# On mixed graphs whose Hermitian spectral radii are at most 2 * 

Bo-Jun Yuan ${ }^{a}$, Yi Wang ${ }^{a}{ }^{\dagger}$, Shi-Cai Gong ${ }^{b}$, Yun Qiao ${ }^{a}$<br>a. School of Mathematical Sciences, Anhui University, Hefei, 230601, P. R. China<br>b. School of Science, Zhejiang University of Science and Technology, Hangzhou, 310023, P. R. China


#### Abstract

A mixed graph is a graph with undirected and directed edges. Guo and Mohar in 2017 determined all mixed graphs whose Hermitian spectral radii are less than 2. In this paper, we give a sufficient condition which can make Hermitian spectral radius of a connected mixed graph strictly decreasing when an edge or a vertex is deleted, and characterize all mixed graphs with Hermitian spectral radii at most 2 and with no cycle of length 4 in their underlying graphs.


Keywords: $C_{4}$-free graph, Mixed graph, Hermitian spectral radius
AMS Mathematics Subject Classification: 05C50.

## 1 Introduction

Characterizing the structure of a graph by the eigenvalue spectrum of an associated matrix with the graph is a basic problem in spectral graph theory. Since restrictions on the spectral radii of graphs with respect to their adjacency matrices often force those to have very special structures, it is always a hot topic to characterize the graphs whose spectral radii are bounded above. Smith [12] determined all graphs whose spectral radii are at most 2. This work stimulated the interest of the researchers. There are a lot of results in the literature concerning the topic. Brouwer and Neumaier [1] characterized the graphs whose spectral radii are contained in the interval $(2, \sqrt{2+\sqrt{5}}$ ] and later, Woo and Neumaier [14] described the structure of graphs whose spectral radii are bounded above by $\frac{3}{2} \sqrt{2}$.

Studying the same problem on digraphs has received less attention. Xu and Gong [16] investigated digraphs whose spectral radii with respect to their skew adjacency matrices do

[^0]not exceed 2. Guo and Mohar [8] determined all mixed graphs whose spectral radii with respect to their Hermitian adjacency matrices are less than 2. Their work shows that 2 is the smallest limit point of the Hermitian spectral radii of connected mixed graphs. In the present paper, we characterize all $C_{4}$-free mixed graphs whose Hermitian spectral radii do not exceed 2.

A graph containing undirected edges and directed edges is called a mixed graph. Clearly, mixed graphs are natural generalizations of both simple graphs and digraphs. Indeed, a mixed graph $D$ can be obtained from a simple graph $G$ by orienting a subset of its edge set. We call $G$ as the underlying graph of $D$ and denote it by $G(D)$. Formally, a mixed graph $D$ is comprised of the vertex set $V(D)$, which is the same as the vertex set $V(G)$, and the edge set $E(D)$, which consists of two parts: undirected edge set $E_{0}(D)$ and directed edge set $E_{1}(D)$. To distinguish undirected and directed edges, we denote an undirected edge between the vertices $u$ and $v$ by $\{u, v\}$ and a directed edge from $u$ to $v$ by $(u, v)$. If there is no danger of confusion, we write $u v$ instead of $\{u, v\}$ or $(u, v)$.

Let $D$ be a mixed graph of order $n$. For a vertex $v$ of $D$, we define the set of neighbors of $v$ as $N(v)=\{u \in V(D) \mid u v \in E(G(D))\}$. The degree of $v$ is defined as $d(v)=|N(v)|$. A mixed graph is said to be a mixed tree (respectively, unicyclic mixed graph) if its underlying graph is a tree (respectively, unicyclic graph). A mixed subgraph $H$ of $D$ is called elementary if each connected component of $H$ is either a mixed edge or a mixed cycle. A mixed graph $D$ is called $C_{4}$-free if $G(D)$ contains no cycle of length 4 as a subgraph.

The Hermitian adjacency matrix of $D$ is defined as $H(D)=\left[h_{u v}\right]$ with

$$
h_{u v}= \begin{cases}1 & \text { if }\{u, v\} \in E_{0}(D) \\ \mathrm{i} & \text { if }(u, v) \in E_{1}(D) \\ -\mathrm{i} & \text { if }(v, u) \in E_{1}(D) \\ 0 & \text { otherwise }\end{cases}
$$

where i is the unit imaginary number. Since $H(D)$ is Hermitian, the eigenvalues of $H(D)$ are real and can be arranged as $\lambda_{1}(D) \geqslant \cdots \geqslant \lambda_{n}(D)$. The eigenvalues and spectrum of $H(D)$ are called the Hermitian eigenvalues and Hermitian spectrum of $D$, respectively. The Hermitian spectral radius of $D$ is defined as $\rho(D)=\max \left\{\left|\lambda_{1}(D)\right|, \ldots,\left|\lambda_{n}(D)\right|\right\}$. The characteristic polynomial of $H(D)$ is denoted by $\Phi(D, \lambda)$ and is called the Hermitian characteristic polynomial of $D$. These terminologies were introduced by Liu and Li [10] in the study of graph energy and independently by Guo and Mohar 7. In the paper [10, authors investigate the properties of characteristic polynomials of mixed graphs and cospectral problems among mixed graphs. The latter paper contains an introduction to the properties of Hermitian spectrum, and discusses similarities and differences from the case of undirected
graph. Recently, the Hermitian spectrum has been the subject of several publications. For more details about the Hermitian spectrum, one can see the literature [2, 3, 4, 6, 9, 11, 13] and references therein.

