# A SIZE CONDITION FOR DIAMETER TWO ORIENTABLE GRAPHS 

GARNER COCHRAN, ÉVA CZABARKA, PETER DANKELMANN, AND LÁSZLÓ SZÉKELY


#### Abstract

It was conjectured by Koh and Tay [Graphs Combin. 18(4) (2002), 745-756] that for $n \geq 5$ every simple graph of order $n$ and size at least $\binom{n}{2}$ $n+5$ has an orientation of diameter two. We prove this conjecture and hence determine for every $n \geq 5$ the minimum value of $m$ such that every graph of order $n$ and size $m$ has an orientation of diameter two. diameter and oriented diameter and orientation and oriented graph and distance and size


## 1. Introduction

This paper is concerned with the diameter of orientations of graphs. Given a graph $G$, an orientation $O_{G}$ of $G$ is a digraph obtained from $G$ by assigning a direction to every edge of $G$. The distance between two vertices $u$ and $v$ in a graph or digraph $H$, denoted by $d_{H}(u, v)$, is the minimum length of a $(u, v)$-path in $H$; it is infinite if there is no such path. The diameter of $H$ is the largest of the distances between all pairs of vertices, it is denoted by $\operatorname{diam}(H)$. The well-known Robbin's Theorem [10] states that a connected graph has an orientation of finite diameter if and only if it is bridgeless. The oriented diameter of a graph is the minimum diameter of an orientation of $G$. Chvátal and Thomassen [2] showed that there is a function $f$ such that every bridgeless graph of diameter $d$ has an orientation of diameter at most $f(d)$. The determination of the exact values of this function appears extremely difficult. Chvátal and Thomassen [2] showed that every bridgeless graph of diameter two has an orientation of diameter at most six, and that this value is attained by the Petersen graph, so $f(2)=6$. Already the value $f(3)$ is not known. Egawa and Iida 4] and, independently, Kwok, Liu and West [9] showed that the oriented diameter of a bridgeless graph of diameter three is at most 11. In 9 an example of a graph of diameter 3 and oriented diameter 9 was given. Hence $9 \leq f(3) \leq 11$. It was shown by Bau and Dankelmann 1 that every bridgeless graph of order $n$ and minimum degree $\delta$ has an orientation of diameter at most $\frac{11 n}{\delta+1}+O(1)$. Surmacs [11] improved this bound to $\frac{7 n}{\delta+1}+O(1)$. An upper bound on the oriented diameter terms of maximum degree was given by Dankelmann, Guo and Surmacs 3].

Chvátal and Thomassen [2] further showed that the problem of deciding whether a given graph has an orientation of diameter two is NP-complete. Even for complete multipartite graphs the problem which such graphs have an orientation of diameter

[^0]two has not been solved completely, see [6, 7]. Hence it is desirable to have sufficient conditions that guarantee that a given graph has an orientation of diameter two.

In this paper we relate the existence of an orientation of diameter two of a graph of given order to its size. Füredi, Horák, Pareek and Zhu [5] gave an asymptotically sharp lower bound on the number of edges in a graph of given order that admits an orientation of diameter two. The purpose of this paper is to determine for every $n \geq 5$ the minimum value $m(n)$ such that every simple graph of order $n$ and size at least $m(n)$ has an orientation of diameter two.

For $n \geq 5$, the graph $G_{n}$, obtained from a complete graph on $n-1$ vertices by adding a new vertex $v$ and edges joining $v$ to three vertices in the complete graph, does not have an orientation of diameter two. Indeed, suppose to the contrary that $G_{n}$ has an orientation $O_{n}$ of diameter two. Then $v$ has either two in-neighbors and one out-neighbor, or vice versa. We may assume the former. Let $u$ be the out-neighbor and $y_{1}, y_{2}$ be the two in-neighbors of $v$ in $O_{n}$. Since every vertex is at distance at most two from $v$ in $O_{n}$, for every vertex $w \in V\left(G_{n}\right)-\{u, v\}$ the edge $u w$ is oriented from $u$ to $w$. Hence, if $x \in V\left(G_{n}\right)-\left\{u, v, y_{1}, y_{2}\right\}$ any ( $x, u$ )-path in $O_{n}$ goes through $v$ and has thus length at least three, a contradiction to $O_{n}$ having diameter two. Hence $G_{n}$ has no orientation of diameter two. It follows that $m(n) \geq m\left(G_{n}\right)+1=\binom{n}{2}-n+5$ for $n \geq 5$. This was observed by Koh and Tay [8, who conjectured that this construction is best possible, and so $m(n)=\binom{n}{2}-n+5$ for $n \geq 5$. It is the aim of this paper to show that this conjecture is true by proving the following theorem.

