form $2^{\lfloor d/2\rfloor-1}$ distinct crossing pairs of (d-1)-simplices, where each (d-1)-simplex contains exactly one vertex from each part of H'. This implies that $\overline{cr}_d(H')\geq 2^{\lfloor d/2\rfloor-1}$. Note that each crossing pair of hyperedges corresponding to these (d-1)-simplices is contained in $(n-2)^{\lceil d/2\rceil+1}$ distinct induced sub-hypergraphs of H, each

of which is isomorphic to H'. Moreover, there are $\binom{n}{3}^{\lceil d/2 \rceil + 1} \binom{n}{2}^{\lfloor d/2 \rfloor - 1}$

distinct induced sub-hypergraphs of H, each of which is isomorphic to H'. This implies $\overline{cr}_d\left(K_{d\times n}^d\right) \geq 2^{\lfloor d/2\rfloor-1} \binom{n}{3}^{\lceil d/2\rceil+1} \binom{n}{2}^{\lfloor d/2\rfloor-1} / (n-2)^{\lceil d/2\rceil+1} = n^d(n-1)^d/6^{\lceil d/2\rceil+1} = \Omega\left((8/3)^{d/2}\right)(n/2)^d\left((n-1)/2\right)^d$.

In Section 3, we improve the lower bound on $\overline{cr}_d(K_{d\times n}^d)$ as follows. To the best of our knowledge, this is the first non-trivial lower bound on this number.

Theorem 1 $\overline{cr}_d\left(K_{d\times n}^d\right) = \Omega\left(2^d\right)\left(n/2\right)^d\left((n-1)/2\right)^d$ for $n \geq 3$ and sufficiently large d.

2 Techniques Used

Let $A = \langle a_1, a_2, \ldots, a_m \rangle$ be a sequence of $m \geq d+1$ points in \mathbb{R}^d such that the points in A affinely span \mathbb{R}^d . Let the coordinate of the i^{th} point a_i be $(x_1^i, x_2^i, \ldots, x_d^i)$. To compute a Gale transform [5] D(A) of A, we consider the following matrix M(A).

$$M(A) = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Note that the rank of M(A) is d+1, since the points in A affinely span \mathbb{R}^d . The Rank-nullity theorem [11] implies that the dimension of the null space of M(A) is m-d-1. Let $\{(b_1^1,b_2^1,\ldots,b_m^1),(b_1^2,b_2^2,\ldots,b_m^2),\ldots,(b_1^{m-d-1},b_2^{m-d-1},\ldots,b_m^{m-d-1})\}$ be a basis of this null space. The Gale transform D(A) corresponding to this basis is the sequence of m vectors $\langle (b_1^1,b_1^2,\ldots,b_1^{m-d-1}),(b_2^1,b_2^2,\ldots,b_2^{m-d-1}),\ldots,(b_m^1,b_m^2,\ldots,b_m^{m-d-1})\rangle$. Note that D(A) can also be considered as a point sequence in \mathbb{R}^{m-d-1} , which we denote by $\langle g_1,g_2,\ldots,g_m\rangle$. In the following, we mention some properties of D(A) that are used in Section 3. For the sake of completeness, we give a proof of Lemma 3. On the other hand, the proof of Lemma 2 is straightforward and we omit it here.

Lemma 2 [7] If the points in A are in general position in \mathbb{R}^d , each collection of m-d-1 vectors in D(A) spans \mathbb{R}^{m-d-1} .

Lemma 3 [7] Let h be a linear hyperplane, i.e., a hyperplane passing through the origin, in \mathbb{R}^{m-d-1} . Let $D^+(A) \subset D(A)$ and $D^-(A) \subset D(A)$ denote two sets of vectors such that $|D^+(A)|, |D^-(A)| \geq 2$ and the vectors in $D^+(A)$ and $D^-(A)$ lie in the opposite open half-spaces h^+ and h^- created by h, respectively. Then, the convex hull of the point set $A_a = \{a_i | a_i \in A, g_i \in D^+(A)\}$ and the convex hull of the point set $A_b = \{a_j | a_j \in A, g_j \in D^-(A)\}$ intersect.