In this paper, we deal with Hermitian spectral radii of mixed graphs. A mixed graph $D$ is called $C_{4}$-free if $G(D)$ contains no $C_{4}$ as a subgraph. We characterize all $C_{4}$-free mixed graphs whose Hermitian spectral radii are at most 2. The rest of the paper is organized as follows: In Section 2, we will introduce some notations and preliminary results on characteristic polynomials of mixed graphs. In Section 3, we will give a sufficient condition which can make Hermitian spectral radius of a connected mixed graph strictly decreasing when an edge or a vertex is deleted. In Section 4, we will determine all $C_{4}$-free mixed graphs whose Hermitian spectral radii do not exceed 2.

## 2 Notations and Preliminaries

Let $H(D)=\left[h_{i j}\right]$ be the Hermitian adjacency matrix of the mixed graph $D$. The value of a mixed walk $W: v_{1}, v_{2}, \ldots, v_{\ell}$ is defined to be $h_{12} h_{23} \cdots h_{(\ell-1) \ell}$ and is denoted by $h(W)$. For a closed mixed walk $W$, we first fix an arbitrary direction for $W$ before calculating its value. One can verify that if the value of a closed mixed walk is $\alpha$ in a direction, then for the reversed direction its value is $\bar{\alpha}$, the conjugate number of $\alpha$. We say a mixed cycle $C$ to be real (respectively, imaginary) if $h(C)= \pm 1$ (respectively, $\pm \mathrm{i}$ ). It is clear that a mixed cycle is real (respectively, imaginary) if and only if the number of its directed edges is even (respectively, odd). Indeed, the value of a real cycle $C$ is independent of its chosen orientation. Furthermore, a mixed cycle $C$ is called positive (respectively, negative) if $h(C)=1$ (respectively, -1 ). Clearly, a mixed cycle is positive (respectively, negative) if and only if the difference between the number of its forward and backward directed edges with respect to an arbitrary direction is congruence to 0 (respectively, 2) modulo 4.

We here recall the following theorem which can be considered as an analogue of Sachs' Coefficient Theorem [5, Page 32].

Theorem 2.1 [7, 10] Let $D$ be a mixed graph of order $n$ with the Hermitian characteristic polynomial $\Phi(D, \lambda)=\sum_{i=0}^{n} c_{i} \lambda^{n-i}$. Denote by $\mathcal{E}_{i}$ the set of the elementary subgraphs of $D$ of order $i$ whose all mixed cycles are real. Then for $i=1, \ldots, n$,

$$
c_{i}=\sum_{H \in \mathcal{E}_{i}}(-1)^{t(H)+s(H)} 2^{r(H)},
$$

where $t(H), s(H)$, and $r(H)$ are respectively the number of connected components, the number of negative mixed cycles, and the number of mixed cycles in $H$.

The following two corollaries can be considered as immediate consequences of Theorem 2.1 .

Corollary 2.2 [10] If a mixed graph $D$ contains no real mixed odd cycles, then the Hermitian spectrum of $D$ is symmetric about 0 .

Corollary 2.3 [10] If all the mixed cycles in a mixed graph $D$ are positive, then the Hermitian spectra of $D$ and $G(D)$ are the same. In particular, $F$ and $G(F)$ have the same Hermitian spectrum for any mixed forest $F$.

In the following, we apply Theorem 2.1 to determine all mixed cycles with Hermitian spectral radii 2 .

Corollary 2.4 Let $D$ be a mixed cycle. Then $\rho(D)=2$ if and only if either $D$ is a positive mixed cycle or $D$ is a negative mixed odd cycle.