Theorem 1.1. Let $G$ be a simple graph of order $n$, where $n \geq 5$, and size at least $\binom{n}{2}-n+5$. Then $G$ has an orientation of diameter two.

Our proof of Theorem 1.1 consists of a sequence of lemmata. An outline of the proof is as follows. We suppose to the contrary that the theorem is false and that $G$ is a counterexample of minimum order, and among those, minimum size. Our proof focuses on the complement $\bar{G}$ of $G$, defined as the graph on the same vertex set as $G$, where two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in $G$.

In Section 3 we give some sufficient conditions for graphs to have an orientation of diameter two, and we present several graphs that have an orientation of diameter two. In Section4 we present some properties of the graph $\bar{G}$ that will be useful later; in particular we show that each component of $\bar{G}$ contains neither three independent vertices nor two non-adjacent vertices that share more than one neighbour. These results, together with some results in Section 5 on the components of $\bar{G}$ that are trees, will be used in Section 6 to show that the components of $\bar{G}$ are short paths, and possibly an additional component that is one of four types of graphs on at most 6 vertices. In Section 7 we complete the proof by showing that the presence of any of these four types of graphs either allows us to apply certain reductions to the graph $G$ to obtain a smaller counterexample $G^{\prime}$, or that $G$ is one of the graphs in the list of graphs with an orientation of diameter two presented in Section 3, so $G$ is not a counterexample. Finally, we conclude the proof by dealing with the case that all components of $\bar{G}$ are trees.

## 2. Notation

All graphs and digraphs in this paper have neither loops nor multiple edges, i.e. they are unoriented or oriented simple graphs. Let $G$ be a graph of order $n=n(G)$ and size $m=m(G)$. We define $G_{1}=\left(V_{1}, E_{1}\right)$ to be a subgraph of $G_{2}=\left(V_{2}, E_{2}\right)$ when $V_{1} \subseteq V_{2}$ and $E_{1} \subseteq E_{2}$. We denote this as $G_{1} \unlhd G_{2}$. We define the excess of $G$ by $\operatorname{ex}(G)=m(G)-n(G)$. We find it convenient to consider $G$ and $\bar{G}$ as obtained by colouring the edges of a complete graph on $n$ vertices either red or blue, with the edges of $G$ being the red, and the edges of $\bar{G}$ as blue edges. Accordingly, we usually denote $G$ as $R$, and $\bar{G}$ as $B$. We denote the vertex set common to $R$ and $B$ by $V$. If $W \subseteq V$, then the red and blue subgraph induced by $W$ in $R$ and $B$, respectively, is denoted by $R[W]$ and $B[W]$.

Let $u, v$ be vertices of a graph $G$ or digraph $O_{G}$. If $u v \in E(G)$ then we say that $u$ and $v$ are adjacent in $G$ and that $u$ is a neighbor of $v$. The set of all neighbors of $v$ is the neighborhood of $v$ in $G$, denoted by $N_{G}(v)$. The closed neighborhood $N_{G}[v]$ of $v$ in $G$ is defined as $N_{G}(v) \cup\{v\}$. If $\overrightarrow{u v}$ is a directed edge of $O_{G}$, then we say that $v$ is an out-neighbor of $u$ and that $u$ is an in-neighbor of $v$. The degree of vertex $v$ in $G$ is the number of neighbors of $v$, it is denoted by $\operatorname{deg}_{G}(v)$.