Proof Let us assume that the hyperplane h is given by the equation $\sum_{i=1}^{m-d-1} \alpha_i x_i = 0$ such that $\alpha_i \neq 0$ for at least one i, and h^+ (h^-) is the positive (negative) open half-space created by it. Let $D^0(A) = \{g_k | g_k \in D(A), g_k \text{ lies on } h\}$. This implies that there exists a vector $(\mu_1, \mu_2, \dots, \mu_m) = \alpha_1(b_1^1, b_2^1, \dots, b_m^1) + \alpha_2(b_1^2, b_2^2, \dots, b_m^2) + \dots + \alpha_{m-d-1}(b_1^{m-d-1}, b_2^{m-d-1}, \dots, b_m^{m-d-1})$ such that $\mu_i > 0$ for each $g_i \in D^+(A)$, $\mu_j < 0$ for each $g_j \in D^-(A)$ and $\mu_k = 0$ for each $g_k \in D^0(A)$. Since this vector $(\mu_1, \mu_2, \dots, \mu_m)$ lies in the null space of M(A), it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

From the equation above, we obtain the following.

$$\sum_{i:g_i \in D^+(A)} \mu_i a_i = \sum_{j:g_j \in D^-(A)} -\mu_j a_j, \sum_{i:g_i \in D^+(A)} \mu_i = \sum_{j:g_j \in D^-(A)} -\mu_j$$

Rearranging the equations above, we obtain the following.

$$\sum_{i:g_i \in D^+(A)} \frac{\mu_i}{\sum_{i:g_i \in D^+(A)} \mu_i} a_i = \sum_{j:g_j \in D^-(A)} \frac{\mu_j}{\sum_{j:g_j \in D^-(A)} \mu_j} a_j$$

$$\sum_{i:g_i \in D^+(A)} \frac{\mu_i}{\sum_{i:g_i \in D^+(A)} \mu_i} = \sum_{j:g_j \in D^-(A)} \frac{\mu_j}{\sum_{j:g_i \in D^-(A)} \mu_j} = 1$$

It shows that $Conv(A_a)$ and $Conv(A_b)$ intersect.

Let us also state Ham-Sandwich theorem here.

Ham-Sandwich Theorem. [10,14] There exists a (d-1)-dimensional hyperplane h which simultaneously bisects d finite point sets P_1, P_2, \ldots, P_d in \mathbb{R}^d , such that each of the open half-spaces created by h contains at most $\lfloor |P_i|/2 \rfloor$ points of P_i for each i in the range $1 \leq i \leq d$.

3 Lower Bound on the $d\text{-}\mbox{Dimensional Rectilinear Crossing Number of }K^d_{d\times n}$

In this section, we use Gale transform of a sequence of points and Ham-Sandwich theorem to improve the previously observed lower bound on the d-dimensional rectilinear crossing number of $K^d_{d\times n}$ for $n\geq 3$ and sufficiently large d.

Proof of Theorem 1. Let us consider the hypergraph $H = K_{d \times n}^d$ such that

