GENERALIZATION OF SOME RESULTS ON LIST COLORING AND DP-COLORING

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Abstract

Let G be a graph and let $f_i, i \in \{1, \ldots, s\}$, be a function from $V(G)$ to the set of nonnegative integers. In [23], the concept of DP-F-coloring, a generalization of DP-coloring and variable degeneracy, was introduced. We use DP-F-coloring to define DPG-$[k,t]$-colorable graph and modify the proofs in [22, 24, 25] to obtain more results on list coloring, DP-coloring, list-forested coloring, and variable degeneracy.

1. Introduction

Every graph in this paper is finite, simple, and undirected. We let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of a graph G. Let $d_G(v)$ denote the degree of a vertex v in a graph G. If no confusion arises, we simply use $d(v)$ instead of $d_G(v)$. Let S be a subset of $V(G)$. A subgraph of G induced by S is denoted by $G[S]$. If a plane graph G contains a cycle C, we use $int(C)$ (respectively, $ext(C)$) for the subgraph induced by vertices on C and inside C (respectively, outside C).

Let f be a function from $V(G)$ to the set of positive integers. A graph G is *strictly f-degenerate* if every subgraph G' has a vertex v with $d_{G'}(v) < f(v)$. Equivalently, G is strictly f-degenerate if and only if vertices of G can be ordered so that each vertex has less than $f(v)$ neighbors in the lower order. Let k be a positive integer. A graph G is *strictly k-degenerate* if and only if G is strictly f-degenerate where $f(v) = k$ for each vertex v. Thus a strictly 1-degenerate graph is an edgeless graph and a strictly 2-degenerate graph is a forest. Equivalently, G is strictly k-degenerate if and only if vertices of G can be ordered so that each vertex has less than k neighbors in the lower order.

Let $f_i, i \in \{1, \ldots, s\}$, be a function from $V(G)$ to the set of nonnegative integers. An $(f_1, \ldots, f_s)$-*partition* of a graph G is a partition of $V(G)$ into $V_1, \ldots, V_s$ such that an induced subgraph $G[V_i]$ is strictly f_i-degenerate for each $i \in \{1, \ldots, s\}$. A $(k_1, \ldots, k_s)$-*partition* where k_i is a constant for each $i \in \{1, \ldots, s\}$ is an $(f_1, \ldots, f_s)$-partition such that $f_i(v) = k_i$ for each vertex v. We say that G is $(f_1, \ldots, f_s)$-*partitionable* if G has an $(f_1, \ldots, f_s)$-partition. By Four Color Theorem [2], every planar graph is $(1,1,1,1)$-partitionable. On the other hand, Chartrand and Kronk [11] constructed planar graphs which are not $(2,2)$-partitionable. Even stronger, Wegner [28] showed that there exists a planar graph which is not $(2,1,1)$-partitionable. Thus it is of interest to find sufficient conditions for planar graphs to be $(1,1,1,1)$-, $(2,1,1)$-, or $(2,2)$-partitionable.

Borodin and Ivanova [7] obtained a sufficient condition that implies $(1,1,1,1)$-, $(2,1,1)$-, or $(2,2)$-partitionability as follows.

Theorem 1.1. *(Theorem 6 in [7]) Every planar graph without 4-cycles adjacent to 3-cycles is $(f_1, \dots, f_s)$-partitionable if $f_1(v) + \cdots + f_s(v) \geq 4$ for each vertex v, and $f_i(v) \in \{0, 1, 2\}$ for each v and i.*

The *vertex-arboricity* $va(G)$ of a graph G is the minimum number of subsets in which $V(G)$ can be partitioned so that each subset induces a forest. This concept was introduced by Chartrand, Kronk, and Wall [10] as *point-arboricity*. They proved that $va(G) \leq 3$ for every planar graph G. Later, Chartrand and Kronk [11] proved that this bound is sharp by providing an example of a planar graph G with $va(G) = 3$. It was shown that determining the vertex-arboricity of a graph is NP-hard by Garey and Johnson [15] and determining whether $va(G) \leq 2$ is NP-complete for maximal planar graphs G by Hakimi and Schmeichel [16]. Raspaud and Wang [21] showed that $va(G) \leq \lceil \frac{k+1}{2} \rceil$ for every k-degenerate graph G. It was proved that every planar graph G has $va(G) \leq 2$ when G is without k-cycles for $k \in \{3, 4, 5, 6\}$ (Raspaud and Wang [21]), without 7-cycles (Huang, Shiu, and Wang [17]), without intersecting 3-cycles (Chen, Raspaud, and Wang [12]), without chordal 6-cycles (Huang and Wang [18]), or without intersecting 5-cycle (Cai, Wu, and Sun [9]).