By $K_{n}, P_{n}, C_{n}$, and $K_{a, b}$ we mean the complete graph on $n$ vertices, the path on $n$ vertices, the cycle on $n$ vertices, and the complete bipartite graph whose partite sets have $a$ and $b$ vertices, respectively. If $G$ and $H$ are graphs, then $G \cup H$ is the disjoint union of $G$ and $H$. If $a$ is a positive integer, then $a G$ is the disjoint union of $a$ copies of $G$, so the edgeless graph on $n$ vertices is denoted by $n K_{1}$.

If $U$ and $W$ are disjoint subsets of $V$ then $U \rightarrow W$ indicates that for all $x \in U$ and $y \in W$ that are adjacent in $R$ we orient the edge $x y$ as $\overrightarrow{x y}$, i.e., from $x$ to $y$. We write $u \rightarrow W$ instead of $\{u\} \rightarrow W$, and similarly $U \rightarrow w$ and $u \rightarrow w$ instead of $U \rightarrow\{w\}$ and $\{u\} \rightarrow\{w\}$.

If $A, B$ are sets of vertices in $H$, then their distance, $d_{H}(A, B)$, is defined as the Hausdorff distance $\min _{u \in A, v \in B} d_{H}(u, v) . \quad d_{H}(u, B)$ and $d_{H}(A, v)$ are defined analogously.

As usual, $[n]=\{1,2,3, \ldots, n\}$ and for a set $A$ and $k \in \mathbb{N},\binom{A}{k}$ is the collection of $k$-element subsets of $A$.

Definition 2.1. Let $k, \ell \in \mathbb{Z}^{+}$. A $(k, \ell)$-dumbbell, denoted by $D_{k, \ell}$, is a graph of order $k+\ell$ obtained from the disjoint union of two complete graphs $K_{k}$ and $K_{\ell}$ by adding an edge joining a vertex of $K_{k}$ to a vertex of $K_{\ell}$. $A$ short ( $k, \ell$ )-dumbbell, denoted by $S_{k, \ell}$, is a graph of order $k+\ell-1$ obtained from the disjoint union of two complete graphs $K_{k}$ and $K_{\ell}$ by identifying a vertex of $K_{k}$ and a vertex of $K_{\ell}$. A $(k, \ell)$-dumbbell is proper if it not a tree, i.e., if $\max (k, \ell) \geq 3$. A short $(k, \ell)$ dumbbell is proper if it is neither complete, nor a tree, nor a dumbbell, i.e., if $\min (k, \ell) \geq 3$.

Note that a $(k, \ell)$-dumbbell is a tree if and only if $\max (k, \ell) \leq 2$, The dumbbells that are trees are paths $P_{i}$ on $2 \leq i \leq 4$ vertices. A short $(k, \ell)$-dumbbell is a dumbbell or a complete graph if and only if $\min (k, \ell) \leq 2$.

## 3. Sufficient conditions for a diameter two orientation

In this section we present a few sufficient conditions for the existence of a diameter two orientation of a graph. Using these conditions we obtain a list of several
graphs that have diameter two orientations. This list will be used extensively in later sections.

Definition 3.1. Let $W \subseteq V$. An orientation $O_{W}$ of $R[W]$ is good if there exists a partition of $W$ into two sets $U_{1}$ and $V_{1}$, which we call the partition classes of $W$ (or of $O_{W}$ ), such that
(i) $d_{O_{W}}(x, y) \leq 2$ whenever $x$ and $y$ are both in $U_{1}$ or both in $V_{1}$.

If in addition
(ii) every vertex in $U_{1}$ has an in-neighbor and an out-neighbor in $V_{1}$ and vice versa, then $O_{W}$ is a non-trivial good orientation. If $R[W]$ has a (non-trivial) good orientation, then we sometimes say simply that $W$ has a (non-trivial) good orientation.