Proof. Let $n=|V(D)|$ and $C=G(D)$. We know that $\rho(C)=2$. In addition, $\Phi(C,-2)=0$ if and only if $n$ is even. So, using the Perron-Frobenius theorem, $\rho(D) \leqslant \rho(C)=2$. Define

$$
t=\left\{\begin{array}{ll}
1 & \text { if } D \text { is real; } \\
0 & \text { otherwise }
\end{array} \quad \text { and } \quad s= \begin{cases}1 & \text { if } D \text { is negative; } \\
0 & \text { otherwise }\end{cases}\right.
$$

By Theorem 2.1, $\Phi(D, \lambda)-\Phi(C, \lambda)=(-1)^{s+1} 2 t+2$. This means that $\Phi(D, 2)=0$ if and only if $t=1$ and $s=0$. If $n$ is even, then the equality $\Phi(D, \lambda)-\Phi(C, \lambda)=(-1)^{s+1} 2 t+2$ shows that $\Phi(D,-2)=0$ if and only if $t=1$ and $s=0$. If $n$ is odd, then it follows from Theorem[2.1]that $\Phi(D, \lambda)+\Phi(C,-\lambda)=(-1)^{s+1} 2 t-2$. Hence, if $n$ is odd, then $\Phi(D,-2)=0$ if and only if $t=1$ and $s=1$.

We prove the following theorem as a consequence of Theorem 2.1.
Theorem 2.5 Let $D$ be a mixed graph and $e=u v \in E(D)$. Let $\mathcal{C}_{e}$ be the set of all real mixed cycles in $D$ containing $e$. Then

$$
\Phi(D, \lambda)=\Phi(D-e, \lambda)-\Phi(D-u-v, \lambda)-2 \sum_{C \in \mathcal{C}_{e}} h(C) \Phi(D-C, \lambda) .
$$

Proof. Let $n=|V(D)|$ and $\Phi(D, \lambda)=\sum_{\ell=0}^{n} c_{\ell} \lambda^{n-\ell}$ and fix $i \in\{1, \ldots, n\}$. Let $\mathcal{E}_{i}$ be the set of all elementary subgraphs of $D$ of order $i$ whose all mixed cycles are real. For any given edge $e, \mathcal{E}_{i}$ can be divided into the following subsets:

$$
\begin{aligned}
& \mathcal{E}_{i}^{1}=\left\{H \in \mathcal{E}_{i} \mid e \notin E(H)\right\} ; \\
& \mathcal{E}_{i}^{2}=\left\{H \in \mathcal{E}_{i} \mid e \text { is a single edge of } H\right\} ; \\
& \mathcal{E}_{i}^{3}=\left\{H \in \mathcal{E}_{i} \mid e \text { is contained in a positive mixed cycle } P_{H} \text { of } H\right\} ; \\
& \mathcal{E}_{i}^{4}=\left\{H \in \mathcal{E}_{i} \mid e \text { is contained in a negative mixed cycle } N_{H} \text { of } H\right\} .
\end{aligned}
$$

By Theorem 2.1, we have

$$
\begin{aligned}
& \sum_{H \in \mathcal{E}_{i}^{1}}(-1)^{t(H)+s(H)} 2^{r(H)}=c_{i}(D-e), \\
& \sum_{H \in \mathcal{E}_{i}^{2}}(-1)^{t(H)+s(H)} 2^{r(H)}=-\sum_{H \in \mathcal{E}_{i}^{2}}(-1)^{t(H-u-v)+s(H-u-v)} 2^{r(H-u-v)} \\
&=-c_{i-2}(D-u-v), \\
& \sum_{H \in \mathcal{E}_{i}^{3}}(-1)^{t(H)+s(H)} 2^{r(H)}=-2 \sum_{H \in \mathcal{E}_{i}^{3}}(-1)^{t\left(H-P_{H}\right)+s\left(H-P_{H}\right)} 2^{r\left(H-P_{H}\right)} \\
&=-2 \sum_{C \in \mathcal{C}_{e}^{+}} c_{i-|V(C)|}(D-C),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{H \in \mathcal{E}_{i}^{4}}(-1)^{t(H)+s(H)} 2^{r(H)} & =2 \sum_{H \in \mathcal{E}_{i}^{4}}(-1)^{t\left(H-N_{H}\right)+s\left(H-N_{H}\right)} 2^{r\left(H-N_{H}\right)} \\
& =2 \sum_{C \in \mathcal{C}_{e}^{-}} c_{i-|V(C)|}(D-C),
\end{aligned}
$$

where $\mathcal{C}_{e}^{+}$(respectively, $\mathcal{C}_{e}^{-}$) is the set of all positive (respectively, negative) mixed cycles in $D$ containing $e$. Now, it follows from Theorem 2.1) and $\mathcal{E}_{i}=\mathcal{E}_{i}^{1} \cup \cdots \cup \mathcal{E}_{i}^{4}$ that

$$
c_{i}(D)=c_{i}(D-e)-c_{i-2}(D-u-v)-2 \sum_{C \in \mathcal{C}_{e}} h(C) c_{i-|V(C)|}(D-C),
$$

which in turn implies that

$$
\begin{aligned}
\sum_{i=0}^{n} c_{i}(D) \lambda^{n-i} & =\sum_{i=0}^{n} c_{i}(D-e) \lambda^{n-i}-\sum_{i=2}^{n} c_{i-2}(D-u-v) \lambda^{n-i} \\
& -2 \sum_{C \in \mathcal{C}_{e}} h(C) \sum_{i=|V(C)|}^{n} c_{i-|V(C)|}(D-C) \lambda^{n-i}
\end{aligned}
$$

This means that $\Phi(D, \lambda)=\Phi(D-e, \lambda)-\Phi(D-u-v, \lambda)-2 \sum_{C \in \mathcal{C}_{e}} h(C) \Phi(D-C, \lambda)$.
The next result can be proved by applying Theorem 2.5 for the mixed edges incident to a vertex repeatedly one by one.