The concept of list coloring was independently introduced by Vizing [26] and by Erdős, Rubin, and Taylor [14]. A *k-assignment* L of a graph G assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex v of G. A graph G is *L-colorable* if there is a proper coloring c where $c(v) \in L(v)$ for each vertex v. If G is L-colorable for each k-assignment L, then we say G is *k-choosable*. The *list chromatic number* of G, denoted by $\chi_l(G)$, is the minimum number k such that G is k-choosable.

Borodin, Kostochka, and Toft [8] introduced list vertex arboricity which is list version of vertex arboricity. We say that G has an *L-forested-coloring* f for a set $L = \{L(v) | v \in V(G)\}$ if one can choose $f(v) \in L(v)$ for each vertex v so that a subgraph induced by vertices with the same color is a forest. We say that G is *list vertex k-arborable* if G has an L-forested-coloring for each k-assignment L. The *list vertex arboricity* $a_l(G)$ is defined to be the minimum k such that G is list vertex k-arborable. Obviously, $a_l(G) \geq va(G)$ for every graph G.

It was proved that every planar graph G is list vertex 2-arborable when G is without k-cycles for $k \in \{3, 4, 5, 6\}$ (Xue and Wu [29]), with no 3-cycles at distance less than 2 (Borodin and Ivanova [5]), or without 4-cycles adjacent to 3-cycles (Borodin and Ivanova [7]).

Borodin, Kostochka, and Toft [8] observed that the notion of $(f_1, \dots, f_s)$-partition can be applied to problems in list coloring and list vertex arboricity. Since v cannot have less than zero neighbor, the condition that $f_i(v) = 0$ is equivalent to v cannot be colored by i. In other words, i is not in the list of v. Thus the case of $f_i \in \{0, 1\}$ corresponds to list coloring, and one of $f_i \in \{0, 2\}$ corresponds to L-forested-coloring. Note that Theorem 1.1 implies that planar graphs without 3-cycles adjacent to 4-cycles are 4-choosable and list vertex 2-arborable.

Dvořák and Postle [13] introduced a generalization of list coloring in which they called a *correspondence coloring*. Following Bernshteyn, Kostochka, and Pron [4], we call it a *DP-coloring*.

Definition 1. Let L be an assignment of a graph G. We call H a *cover* of G if it satisfies all the followings:
(i) The vertex set of H is $\bigcup_{u \in V(G)}(\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$;
(ii) $H[\{u\} \times L(u)]$ is a complete graph for each $u \in V(G)$;
(iii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (may be empty);
(iv) If $uv \notin E(G)$, then no edges of H connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.
Let (G, H) denote a graph G with a cover H.

Definition 2. A *representative set* of (G, H) is a set of vertices of size $|V(G)|$ containing exactly one vertex from each $\{v\} \times L(v)$. A DP-coloring of (G, H) is a representative set R that $H[R]$ has no edges. We say that a graph G is *DP-k-colorable* if (G, H) has a DP-coloring for each cover H of G with a k-assignment L. The *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$, is the minimum number k such that G is DP-k-colorable.

If we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$, then (G, H) has a DP-coloring if and only if G is L-colorable. Thus DP-coloring is a generalization of list coloring. Moreover, $\chi_{DP}(G) \geq \chi_l(G)$. For example, Alon and Tarsi [1] showed that every planar bipartite graph is 3-choosable, while Bernshteyn and Kostochka [3] obtained a bipartite planar graph G with $\chi_{DP}(G) = 4$.

Dvořák and Postle [13] observed that $\chi_{DP}(G) \leq 5$ for every planar graph G. This extends a seminal result by Thomassen [25] on list colorings. On the other hand, Voigt [27] gave an example of a planar graph which is not 4-choosable (thus not DP-4-colorable). Kim and Ozeki [19] showed that planar graphs without k-cycles are DP-4-colorable for each $k = 3, 4, 5, 6$. Kim and Yu [20] extended the result on 3- and 4-cycles by showing that planar graphs without 3-cycles adjacent to 4-cycles are DP-4-colorable.

Later, the concept of DP-coloring and improper coloring is combined by allowing a representative set R to yield $H[R]$ with edges but requiring $H[R]$ to satisfy some degree conditions such as degeneracy [23] or maximum degree [24].

Definition 3. A *DP-forested-coloring* of (G, H) is a representative set R such that $H[R]$ is a forest. We say that a graph G is *DP-vertex-k-arborable* if (G, H) has a DP-forested-coloring for each k-assignment L and each cover H of G.

If we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$, then (G, H) has a DP-forested-coloring if and only if G has an L-forested-coloring.

From now on, we assume G is a graph with a k-assignment of colors L such that $\bigcup_{v \in V(G)} L(v) \subseteq \{1, \ldots, s\}$ and H is a cover of G. Assume furthermore that $F = (f_1, \ldots, f_s)$ and f_i, where $1 \leq i \leq s$, is a function from $V(G)$ to the set of nonnegative integers. The concept of DP-coloring is combined with $(f_1, \ldots, f_s)$-partition in [23] as follows.