The following lemma is based on a construction of digraphs of diameter two with no 2-cycles having close to the minimum number or edges by Füredi, Horák, Pareek and Zhu [5].
Lemma 3.1. Let $a, b \in \mathbb{N}$ with $2 \leq a \leq b \leq\binom{ a}{\lfloor a / 2\rfloor}$. If $R[W]$ contains $K_{a, b}$ as a spanning subgraph, then $R[W]$ has a non-trivial good orientation. If $R[W]$ is isomorphic to $K_{1,1}$, then $R[W]$ has a good orientation.

Proof. Any orientation of $K_{1,1}$ is vacuously good, so it suffices to show that $K_{a, b}$ has a non-trivial good orientation for all $2 \leq a \leq b \leq\binom{ a}{\lfloor a / 2\rfloor}$.

Let the partite classes of $K_{a, b}$ be $U_{1}=\left\{x_{1}, \ldots, x_{a}\right\}$ and $V_{1}=\left\{y_{1}, \ldots, y_{b}\right\}$ and set $c=\left\lfloor\frac{a}{2}\right\rfloor-1$. Consider an injection $f:[b] \rightarrow\binom{[a]}{c+1}$ such that for $i \in[a] \subseteq[b]$ we have $f(i)=\{i, \ldots, i+c\}$, where numbers in $f(i)$ are taken modulo $a$. Such an injection exists by the conditions on $a, b$ and $c$. Orient the edge $y_{i} x_{j}$ as $\overrightarrow{y_{i} x_{j}}$ if $j \in f(i)$, and as $\overrightarrow{x_{j} y_{i}}$ otherwise. For $i \neq k, i, k \in[b]$, both $f(i) \backslash f(k)$ and $f(k) \backslash f(i)$ are nonempty, ensuring a directed path of length 2 in both directions between $y_{i}$ and $y_{k}$.

Now take $i, k$ such that $1 \leq i<k \leq a$. If $k-i \leq c$, let $\ell \in[a]$ such that $\ell \equiv k+c$ $\bmod a$; we have that $i \in f(i) \backslash f(k)$ an $\ell \in f(k) \backslash f(i)$. If $k-i>c$, let $\ell=i+c$; we have that $k \in f(k) \backslash f(i)$ and $\ell \in f(i) \backslash f(k)$. This ensures a directed path of length 2 in both directions between $x_{i}$ and $x_{k}$. So $K_{a, b}$ has a good orientation.

As every vertex $y_{i} \in V_{1}$ has $\left\lfloor\frac{a}{2}\right\rfloor$ in-neighbors and $\left\lceil\frac{a}{2}\right\rceil$ out-neighbors in $U_{1}$, it has at least one of each. For each $x_{i} \in U_{1}$, the arc $\overrightarrow{y_{i} x_{i}}$ exists, and the arc $\overrightarrow{x_{i} y_{i-1}}$ exists. Hence $K_{a, b}$ has a non-trivial good orientation.
Definition 3.2. Let $\ell \geq k$ be positive integers. We define $K_{\ell} \boxplus K_{k}$ as the disjoint union of $K_{\ell}$ and $K_{k}$ together with a set of edges $M^{\star}$ that match every vertex of $K_{k}$ to a vertex of $K_{\ell}$.
Lemma 3.2. Let $a, b \in \mathbb{N}$ with $3 \leq a \leq b \leq 2 a$. If $R[W]$ contains $K_{a, b}$ as $a$ spanning subgraph with partite sets $X$ and $Y$ such that $B[Y] \unlhd K_{a} \boxplus K_{b-a}$, then $R[W]$ has a non-trivial good orientation.
Proof. Let $W=X \cup Y$ where $X=\left\{x_{1}, \ldots, x_{a}\right\}, Y=\left\{y_{1}, \ldots, y_{b}\right\}$. It suffices to prove that $R[W]$ has a non-trivial good orientation when the edges of $B$ are the union of the edges of the complete graphs on $X,\left\{y_{1}, \ldots, y_{a}\right\}$ and $\left\{y_{a+1}, \ldots, y_{b}\right\}$ together with the edges $\left\{y_{i} y_{a+i}: i \in[b-a]\right\}$.