Corollary 2.6 Let $D$ be a mixed graph and $v \in V(D)$. Let $\mathcal{C}_{v}$ be the set of all real mixed cycles in $D$ containing $v$. Then

$$
\Phi(D, \lambda)=\lambda \Phi(D-v, \lambda)-\sum_{u v \in E(D)} \Phi(D-u-v, \lambda)-2 \sum_{C \in \mathcal{C}_{v}} h(C) \Phi(D-C, \lambda) .
$$

## 3 The Hermitian spectral radii of mixed graphs

In this section, we present some results on Hermitian spectral radii of mixed graphs for later use. From the Perron-Frobenius theorem, we know that the spectral radius of a connected undirected graph strictly decreases by deleting a vertex or an edge from the graph. However, the fact does not hold for Hermitian spectral radius of a connected mixed graph. We give a sufficient condition in the following theorem generalizing Theorem 3.2 in [15].

Theorem 3.1 Let $D$ be a connected mixed graph all whose real mixed cycles are positive mixed even cycles. Then $\rho(D)>\rho(D-u)$ and $\rho(D)>\rho(D-e)$ for every vertex $u \in V(D)$ and edge $e \in E(D)$.

Proof. We prove the assertion by induction on $m=|E(D)|$. The assertion clearly holds for $m=1$. Suppose that the assertion is valid for all connected mixed graphs of size less than $m$. Consider a connected mixed graph $D$ of size $m$ and assume that $e=u v$ is an arbitrary edge of $D$. By Corollary 2.2, the Hermitian spectrum of $D-e$ is symmetric about 0 , so $\rho=\rho(D-e)$ can be considered as the largest Hermitian eigenvalue of $D-e$. We first establish that $\Phi(D-u-v, \rho)>0$. For this, we consider the following two cases.
Case 1. The edge $e$ is a cut edge.
Denote the connected components of $D-e$ by $D_{1}$ and $D_{2}$. Assume without loss of generality that $u \in V\left(D_{1}\right)$ and $v \in V\left(D_{2}\right)$. By the induction hypothesis, $\rho\left(D_{1}\right)>\rho\left(D_{1}-u\right)$ and $\rho\left(D_{2}\right)>\rho\left(D_{2}-v\right)$. Therefore,

$$
\rho(D-u-v)=\max \left\{\rho\left(D_{1}-u\right), \rho\left(D_{2}-v\right)\right\}<\max \left\{\rho\left(D_{1}\right), \rho\left(D_{2}\right)\right\}=\rho(D-e)=\rho .
$$

This means that $\Phi(D-u-v, \rho)>0$.
Case 2. The edge $e$ is not a cut edge.
By the induction hypothesis and the interlacing theorem, $\rho=\rho(D-e)>\rho(D-u) \geqslant$ $\rho(D-u-v)$ which means that $\Phi(D-u-v, \rho)>0$.

Let $\mathcal{C}_{e}$ be the set of all real mixed cycles in $D$ containing $e$. For any $C \in \mathcal{C}_{e}, D-C$ is an induced subgraph of $D-u-v$ and so by the interlacing theorem, $\rho(D-C) \leqslant \rho(D-u-v)<\rho$
which yields that $\Phi(D-C, \rho)>0$. Since all real mixed cycles of $D$ are positive, $h(C)=1$ for each $C \in \mathcal{C}_{e}$. By Theorem 2.5,

$$
\Phi(D, \rho)=\Phi(D-e, \rho)-\Phi(D-u-v, \rho)-2 \sum_{C \in \mathcal{C}_{e}} h(C) \Phi(D-C, \rho)<0,
$$

proving that $\rho(D)>\rho=\rho(D-e)$. Applying the interlacing theorem, $\rho(D)>\rho(D-e) \geqslant$ $\rho(D-u)$.

Remark. The conditions in Theorem 3.1 can not be omitted. Consider two mixed graphs $D_{1}$ and $D_{2}$ whose labeling are shown in Fig. 1. Notice that $D_{1}$ contains a real mixed odd cycle $v_{2} v_{3} v_{4} v_{2}$ and a negative mixed even cycle $v_{1} v_{2} v_{3} v_{4} v_{1}$, and $D_{2}$ contains a negative mixed even cycle $u_{1} u_{2} u_{3} u_{4} u_{1}$, no real mixed odd cycle. By an easy calculation, it follows that $\rho\left(D_{1}\right)=\rho\left(D_{1}-v_{3}\right)=2, \rho\left(D_{1}-v_{3} v_{4}\right) \sim 2.170>2 ; \rho\left(D_{2}\right)=\rho\left(D_{1}-u_{2}\right)=\sqrt{3}$, $\rho\left(D_{2}-u_{1} u_{3}\right)=2>\sqrt{3}$.