Definition 4. A *DP-F-coloring* R of (G, H) is a representative set which can be ordered so that each element (v, i) in R has less than $f_i(v)$ neighbors in the lower order. Such order is called a *strictly F-degenerate order*. We say that G is *DP-F-colorable* if (G, H) has a DP-F-coloring for every cover H.

If we define edges on H to match exactly the same colors for each $uv \in E(G)$, then G has an $(f_1,\dots,f_s)$-partition if and only if (G,H) has a DP-F-coloring. Thus an $(f_1,\dots,f_s)$-partition is a special case of a DP-F-coloring. Observe that a DP-F-coloring where $f_i(v) \in \{0,1\}$ for each i and each vertex v is equivalent to a DP-coloring. Furthermore, a DP-F-coloring where $f_i(v) \in \{0,1,2\}$ for each i and each vertex v is equivalent to a DP-forested-coloring. We show in this work that the condition $f_i(v) \in \{0,1\}$ (DP-coloring) may be relaxed to $f_i(v) \in \{0,1,2\}$ to obtain a more general result. For conciseness, we define the following definition.

Definition 5. Let $|f(v)|$ denote $f_1(v)+\cdots+f_s(v)$. A graph G is *DPG-$[k,t]$-colorable* if (G,H) has a DP-F-coloring for every cover H and f such that $|f(v)| \geq k$ and $f_i(v) \leq t$ for every vertex v and every i with $1 \leq i \leq s$.

Lemma 1.2. *Let $C(i)$ denote the set of vertices colored i in G. If G is DPG-$[k,2]$-colorable, then we have the followings:*
(1) G is DP-k-colorable and thus k-choosable.
(2) G is DP-vertex-$\lceil k/2 \rceil$-arborable.
(3) Let $2d > k$. If L is a d-assignment for G where $d \leq k$ and $1,2,\dots,2d-k$ are colors, then we can find an L-foreted-coloring such that $C(i)$ is an independent set for each $i \in \{1,\dots,2d-k\}$.

Proof. Let G be a DPG-$[k,2]$-colorable graph.
(1) Let L be a k-assignment of G. Define $f_i(v)=1$ if $i \in L(v)$, otherwise $f_i(v)=0$. Note that (G,H) has a DP-k-coloring if and only if (G,H) has a DP-F-coloring. Since G is DPG-$[k,2]$-colorable, (G,H) has a DP-k-coloring for every cover H.
(2) Let L be a $\lceil k/2 \rceil$-assignment of G. Define $f_i(v)=2$ if $i \in L(v)$, otherwise $f_i(v)=0$. Note that (G,H) has a DP-forested-coloring if and only if (G,H) has a DP-F-coloring. Since G is a DPG-$[k,2]$-colorable graph, (G,H) has a DP-forested-coloring for every cover H and every $\lceil k/2 \rceil$-assignment of G.
(3) Let L be a d-assignment of G. Define $f_i(v)=1$ when $i \in L(v)$ and $1 \leq i \leq 2d-k$, $f_i(v)=2$ when $i \in L(v)$ and $i \geq 2d-k+1$, and $f_i(v)=0$ otherwise. Let edges on H match exactly the same colors. Note that G has an L-forested-coloring with $C(i)$ is an independent set for $1 \leq i \leq 2d-k$ if and only if (G,H) has a DP-F-coloring. Since G is DPG-$[k,2]$-colorable, we have the desired result.

□

We use the concept of DPG-$[k,2]$-colorable graph to generalize these three results on list coloring and DP-coloring.

Theorem 1.3. [25] *Every planar graph is 5-choosable.*

Theorem 1.4. [24] *Let $\mathcal{A}$ be the family of planar graphs without pairwise adjacent 3-, 4-, and 5-cycles. If $G \in \mathcal{A}$ contains a 3-cycle C, then each precoloring of C can be extended to a DP-4-coloring of G.*

Theorem 1.5. [22] *Let G be a planar graph without cycles of lengths $\{4,a,b,9\}$ where a and b are distinct values from $\{6,7,8\}$. Then G is DP-3-colorable.*

Using DPG-$[k,2]$-colorability, we modify the proof of Theorems 1.3, 1.4, and 1.5 to obtain the following main results.

Theorem 1.6. *Every planar graph G is DPG-$[5,2]$-colorable. In particular, we have the followings.*
(1) G is 5-choosable [25].
(2) G is 5-DP-colorable [13].
(3) If L is a 4-assignment of G with colors i, j, and k, then G has an L-forested-coloring with $C(i), C(j)$, and $C(k)$ are independent sets.
(4) If L is a 3-assignment of G with a color i, then G has an L-forested-coloring with $C(i)$ is an independent set.
(5) G is DP-vertex-3-arborable.
(6) G is $(f_1, \ldots, f_s)$-partitionable if $|f(v)| \geq 5$ and $f_i(v) \in \{0,1,2\}$ for every vertex v and every i with $1 \leq i \leq s$.