its vertices are in general position in \mathbb{R}^d . Let $H'=K^d_{2\times 3+(d-2)\times 2}$ be an induced sub-hypergraph of it containing 3 vertices from each of the first 2 parts and 2 vertices from each of the remaining (d-2) parts of the vertex set of H. Let $V' = \langle v_1, v_2, v_3, \dots, v_{2d+1}, v_{2d+2} \rangle$ be a sequence of the vertices of H' such that $\{v_1, v_2, v_3\}$ belongs to the first part $L_1, \{v_4, v_5, v_6\}$ belongs to the second part L_2 and $\{v_{2k+1}, v_{2k+2}\}$ belongs to the k^{th} part L_k for each kin the range $3 \leq k \leq d$. We consider a Gale transform of V' and obtain a sequence of 2d + 2 vectors $D(V') = \langle p_1, p_2, p_3, \dots, p_{2d+1}, p_{2d+2} \rangle$ in \mathbb{R}^{d+1} . It follows from Lemma 2 that any set containing d+1 of these vectors spans \mathbb{R}^{d+1} . As mentioned before, D(V') can also be considered as a sequence of points in \mathbb{R}^{d+1} . In order to apply Ham-Sandwich theorem in \mathbb{R}^{d+1} , we color the origin with color c_0 , $\{p_1, p_2, p_3\}$ with color c_1 , $\{p_4, p_5, p_6\}$ with color c_2 and $\{p_{2k+1}, p_{2k+2}\}$ with color c_k for each k in the range $3 \leq k \leq d$. It follows from Ham-Sandwich theorem that there exists a hyperplane h such that it passes through the origin and bisects the set colored with c_i for each i in the range $1 \leq i \leq d$. Note that at most d points of D(V') lie on the linear hyperplane h, since any set of d+1 vectors in D(V') spans \mathbb{R}^{d+1} . This implies that there exist at least d+2 points of D(V') that lie in $h^+ \cup h^-$, where h^+ is the positive open half-space and h^- is the negative open half-space created by h. Let $D^+(V')$ and $D^-(V')$ be the two sets of points lying in h^+ and h^- , respectively. It follows from Ham-Sandwich theorem that at most d points of D(V') can lie in either of h^+ and h^- . This implies that $|D^+(V')| \ge 2$ and $|D^-(V')| \ge 2$. Moreover, note that 2 points having the same color cannot lie in the same open half-space. Lemma 3 implies that there exist a (u-1)-simplex $Conv(V'_a)$ spanned by some vertices $V'_a \subset V'$ and a (w-1)-simplex $Conv(V'_b)$ spanned by some vertices $V'_b \subset V'$ such that the following conditions are satisfied.

- (I) $V_a' \cap V_b' = \emptyset$
- (II) $Conv(V'_a)$ and $Conv(V'_b)$ cross.
- (III) $2 \le |V'_a|, |V'_b| \le d, |V'_a| + |V'_b| \ge d + 2$ (IV) $|V'_a \cap L_i| \le 1$ for each i in the range $1 \le i \le d$ (V) $|V'_b \cap L_i| \le 1$ for each i in the range $1 \le i \le d$

Lemma 1 implies that the crossing between $Conv(V'_a)$ and $Conv(V'_b)$ can be extended to a crossing pair of (d-1)-simplices spanned by any two disjoint vertex sets $U', W' \subset V'$ satisfying |U'| = |W'| = d and $U' \supseteq V'_a$ and $W' \supseteq V'_b$, respectively. In fact, it is always possible to add vertices to V_a' and V_b' in such a way that the following conditions hold for U' and W'.

- (I) $U' \cap W' = \emptyset$
- (II) Conv(U') and Conv(W') cross.
- (III) |U'| = |W'| = d
- (IV) $|U' \cap L_i| = 1$ for each i in the range $1 \le i \le d$
- (V) $|W' \cap L_i| = 1$ for each i in the range $1 \le i \le d$

The argument above establishes the fact that $\overline{cr}_d(H') \geq 1$. Note that H distinct induced sub-hypergraphs, each of which is isomorphic to H'. Since each crossing pair of hyperedges is contained in $(n-2)^2$ distinct induced sub-hypergraphs of H, each of which is isomorphic to H', we obtain $\overline{cr}_d\left(K_{d\times n}^d\right) \geq \binom{n}{3}^2\binom{n}{2}^{d-2} \bigg/(n-2)^2 = (1/9)n^d(n-1)^d/2^d = \Omega\left(2^d\right)\left(n/2\right)^d\left((n-1)/2\right)^d$.

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