Fig. 1: Mixed graphs $D_{1}$ and $D_{2}$.

Lemma 3.2 [8] Suppose that a mixed graph $M$ is obtained from a connected mixed graph $N$ by attaching a new vertex to a vertex $u$ in $N$. If $x$ is an eigenvector of $N$ whose eigenvalue $\lambda$ satisfies $|\lambda|=\rho(N)$ and $x_{u} \neq 0$, then $\rho(M)>\rho(N)$.

Lemma 3.3 Let $D$ be a connected unicyclic mixed graph containing a mixed cycle $C$. If $\rho(D)=\rho(C)=2$, then $D=C$.

Proof. Let $C=C_{n}$. Applying Lemma 3.2, it is sufficient to show that any eigenvector $x$ corresponding to an eigenvalue $\lambda$ of $C$ with $|\lambda|=\rho(C)=2$ has no zero components. If $x_{u}=0$ for some $u \in V(C)$, then the vector obtained from $x$ by deleting the $u$ th component of $x$ is an eigenvector of $C-u$ corresponding to $\lambda$. But, this is impossible, since the

Hermitian spectral radius of $C-u$ is less than 2 in view of Corollary 2.3, Towards a contradiction, suppose that $D \neq C$. Since $D$ is a connected unicyclic mixed graph, there exists a vertex $v \in V(D-C)$ adjacent to exactly one vertex on $C$. If $H$ is the induced subgraph of $D$ on $V(C) \cup\{v\}$, then Lemma 3.2 implies that $\rho(H)>\rho(C)=2$, which contradicts $\rho(H) \leqslant \rho(D)=2$.

## $4 \quad C_{4}$-free mixed graphs whose spectral radii do not exceed 2

In this section, we determine all $C_{4}$-free mixed graphs whose spectral radii do not exceed 2 . We first introduce some families of graphs to use later. Let $P_{n}$ and $C_{n}$ denote respectively the path and the cycle on $n$ vertices. A star-like tree $S\left(n_{1}, \ldots, n_{k}\right)$ is an undirected tree with a vertex $v$ such that $S\left(n_{1}, \ldots, n_{k}\right)-v=P_{n_{1}} \cup \cdots \cup P_{n_{k}}$. Denote by $Y(r, s, t)$ the tree consisting of the path $P_{r+s+t-1}$ whose vertices are ordered as $v_{1}, \ldots, v_{r+s+t-1}$ with two extra pendant edges affixed at $v_{r}$ and $v_{r+s}$. A dumbbell graph, denoted by $D(r, s, t)$, is an undirected graph consisting of two vertex disjoint cycles $C_{r}, C_{s}$, and a path $P_{t}$ joining the cycles having only its endpoints in common with them. A theta graph, denoted by $\theta(r, s, t)$, is an undirected graph consisting of three internally disjoint paths $P_{r}, P_{s}, P_{t}$ with the same endpoints.

Definition 4.1 Consider the cycle $C_{n}$ as $v_{1} v_{2} \cdots v_{n} v_{1}$. Denote by $C_{n}\left(k_{1}, \ldots, k_{n}\right)$ the undirected graph obtained from $C_{n}$ by identifying $v_{i}$ with a pendent vertex of $P_{k_{i}+1}$ for $i=$ $1, \ldots, n$. We write $C_{n}\left(k_{1}, \ldots, k_{t}\right)$ instead of $C_{n}\left(k_{1}, \ldots, k_{t}, 0, \ldots, 0\right)$ for simplicity whenever $k_{t+1}=\cdots=k_{n}=0$.

The following theorem characterizes the undirected graphs whose spectral radii do not exceed 2.

Theorem 4.2 [12] All undirected graphs whose spectral radii do not exceed 2 are isomorphic to one of the following undirected graphs or their subgraphs.
(i) $C_{n}$ for any integer $n \geqslant 3$;
(ii) $Y(2, n-5,2)$ for any integer $n \geqslant 5$;
(iii) $S(1,2,5), S(1,3,3), S(2,2,2)$.

Definition 4.3 Let $G$ be an undirected graph. Denote by $G^{+}$(respectively, $G^{-}, G^{*}$ ) the family of mixed graphs with $G$ as their underlying graph, whose all mixed cycles are positive
(respectively, negative, imaginary). Denote by $G^{\hat{+}}$ (respectively, $G^{-}, G^{\widehat{*}}$ ) the family of mixed graphs contained in $G^{+}$(respectively, $G^{-}, G^{*}$ ) and their induced mixed subgraphs.