Theorem 1.7. *Let $G \in \mathcal{A}$ contains a 3-cycle C_0. Let $|f(v)| \geq k$ and $f_i(v) \leq 2$ for $1 \leq i \leq s$. Then every DP-F-coloring on C_0 can be extended to a DP-F-coloring on G. In particular, we have the followings.*
(1) G is DP-4-colorable [24].
(2) If L is a 3-assignment of G with colors i and j, then G has an L-forested-coloring with $C(i)$ and $C(j)$ are independent sets.
(3) G is DP-vertex-2-arborable.
(4) G is $(f_1, \ldots, f_s)$-partitionable if $|f(v)| \geq 4$ and $f_i(v) \in \{0,1,2\}$ for every vertex v and every i with $1 \leq i \leq s$.
Note that (1), (2), and (3) still hold even when G has a corresponding precoloring on C_0.

Theorem 1.8. *Let G be a planar graph without cycles of lengths $\{4, a, b, 9\}$ where a and b are distinct values from $\{6,7,8\}$. Then G is DPG-$[3,2]$-colorable. In particular, we have the followings.*
(1) G is DP-3-colorable [22].
(2) G is DP-vertex-2-arborable.
(3) If L is a 2-assignment of G with a color i, then G has an L-forested-coloring with $C(i)$ is an independent set.
(4) G is $(f_1, \ldots, f_s)$-partitionable if $|f(v)| \geq 3$ and $f_i(v) \in \{0,1,2\}$ for every vertex v and every i with $1 \leq i \leq s$.

2. Helpful Tools

Some definitions and lemmas which are used to prove the main results are presented in this section. Since we focus on DP-$[k,2]$-colorability, we assume from now on that $f_i(v) \in \{0,1,2\}$ for every vertex v and every i with $1 \leq i \leq s$. Furthermore, a DP-F-precoloring on a subgraph G' is assumed to be a DP-F-coloring restrict on (G', H') where H' is a cover H restrict to G'.

Definition 6. Let R' be a DP-F-precoloring on an induced subgraph G' of G. The *residual function* $f^* = (f_1^*, \ldots, f_s^*)$ for $G - G'$ is defined by

$$f_i^*(v) = \max\{0, f_i(v) - |\{(x,j) \in R' : (v,i)(x,j) \in E(H)\}|\}$$

for each $v \in V(G) - V(G')$.

For conciseness, we simply say R_2 is a DP-F^*-coloring of $G - G'$ instead of that of $(G - G', H - H')$. From the above definition, we have the following fact.

Lemma 7. *Let R' be a DP-F-precoloring of an induced subgraph G' of G and let $F^* = (f_1^*, \ldots, f_s^*)$ be a residual function of $G - G'$. If $G - G'$ has a DP-F^*-coloring, then (G, H) has a DP-F-coloring.*

Proof. Let R_1 be a DP-F-precoloring of G' with a strictly F-degenerate order S_1 and R_2 be a DP-F^*-coloring of $G - G'$ with a strictly F^*-degenerate order S_2. Then $R_1 \cup R_2$ is a representative set of (G, H). We claim that the order S obtained from S_1 followed by S_2 is a strictly F-degenerate order of $R_1 \cup R_2$. Consequently, $R_1 \cup R_2$ is a DP-F-coloring of (G, H). For $(v, i) \in R_1$, the neighbors in the lower order of S and that of S_1 are the same. By the construction of S_1, (v, i) has less than $f_i(v)$ neighbors in the lower order of S. Consider $(v, i) \in R_2$. Suppose (v, i) has d neighbors in R_1. Note that $f_i^*(v) \geq 1$, otherwise (v, i) cannot be chosen in R_2. It follows that $f_i^*(v) = f_i(v) - d$ by the definition of f_i^*. Since (v, i) has less than $f_i^*(v)$ neighbors in R_2 in the lower order of S, (v, i) has less than $f_i^*(v) + d = f_i(v)$ neighbors in the lower order of S. Thus S is a strictly F-degenerate order. □

Similarly, a partial DP-F-coloring R' with a strictly F-degenerate order S can be extended by a greedy coloring on a vertex v with $|f^*(v)| \geq 1$. We add (v, i) with $f_i(v) \geq 1$ to R'. It can be seen that S followed by (v, i) is a strictly F-degenerate order.

The term *minimal counterexample* is used for (G, H) that is a counterexample and $|V(G)|$ is minimized.

Lemma 2.1. *If (G, H) is a minimal counterexample to Theorem 1.8, then every vertex has degree at least* 3.