We will provide an appropriate orientation of the red edges.
For $i \in[a]$, orient the edges $x_{i} y_{i}$ as $\overrightarrow{x_{i} y_{i}}$. For $i, j \in[a]$, where $i \neq j$, orient the edges $x_{i} y_{j}$ as $\overrightarrow{y_{j} x_{i}}$. Note that, as $a>2$, this already ensures that for all $i, j \in[a]$,
there is a path of length at most two from $x_{i}$ to $x_{j}$ and from $y_{i}$ to $y_{j}$, and vertices in $\left\{x_{1}, \ldots, x_{1}, y_{1}, \ldots, y_{a}\right\}$ have both an in-neighbor and an out-neighbor in $R$.

For $i \in[b-a]$, orient the edges $x_{i} y_{a+i}$ as $\overrightarrow{y_{a+i} x_{i}}$. For $i, j \in[b-a], i \neq j$, orient the edges $x_{i} y_{a+j}$ as $\overrightarrow{x_{i} y_{a+j}}$. This ensures that for all $i, j \in[b-a]$ and $j \in[a] \backslash\{i\}$ there is an oriented path of length at most two from $y_{a+i}$ to $y_{a+j}$ and from $y_{a+i}$ to $y_{i}\left(\right.$ through $\left.x_{i}\right)$; and all vertices of $W$ have an in-neighbor and an out-neighbor in $R$.

For $i \in[a] \backslash[b-a]$ and $j \in[b-a]$, orient the edges $x_{i} y_{a+j}$ as $\overrightarrow{x_{i} y_{a+j}}$. This ensures that for all $j \in[b-a]$ and $k \in[a]$ there is an oriented path from $y_{k}$ to $y_{a+j}$ (through an $x_{\ell}$ where $\ell \in[a] \backslash\{k, j\}$ ).

Finally, for $i, j \in[b-a]$, with $i \neq j$, orient the edges $y_{a+i} y_{j}$ as $\overrightarrow{y_{a+i} y_{j}}$. The resulting orientation of $R[W]$ is non-trivially good.

Corollary 3.1. For a vertex set $W \subseteq V$, if $B[W]$ is a disjoint union of paths and the components of $B[W]$ can be partitioned into sets $X$ and $Y$ such that $|X|=a$ and $|Y|=b$ for some $3 \leq a \leq b \leq 2 a$, then $R[W]$ has a non-trivial good orientation.
Proof. Let $B[W]$ be the disjoint union of paths which can be partitioned into sets $X$ and $Y$ such that $|X|=a$ and $|Y|=b$ where $3 \leq a \leq b \leq 2 a$. Then $R[W]$ has $K_{a, b}$ as spanning subgraph with partite sets $X$ and $Y$. Moreover, $Y$ can be partitioned into two sets $Y_{a}$ and $Y_{b-a}$ of cardinality $a$ and $b-a$ respectively, such that $B[Y]$ contains at most one edge joining a vertex in $Y_{a}$ to a vertex in $Y_{b-a}$. Hence, $B[Y] \unlhd P_{b} \unlhd K_{a} \boxplus K_{b-a}$.

Lemma 3.3. Assume that $V$ can be partitioned into two disjoint sets $W$ and $Z$ so that there is no edge in $B$ joining a vertex in $W$ to a vertex in $Z$. Furthermore, assume that $R[W]$ has a non-trivial good orientation, and one of the following holds for $Z$ :
(i) $Z$ has a non-trivial good orientation, or
(ii) $|Z|=3$ and the vertices in $Z$ are isolated in $B$, or
(iii) $|Z|=2$,
then $R$ has an orientation of diameter 2 .
Proof. Let $O_{W}$ be a non-trivial good orientation of $R[W]$ with a corresponding partition of $W$ into sets $U_{1}$ and $V_{1}$. We will extend it to a non-trivial good orientation of $V$.