Remark. Let $G$ be an undirected graph. Using Theorem 2.1, it is easy to see that all mixed graphs in $G^{+}\left(G^{-}, G^{*}\right)$ have the same Hermitian spectrum.

The following consequence is obtained from Corollary 2.4, Lemma 3.3 and the interlacing theorem.

Corollary 4.4 Any graph in one of the following families has the Hermitian spectral radius greater than 2 .
(i) $C_{n}(1)^{+}$for any integer $n \geqslant 3$;
(ii) $C_{n}(1)^{-}$for any odd number $n \geqslant 3$.

The girth of a mixed graph $D$ is the minimum length of cycles in $G(D)$. The following theorem generalizes the analogue result for oriented graphs appeared in [16].

Lemma 4.5 Let $D$ be a connected $C_{4}$-free mixed graph with $\rho(D) \leqslant 2$. If $D$ is neither a mixed tree nor a mixed cycle, then $D$ is isomorphic to a mixed graph contained in one of the following families.
(i) $C_{3}(2)^{\widehat{*}}, C_{6}(1,0,1,0,1)^{\hat{-}}, C_{6}(2,0,0,2)^{\widehat{-}}, C_{8}(1,0,0,0,1)^{\hat{=}}$;
(ii) The family of mixed graphs with underlying graph $\theta(3,5,5)$ containing two negative mixed cycles $C_{6}$, and their induced mixed subgraphs.

Proof. Let $m$ be the girth of $D$ and let $C$ be a mixed cycle of length $m$ in $D$ as $u_{1} u_{2} \ldots u_{m} u_{1}$. We identify $G(C)$ with $C_{m}$. If $m \geqslant 5$, then $G(D)$ contains an induced subgraph isomorphic to $C_{m}(1)$, since $D$ is connected and is not a mixed cycle. If $m \geqslant 9$, then $C_{m}(1)-u_{5}$ contains $S(1,3,4)$ as an induced subgraph. This is a contradiction, since $\rho(S(1,3,4))>2$ by Theorem4.2, and $\rho(S(1,3,4)) \leqslant \rho(D) \leqslant 2$ by Corollary 2.3 and the interlacing theorem. If $m \in\{5,7\}$, then it follows from Corollary 4.4 that $C$ is imaginary. By the Remark in Section 4 and an easy calculation, it is easy to show that the Hermitian spectral radius of any mixed graph in $C_{5}(1)^{*} \cup C_{7}(1)^{*}$ is greater than 2 , a contradiction. Therefore, $m \in\{3,6,8\}$. Case 1. $m=3$.