Proof. Suppose to the contrary that a vertex v has degree at most 2. By minimality, $G - v$ has a DP-F-coloring. Now, $|f^*(v)| \geq |f(v)| - d(v) \geq 3 - 2 = 1$. Thus we can apply a greedy coloring to v to complete the coloring. □

With a similar proof, one obtain the following lemma.

Lemma 2.2. *If (G, H) and a precolored* 3*-cycle C_0 is a minimal counterexample to Theorem 1.7, then every vertex not on C_0 has degree at least* 4.

Lemma 2.3. *Let G be a graph containing a subgraph K with the following property: if H is a cover of G and f has $f(v)| \geq k$ for every vertex v, then each DP-F-coloring of K can be extended to that of (G, H). Suppose R_1 is a DP-F-coloring of K. Then there exists a DP-F-coloring of (G, H) with a strictly F-degenerate S such that the $|R_1|$ lowest-ordered elements are in R_1.*

Proof. Let R_1 be a DP-F-coloring of K with a strictly F-degenerate order S_1. By renaming the colors, we assume that S_1 has the order $(v_1, 1), \dots, (v_t, 1)$. Let H' be a cover of G obtained from H by modifying matchings between colors in R_1 so that R_1 is independent.

Let f' be obtained from f by defining $f'_i(v_1) = \cdots = f'_i(v_t) = 1$ if $1 \le i \le k$, otherwise $f'_i(v_1) = \cdots = f'_i(v_t) = 0$. Note that $|f'(v)| \ge k$ and $f'_i(v) \in \{0, 1, 2\}$ for every vertex v and every i with $1 \le i \le k$. By condition of G and K, (G, H') has a DP-f'-coloring R with a strictly f'-degenerate order S'. Let S be obtained from S' by moving $(v_1, 1), \dots, (v_t, 1)$ to be in the lowest order. We claim that R is a DP-F-coloring with a strictly F-degenerate order S.

It is obvious that R is a representative set of (G, H) and $(v_1, 1), \dots, (v_t, 1)$ are the lowest elements of S. It remains to show that S is a strictly F-degenerate order. Consider $(u, i) \in R$. If $(u, i) \in R_1$, then it has less than $f_i(u)$ neighbors in the lower order of S_1 by the construction. Since the neighbors in the lower order of S_1 and that of S are the same, (u, i) has less than $f_i(u)$ neighbors in the lower order of S.

Assume that $(u, i) \notin R_1$. Suppose to the contrary that (u, i) has at least $f_i(u)$ neighbors in the lower order of S. Since S' is a strictly f'-degenerate order, (u, i) has less than $f'_i(u) = f_i(u)$ neighbors in the lower order of S'. Then an additional neighbor in the lower order of S, say $(v, 1)$, is in R_1 by the construction of S. Moreover, the order of (u, i) in S' is lower than that of $(v, 1)$. It follows that $(v, 1)$ has at least $f'_1(v) = 1$ neighbor in the lower order of a strictly f'-degenerate order S', a contradiction. It follows that (u, i) has less than $f_i(u)$ neighbors in the lower order of S. Thus S is a strictly F-degenerate order and this completes the proof. □

Note that Lemma 2.3 holds regardless of an upper bound on $f_i(v)$.

Lemma 2.4. *Let (G, H) be a minimal counterexample to Theorem 1.7 with a DP-F-precoloring of 3-cycle C_0. Then G has no separating 3-cycles.*

Proof. Suppose to the contrary that G has a separating 3-cycle C. By symmetry, we assume $C_0 \subseteq ext(C)$. By minimality, a DP-F-coloring on C_0 can be extended to a coloring R_1 on $ext(C)$. Let S_1 be a strictly F-degenerate order of R_1. Let $V(C) = \{x, y, z\}$ and $(x, 1), (y, 1), (z, 1) \in R_1$. By minimality, $int(C)$ has a DP-F-coloring R_2 including $(x, 1), (y, 1), (z, 1)$. By Lemma 2.3, R_2 has a strictly F-degenerate order S_2 such that $(x, 1), (y, 1), (z, 1)$ are the lowest order elements.

It is obvious that $R_1 \cup R_2$ is a representative set of (G, H). Let S'_2 be obtained from S_2 by deleting $(x, 1), (y, 1), (z, 1)$. We claim that S obtained from S_1 followed by S'_2 is a strictly F-degenerate order. If $(u, i) \in R_1$, then the neighbors of (u, i) in the lower order of S are the same as that of S_1 by the construction of S. It follows from S_1 is a strictly F-degenerate that (u, i) has less than $f_i(u)$ neighbors in the lower order of S. Note that this case also includes (u, i) is $(x, 1), (y, 1)$ or $(z, 1)$.