Proof of (i): Let $O_{Z}$ be a non-trivial good orientation of $R[Z]$ with a corresponding partition of $Z$ into sets $U_{2}$ and $V_{2}$. We assign the orientation $U_{1} \rightarrow U_{2}$, $U_{2} \rightarrow V_{1}, V_{1} \rightarrow V_{2}$, and $V_{2} \rightarrow U_{1}$. We also include $O_{W}$ and $O_{Z}$ in the orientation. It is easy to verify that this in indeed a non-trivial orientation of diameter 2.

Proof of (ii) and (iii): Let $Z=\left\{y_{1}, \ldots, y_{k}\right\}(k \in\{2,3\})$. If $k=3$, orient $R[Z]$ as $y_{1} \rightarrow y_{2} \rightarrow y_{3} \rightarrow y_{1}$. For the remaining red edges, orient $U_{1} \rightarrow y_{1}$ and $y_{1} \rightarrow V_{1}$, and for $j \in[k] \backslash\{1\}$ orient $y_{j} \rightarrow U_{1}$ and $V_{1} \rightarrow y_{j}$. Orient any remaining red edges arbitrarily. It is easy to verify that this is indeed a non-trivial orientation of diameter two.

Lemma 3.4. The following graphs have an orientation of diameter two:
(1) $\overline{Q \cup 7 K_{1}}$, where $Q \in\left\{K_{4}, D_{4,2}, D_{4,1}\right\}$
(2) $\overline{D_{4,3} \cup 8 K_{1}}$,
(3) $\overline{Q \cup 6 K_{1}}$ and $\overline{Q \cup K_{2} \cup 5 K_{1}}$, where $Q \in\left\{D_{3,3}, S_{3,3}\right\}$
(4) $\overline{Q \cup a P_{1} \cup b P_{2}}$, with $a, b \geq 0$ and $a+b=5$, where $Q \in\left\{D_{3,2}, C_{5}, D_{3,1}, K_{3}\right\}$
(5) $\overline{a P_{1} \cup b P_{2} \cup c P_{3} \cup d P_{4}}$, with $a, b, c, d \geq 0$ and $a+b+c+d=5$.

In particular by case (5) Theorem 1.1 holds for $5 \leq n \leq 7$.
Proof. We either directly give the orientation (for small graphs in case (5)) or find a partition of $V$ into two disjoint sets $W$ and $Z$ for which the conditions of Lemma 3.3 hold. We will do the latter by exhibiting a quadruple $\left(U_{1}, V_{1}, U_{2}, V_{2}\right)$ of subgraphs of $B$ whose vertices partition $V$. This signifies that $Z=V\left(U_{1}\right) \cup V\left(V_{1}\right)$, $B[W]=U_{2} \cup V_{2}$, all edges between $Z$ and $W$ are red, $R[W]$ has a non-trivial good orientation with partition classes $U_{2}$ and $V_{2}$, and either $|Z|=2$ (i.e. both $U_{1}$ and $V_{1}$ are the singleton $K_{1}$ and $\left.B[Z]\right) \in\left\{K_{2}, 2 K_{1}\right\}$ ), or $|Z|=3$ and the vertices in $Z$ are isolated in $B$, or $R[Z]$ has a non-trivial good orientation with partition classes $U_{1}$ and $V_{1}$ (and consequently $B[Z]=U_{1} \cup V_{1}$ ).