Since $D$ is a connected $C_{4}$-free mixed graph which is not a mixed cycle, any triangle in $G(D)$ is contained in an induced subgraph isomorphic to $C_{3}(1)$. It follows from Corollary 4.4 that $C$ and all other triangles in $D$ are imaginary. Towards a contradiction, suppose
that two vertices on $C$ have neighbors in $V(D-C)$. Since $D$ is $C_{4}$-free, $D$ contains an element of $C_{3}(1,1)^{*}$ as an induced subgraph. But, by the Remark in Section 4 and an easy calculation, it is easy to show that the Hermitian spectral radius of any mixed graph in $C_{3}(1,1)^{*}$ is greater than 2 , a contradiction. Without loss of generality, assume that $u_{1}$ is the unique vertex on $C$ having a neighbor outside $C$, say $w_{1}$. Noting that all triangles in $D$ are imaginary and using $\rho\left(K_{1,4}\right)=2$, we conclude from Theorem 3.1 that $d\left(u_{1}\right)=3$. We may assume that $w_{1}$ has a neighbor other than $u_{1}$, since otherwise, $D \in C_{3}(2)^{\widehat{*}}$, we are done. Since $D$ is $C_{4}$-free, $N\left(w_{1}\right) \cap\left(N\left(u_{2}\right) \cup N\left(u_{3}\right)\right)=\varnothing$. Again, By the Remark in Section 4 and a routine calculation, it follows that the Hermitian spectral radius of any mixed graph in $C_{3}(2)^{*}$ is equal to 2 . Using this and noting that $D$ is $C_{4}$-free, we conclude from Theorem 3.1 that $d\left(w_{1}\right)=2$. Let $w_{2}$ be the neighbor of $w_{1}$ other than $u_{1}$. Now, if $w_{2}$ is adjacent to a vertex other than $w_{1}$, then $D$ contains a member $H \in C_{3}(3)^{*}$ as an induced subgraph. This contradicts Theorem 3.1, since the Hermitian spectral radius of any mixed graph in $C_{3}(2)^{*}$ is equal to 2 . Therefore, $D \in C_{3}(2)^{\widehat{*}}$.
Case 2. $m=6$.
As we mentioned in the first paragraph of the proof, $G(D)$ contains $C_{m}(1)$ as an induced subgraph. The Remark and a routine calculation show that the Hermitian spectral radius of any mixed graph in $C_{6}(1)^{*}$ is greater than 2 . This along with Corollary 4.4 forces that $C$ is negative. By Theorem 4.2, $\rho(Y(3,0,3))>2$ and so it follows from the interlacing theorem that any vertex on $C$ has at most 3 neighbors in $D$.
Case 2.1. There is no mixed cycle $C^{\prime} \neq C$ in $D$ with $E(C) \cap E\left(C^{\prime}\right) \neq \varnothing$.
We first claim that $D \in C_{6}\left(k_{1}, \ldots, k_{6}\right)^{-}$for some $k_{1}, \ldots, k_{6}$. Towards a contradiction and without loss of generality, suppose that the subgraph $T$ attached to $u_{1}$ is not a path. So, the induced mixed subgraph of $D$ on $V\left(C-u_{3}\right) \cup V(T)$ contains an induced mixed subgraph with the underlying graph $Y(4, s, 2)$ for some $s \geqslant 1$. This contradicts Theorem 4.2, proving the claim. Now, since $C_{6}(3)-u_{4}=S(3,2,2), C_{6}(1,1)-u_{4}=Y(3,1,2)$, and $C_{6}(2,0,1)-u_{5}=Y(3,2,2)$, it follows from Theorem 4.2 and the interlacing theorem that $D$ contains no induced subgraphs in $C_{6}(3)^{-} \cup C_{6}(1,1)^{-} \cup C_{6}(2,0,1)^{-}$. Therefore, $D \in C_{6}(1,0,1,0,1)^{-} \cup C_{6}(2,0,0,2)^{-}$. It is routine to verify that any mixed graph in $C_{6}(1,0,1,0,1)^{-} \cup C_{6}(2,0,0,2)^{-}$has the Hermitian spectral radius 2 , we are done.
Case 2.2. There is a mixed cycle $C^{\prime} \neq C$ in $D$ with $E(C) \cap E\left(C^{\prime}\right) \neq \varnothing$.
As we mentioned in Case 2.1, $D$ has no induced subgraphs in $C_{6}(3)^{-}$. Further, an easy calculation shows that the Hermitian spectral radius of any mixed graph in $C_{7}(1)^{*}$ is greater than 2. Using these facts and Corollary 4.4, one deduces that $D$ contains no mixed graph in $C_{6}(3)^{-} \cup C_{7}(1)^{+} \cup C_{7}(1)^{-} \cup C_{7}(1)^{*}$ as an induced mixed subgraph. This implies that the
length of $C^{\prime}$ must be 6 . Moreover, if $\left|E(C) \cap E\left(C^{\prime}\right)\right|=1$, then $D$ contains an induced mixed subgraph in $C_{6}(3)^{-}$, a contradiction. Consequently, the underlying graph of the induced mixed subgraph $H$ of $D$ on $V(C) \cup V\left(C^{\prime}\right)$ is either $\theta(5,5,3)$ or $\theta(4,4,4)$. On the other hand, since $C$ and $C^{\prime}$ are negative, then the third mixed cycle in the induced subgraph of $D$ on $V(C) \cup V\left(C^{\prime}\right)$ must be positive. Using Corollary 4.4, any mixed graph in $C_{6}(1)^{+}$has the Hermitian spectral radius greater than 2. This yields that $G(H)=\theta(3,5,5)$. Without loss of generality, assume that $V(C) \cap V\left(C^{\prime}\right)=\left\{u_{1}, u_{2}, u_{3}\right\}$. As we mentioned in Case 2.1, $D$ has no induced subgraphs in $C_{6}(1,1)^{-}$. Since the girth of $D$ is 6 , one concludes that the degree of $u_{2}$ in $D$ must be 2. Furthermore, Corollary 4.4 implies that $D$ contains no mixed graph in $C_{8}(1)^{+}$. This along with the connectivity of $D$ forces that $D=H$, as required.
Case 3. $m=8$.
By the Remark in Section 4 and an easy calculation, it is easy to show that the Hermitian spectral radius of any mixed graph in $C_{8}(1)^{*}$ is greater than 2 . This along with Corollary 4.4 forces that $C$ is negative. Since $C_{8}(2)-u_{5}=S(2,3,3), C_{8}(1,1)-u_{4}=Y(2,1,5)$, $C_{8}(1,0,1)-u_{4}=C_{8}(1,0,0,1)-u_{5}=S(1,3,4)$, Theorem 4.2 along with the interlacing theorem forces that $D \in C_{8}(1,0,0,0,1)^{-}$. Note that the Hermitian spectral radius of any mixed graph in $C_{8}(1,0,0,0,1)^{-}$is equal to 2 by an easy calculation. The result follows.

Now we are in the position to state our main theorem which is obtained by Theorem 4.2 and Lemma 4.5.