Consider $(u, i) \in R_2 - R_1$. Then (u, i) has less than $f_i(v)$ neighbors in the lower order of S_2. It follows that (u, i) has less than $f_i(v)$ neighbors that are in R_2 and in the lower order of S. Since (u, i) is not adjacent to any elements in $R_1 - \{(x, 1), (y, 1), (z, 1)\}$, all neighbors of (u, i) are in R_2. Consequently, (u, i) has less than $f_i(v)$ neighbors in the lower order of S. Thus $R_1 \cup R_2$ is a DP-F-coloring of (G, H), a contradiction. □

Lemma 2.5. *Let $k \geq 3$ and $K \subseteq G$ with $V(K) = \{v_1, \ldots, v_m\}$ such that the followings hold.*
(i) $k - (d_G(v_1) - d_K(v_1)) \geq 3$.
(ii) $d_G(v_m) \leq k$ and neighbors of v_m in K are exactly v_1 and v_{m-1}.
(iii) For $2 \leq i \leq m-1$, v_i has at most $k-1$ neighbors in $G[\{v_1, \ldots, v_{i-1}\}] \cup (G - K)$.
If $|f(v)| \geq k$ for every vertex v, then a DP-F-precoloring of $G - K$ can be extended to that of G.

Proof. Let R_0 be a DP-F-coloring on $G - K$. From Condition (i), $|f^*(v_1)| \geq |f(v)| - (d_G(v_1) - d_K(v_1)) \geq k - (d_G(v_1) - d_K(v_1)) \geq 3$. From Condition (ii), $|f^*(v_m)| \geq |f(v_m)| - (k-2) \geq 2$. We consider only the case $|f^*(v_m)| = 2$ since a strictly F^*-degenerate order of R_2 is also a strictly g-degenerate if $g_i(v) \geq f_i^*(v)$ for every vertex v and i such that $1 \leq i \leq s$. By renaming the colors, we assume that (v_m, j) and (v_i, j), where $i = 1$ and $m-1$, are adjacent for each j. Since $|f^*(v_1)| \geq 3$, we may assume further that $f_1^*(v_1) > f_1^*(v_m)$. By Lemma 7, it suffices to show that K has a DP-F^*-coloring. Consider two cases.

Case 1: $f_1^*(v_m) = 0$.
Choose $(v_1, 1)$ in a coloring. Observe that $|f^*(v_m)|$ remains the same. Apply greedy coloring to $v_2, \ldots, v_{m-1}$, respectively. At this stage $|f^*(v_m)| \geq 1$, thus we can use greedy coloring to v_m to complete a DP-F^*-coloring.

Case 2: $f_1^*(v_m) \geq 1$.
Recall that we consider only $f_i(v) \in \{0, 1, 2\}$ for each vertex v and every i such that $1 \leq i \leq s$. It follows that $f_1^*(v_1) = 2$ and $f_1^*(v_m) = 1$. Since $|f^*(v_m)| = 2$, we assume that $f_2^*(v_m) = 1$. Choose $(v_1, 1)$ in a coloring. We can apply greedy coloring to $v_2, \ldots, v_{m-1}$, respectively. By Condition (iii), $K - v_m$ has a DP-F^*-coloring, say R. By Condition (ii), $(v_m, 2)$ has exactly two neighbors in H restrict to K.

If $(v_{m-1}, 2)$ is not in R, then $(v_m, 2)$ has no neighbors in R. Thus we can add $(v_m, 2)$ to $R$to complete a DP-F^*-coloring. Assume otherwise that $(v_{m-1}, 2) \in R$. Let $(v_i, j_i) \in R$ for $2 \leq i \leq m-2$. By greedy coloring, we have a strictly F^*-degenerate order $S_1 = (v_1, 1)$, (v_2, j_2), $\ldots$, (v_{m-2}, j_{m-2}), $(v_{m-1}, 2)$.

We claim that the order S constructed from $(v_m, 1)$ followed by S is a strictly F^*-degnerate order. It is obvious that $(v_1, 1)$ has less than $f_1^*(v_1) = 2$ neighbors in the lower order. Consider (v_i, j_i) where $2 \leq i \leq m-2$. Since (v_i, j_i) is not adjacent to $(v_m, 1)$ by Condition (ii), (v_i, j_i) has less than $f_{j_i}^*(v_i)$ neighbors in the lower order of S. Since $(v_{m-1}, 2)$ is not adjacent to $(v_m, 1)$, the element $(v_{m-1}, 2)$ has less than $f_2^*(v_{m-1})$. It is obvious that the set of elements in the order of S is a representative set of K. Thus K has a DP-F^*-coloring. This completes the proof. $\square$

3. Proofs of Main Results

Proof of Theorem 1.6. The outline of the proof is similar to that in [25] with additional details on DP-F-coloring. We begin by adding new edges in a plane graph until we obtain a plane graph G such that every bounded face is a triangle. Let $|f(v)| \geq 5$ for each vertex v. Let a cycle $C = v_1 \ldots v_p$ be the boundary of the unbounded face. Using induction on $|V(G)|$, we prove the stronger result that a DP-F-coloring can be achieved even when v_1 and v_p have been precolored and $|f(v_i)| \geq 3$ for $2 \leq i \leq p-1$. Let $\{(v_1, a), (v_p, b)\}$ be a DP-F-precoloring. If $|V(G)| = 3$, the vertex v_2 can be greedily colored. Consider $|V(G)| \geq 4$ for the induction step.