Theorem 4.6 Let $D$ be a connected $C_{4}$-free mixed graph with $\rho(D) \leqslant 2$. Then $D$ is a mixed graph contained in one of the following families.
(i) All mixed graphs with one of the undirected graphs $C_{n}, Y(2, n-5,2)(n \geq 5), S(1,2,5)$, $S(1,3,3), S(2,2,2)$ as their underlying graphs, and their induced mixed subgraphs;
(ii) $C_{3}(2)^{\widehat{*}}, C_{6}(1,0,1,0,1)^{\hat{-}}, C_{6}(2,0,0,2)^{\hat{\sim}}, C_{8}(1,0,0,0,1)^{\wedge}$;
(iii) The family of mixed graphs with underlying graph $\theta(3,5,5)$ containing two negative mixed cycles $C_{6}$, and their induced mixed subgraphs.

By a routine calculation and checking the proof of Lemma 4.5, we get the following corollary as the end of the paper.

Corollary 4.7 Let $D$ be a $C_{4}$-free mixed graph with $\rho(D)=2$. Then $D$ is a mixed graph contained in one of the following families.
(i) All mixed graphs with one of the undirected graphs $Y(2, n-5,2)(n \geq 5), S(1,2,5)$, $S(1,3,3), S(2,2,2)$ as their underlying graphs;
(ii) $C_{n}^{+}$for any integer $n \geqslant 3$ and $C_{n}^{-}$for any odd number $n \geqslant 3$;
(iii) $C_{3}(2)^{*}, C_{6}(1,0,1)^{-}, C_{6}(1,0,1,0,1)^{-}, C_{6}(2)^{-}, C_{6}(2,0,0,1)^{-}, C_{6}(2,0,0,2)^{-}, C_{8}(1)^{-}$, $C_{8}(1,0,0,0,1)^{-}$;
(iv) The family of mixed graphs with underlying graph $\theta(3,5,5)$ containing two negative mixed cycles $C_{6}$.

## References

[1] A. E. Brouwer, A. Neumaier, The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 114/115 (1989) 273-276.
[2] C. Chen, J. Huang, S.-C. Li, On the relation between the H-rank of a mixed graph and the matching number of its underlying graph, Linear Multilinear Algebra 66(9) (2018) 1853-1869.
[3] C. Chen, S.-C. Li, M.-J. Zhang, Relation between the H-rank of a mixed graph and the rank of its underlying graph, Discrete Math. 342(5) (2019) 1300-1309.
[4] X.-L. Chen, X.-L. Li, Y.-Y. Zhang, 3-regular mixed graphs with optimum Hermitian energy, Linear Algebra Appl. 496 (2016) 475-486.
[5] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Theory and Applications, Academic Press, Inc., New York-London, 1980.
[6] G. Greaves, B. Mohar, O. Suil, Interlacing families and the Hermitian spectral norm of digraphs, Linear Algebra Appl. 564 (2019) 201-208.
[7] K. Guo, B. Mohar, Hermitian adjacency matrix of digraphs and mixed graphs, J. Graph Theory 85(1) (2016) 217-248.
[8] K. Guo, B. Mohar, Digraphs with Hermitian spectral radius below 2 and their cospectrality with paths, Discrete Math. 340 (2017) 2616-2632.
[9] D. Hu, X.-L. Li, X.-G. Liu, S.-G. Zhang, The spectral distribution of random mixed graphs, Linear Algebra Appl. 519 (2017) 343-365.
[10] J.-X. Liu, X.-L. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs, Linear Algebra Appl. 466 (2015) 182-207.
[11] B. Mohar, Hermitian adjacency spectrum and switching equivalence of mixed graphs, Linear Algebra Appl. 489 (2019) 324-340.
[12] J. H. Smith, Some properties of the spectrum of a graph, in: Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), Gordon and Breach, New York, 1970, pp. 403-406.
[13] Y. Wang, B.-J. Yuan, S.-D. Li, C.-J. Wang, Mixed graphs with H-rank 3, Linear Algebra Appl. 524 (2017) 22-34.
[14] R. Woo, A. Neumaier, On graphs whose spectral radius is bounded by $\frac{3}{2} \sqrt{2}$, Graphs Combin. 23 (2007) 713-726.
[15] G.-H. Xu, Some inequalities on the skew-spectral radii of oriented graphs, J. Inequal. Appl. (2012) 2012211.
[16] G.-H. Xu, S.-C. Gong, On oriented graphs whose skew spectral radii do not exceed 2, Linear Algebra Appl. 439 (2013) 2878-2887.


[^0]:    *This work was supported by National Natural Science Foundation of China(11771016, 11871073)
    ${ }^{\dagger}$ Corresponding author. E-mail address: wangy@ahu.edu.cn(Y. Wang), ybjmath@163.com(B.-J. Yuan), scgong@zafu.edu.cn(S.-C. Gong), qiaoyunahu@126.com(Y. Qiao)