Case 1: C has a chord v_iv_j with $1 \leq i \leq j-2 \leq p-1$.

Let C_1 be the cycle $v_1v_2 \ldots v_i\ v_jv_{j+1} \ldots v_p$ and let C_2 be the cycle $v_jv_iv_{i+1} \ldots v_{j-1}$. Let $G_1 = int(C_1)$ and let $G_2 = int(C_2)$. By induction hypothesis and Lemma 2.3, G_1 has a DP-F-coloring R_1 with a strictly F-degenerate order S_1 such that two lowest elements are $(v_i, 1)$ and $(v_j, 1)$. It follows from Lemma 2.3 that G_2 has a DP-F-coloring R_2 with a strictly F-degenerate order S_2 with two lowest elements $(v_i, 1)$ and $(v_j, 1)$. Let S_2' be an order obtained from S_2 by removing $(v_i, 1)$ and $(v_j, 1)$. It can be shown as in the proof of Lemma 2.4 that $R_1 \cup R_2$ is a representative set with a strictly F-degenerate order obtained from S_1 followed by S_2'.

Case 2: C has no chords.

Let $v_1, u_1, u_2, \ldots, u_m, v_3$ be the neighbors of v_2 in order. Let U denote $\{u_1, \ldots, u_m\}$ and G' denote $G - \{v_2\}$. Using a DP-F-coloring on v_1 and v_p, we have $|f^*(v_2)| \geq |f(v_2)| - 1 = 2$ for $p \geq 4$ and $|f^*(v_2)| \geq |f(v_2)| - 2 = 1$ for $p = 3$. By renaming the colors, we assume furthermore that (v_2, i) is adjacent to (u, i) for each $u \in U \cup \{v_3\}$ and $1 \leq i \leq s$. Let $f_1^*(v_2) = \max\{f_1^*(v_2), \ldots, f_s^*(v_2)\}$.

Case 2.1: $p = 3$ or $f_1^*(v_2) \geq 2$.

We choose $(v_2, 1)$ in a DP-F-coloring. Let f' be obtained from f by letting $f_1'(u) = 0$ for each $u \in U$. Since $f_1(u) \leq 2$, we have $|f'(u)| \geq 3$ for each $u \in U$. By induction hypothesis and Lemma 2.3, G' has a DP-f'-coloring R' with a strictly f'-degenerate order S' such that (v_1, a) and $(v_{p=3}, b)$ are the first two elements.

Suppose $p = 3$. Let S be obtained from S' by inserting $(v_2, 1)$ as the third element. Since $f_1^*(v_2) \geq 1$ when we have a precoloring $\{(v_1, a), (v_p, b)\}$, the element $(v_2, 1)$ can be chosen by a greedy coloring.

Note that the only neighbors of v_2 are v_1, v_3, and vertices in U. If $u \in U$, then $(u, 1)$ is not in R' since $f_1'(u) = 0$. Thus (v, c) where $v \notin U \cup \{v_1, v_3\}$ has less than $f_c'(v) = f_c(v)$ neighbors in the lower order of S. Thus S is a strictly F-degenerate order of $R' \cup \{(v_2, 1)\}$. It is obvious that $R' \cup \{(v_2, 1)\}$ is a representative set and thus a DP-F-coloring.

Suppose $p = 4$ and $f_1^*(v_2) \geq 2$. After a coloring on G', we have $f_1^*(v_2) \geq 2 - 1$ since the only possible neighbor of $(v_2, 1)$ other than (v_1, a) in the coloring R_1 is $(v_3, 1)$. Thus a greedy coloring can be applied to v_2.

Case 2.2: $p \geq 4$ and $f_1^*(v_2) = 1$.

Since $|f^*(v_2)| \geq 2$ and by symmetry, we assume $f_2^*(v_2) = 1$. Define $g_i(v_2) = f_i^*(v_2)$. Let f' be obtained from f by letting $f_1'(u) = \max\{0, f_1(u) - 1\}$, $f_2'(u) = \max\{0, f_2(u) - 1\}$. Observe that $|f'(u)| \geq 3$ for each $u \in U$. By induction hypothesis, G' has a DP-f'-coloring R' (thus a DP-F-coloring). It follows from Lemma 2.3 that R' has a strictly f'-degenerate order S' with (v_1, a) and (v_p, b) are the two lowest ordered elements.

Let $t = 1$ if $(v_3, 1)$ is not in R', otherwise let $t = 2$. It is obvious that $R = R' \cup \{(v_2, t)\}$ is a representative set. Let S be an order obtained from inserting (v_2, t) as the third element into S'. We claim that S is a strictly F-degenerate order of R.

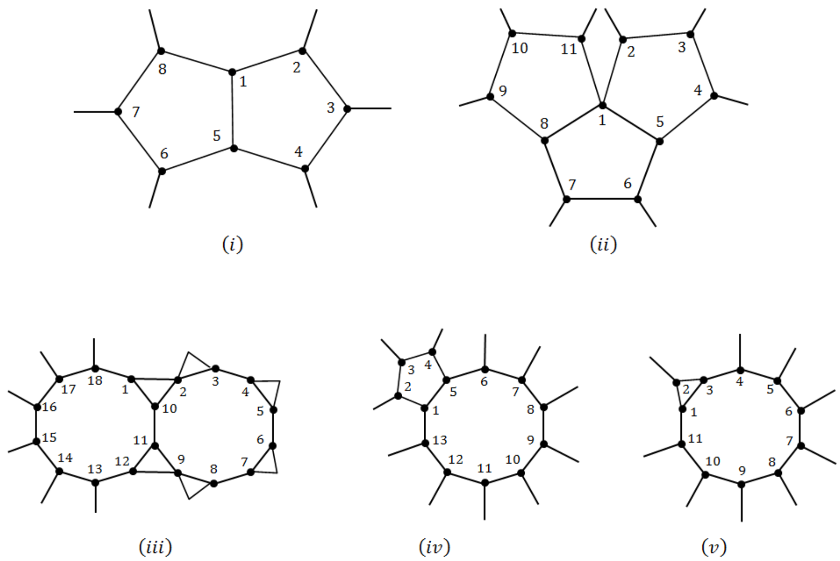

Figure 1: Forbidden configurations in Theorem 1.7

Consider (v_2, t). Since $p \geq 4$, (v_2, t) is not adjacent to (v_p, b). If $t = a$, then $f_t(v_2) = g_t(v_2)+1 = 2$, otherwise, $f_t(v_2) = g_t(v_2) = 1$. In both cases, (v_2, t) has less than $f_t(v_2)$ neighbors in the lower order of S.

Consider (v, c) in R where $v \notin \{v_1, v_2, v_p\}$. We have (v, c) has less than $f'_c(v)$ neighbors other than (v_2, t) in the lower order of S by the construction of S. If (v, c) is adjacent to (v_2, t), then $v \in U$ and $c = t$. Consequently, $f_c(v) = f'_c(v) + 1$. If (v, c) is not adjacent to (v_2, t), then $f_c(v) \geq f'_c(v)$. In both cases, (v, c) has less than $f_c(v)$ neighbors in the lower order of S. Thus S is a strictly F-degenerate of R. This completes the proof.

Modification of the Proof of Theorem 1.7.

For the proof of Theorem 1.7, each configurations that are forbidden to be contained in a minimal counterexample are obtained from the fact that (i) $G \in \mathcal{A}$, (ii) G has no separating 3-cycles (Lemma 2.4) and the following lemma.

Lemma 3.1. *Let $|f(v)| \geq 4$ for each vertex v. Let C be a cycle $x_1 \dots x_m$ with $V(C) \cap V(C_0) = \emptyset$ where C_0 is a precolored 3-cycle. Let $C(l_1, \dots, l_k)$ be obtained from a cycle C with $k-1$ internal chords sharing a common endpoint x_1. Suppose $K = G[C]$ contains $C(l_1, \dots, l_k)$ where x_2 or x_m is not the endpoint of any chord in C. If $d_G(x_1) \leq k+2$ and $d_K(x_1) = k+1$, then there exists $i \in \{2, 3, \dots, m\}$ such that $d(x_i) \geq 5$.*

One can see that Lemma 3.1 is immediate from Lemma 2.5 by assuming an order $x_1, \dots, x_m$ with x_m is not endpoint of any chord. Thus all forbidden configurations required as in the proof of Theorem 1.4 in [24] are obtained. Using Lemma 2.2 about vertex degrees and the discharging method as in [24], one can complete the proof.

Modification of the Proof of Theorem 1.8. All five forbidden configurations of minimal counterexample to Theorem 1.8 (as in Lemma 2.3 of [22]) are in (See Fig. 1). Consider a subgraph K induced by the labeled vertices and order the vertices according to labels. Note that all labeled vertices are different to avoid creating cycles of forbidden lengths. It can be proved by Lemma 2.5 that DP-F-precoloring of $G-K$ can be extended to that of G. Thus a minimal counterexample cannot contains configurations in Fig. 1. Using Lemma 2.1 about vertex degrees and the discharging method as in [22], one can complete the proof.

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