The proofs of each case in the theorem follow.
(1) $B=Q \cup 7 K_{1}$, where $Q \in\left\{K_{4}, D_{4,2}, D_{4,1}\right\}$. As $4 \leq n(Q) \leq 6$, the quadruple $\left(K_{1}, K_{1}, Q, 5 K_{1}\right)$ gives an orientation of diameter two by Lemmata 3.1 and 3.3
(2) $B=D_{4,3} \cup 8 K_{1}$. We use quadruple ( $K_{1}, K_{1}, 6 K_{1}, D_{4,3}$ ). Since $6 K_{1}$ and $D_{4,3}$ form a partition of $B$ into two graphs $U_{2}$ and $V_{2}$, with $n\left(U_{2}\right)=6$ and $n\left(V_{2}\right)=7$, Lemma 3.1 gives that $W$ has a non-trivial good orientation. Since $|Z|=2$, Lemma 3.3 gives a diameter two orientation of $R$.
(3) $B \in\left\{Q \cup 6 K_{1}, Q \cup K_{2} \cup 5 K_{1}\right\}$ where $Q \in\left\{D_{3,3}, S_{3,3}\right\}$. In both cases quadruple ( $K_{1}, K_{1}, 4 K_{1}, Q$ ) gives the required orientation by Lemmata 3.1 and 3.3.
(4) $B=Q \cup a P_{1} \cup b P_{2}$, with $a, b \geq 0$ and $a+b=5$, where
$Q \in\left\{D_{3,2}, C_{5}, D_{3,1}, K_{3}\right\}$. Then $Q=K_{3}$ or $n(Q) \in\{4,5\}$. As $\max (a, b) \geq 3$, there are two paths of the same size. Choose a pair of such paths of minimum order $i$ (so $i \in\{1,2\}$ ), and let $H$ be the union of the remaining three paths. Clearly $3 \leq n(H) \leq 6$.
Consider the quadruple $\left(P_{i}, P_{i}, H, Q\right)$.
If $n(Q)=n(H)=3$ or $n(Q) \neq 3 \neq n(H)$, then by Lemmata 3.1 and 3.3 we have the required orientation.
If $n(H)=3 \neq n(Q)$, notice that $D_{3,2} \unlhd K_{3} \boxplus K_{2}, C_{5} \unlhd K_{3} \boxplus K_{2}$ and $D_{3,1}=$ $K_{3} \boxplus K_{1}$ and use Lemmata 3.2 and 3.3 to find an orientation of diameter two. If $n(H) \neq 3=n(Q)$, then the fact that $H$ is the disjoint union of paths gives that $H \unlhd K_{3} \boxplus K_{n(H)-3}$. Lemmata 3.2 and 3.3 give the required orientation.
(5) $B=a P_{1} \cup b P_{2} \cup c P_{3} \cup d P_{4}$, with $a, b, c, d \geq 0$ and $a+b+c+d=5$.

All cases where $n(G)<8$ (i.e. when $a+2 b+3 c+4 d \leq 7$ ) and the case where $a=4, b=0, c=0$, and $d=1$ were done by computer search. See Figure 1 for the orientations of these graphs.

For $8 \leq n(G) \leq 9$ and we are not in the case $B=P_{4} \cup 4 P_{1}$, we will consider partitions which use Corollary 3.1 and Lemma 3.3. If $B=P_{3} \cup P_{2} \cup$ $3 P_{1}$, consider the partition $\left(K_{1}, K_{1}, 3 P_{1}, P_{3}\right)$. If $B=2 P_{1} \cup 3 P_{2}$, consider the partition $\left(K_{1}, K_{1}, P_{2} \cup P_{1}, P_{2} \cup P_{1}\right)$. If $B=P_{4} \cup P_{2} \cup 3 P_{1}$, consider the partition $\left(K_{1}, K_{1}, 3 P_{1}, P_{4}\right)$. If $B=2 P_{3} \cup 3 P_{1}$, consider the partition $\left(P_{1}, P_{1}, P_{3}, P_{3} \cup P_{1}\right)$. If $B=P_{3} \cup 2 P_{2} \cup 2 P_{1}$, consider the partition ( $P_{1}, P_{1}, P_{3}, 2 P_{2}$ ). If $B=4 P_{2} \cup P_{1}$, consider the partition $\left(K_{1}, K_{1}, 2 P_{2}, 2 P_{2} \cup P_{1}\right)$. This considers all cases where $n(G) \leq 9$.

Let $n(G) \geq 10$. As $\max (a, b, c, d) \geq 2$, we again have two paths of the same length. Let $H$ be the union of two paths $P_{i}$ of the same length where $i$ is chosen to be minimum possible, and the remaining three paths be $P_{j}, P_{k}, P_{\ell}$ where without loss of generality $k \leq \ell \leq j$. We have $2 i+j+k+\ell=n(G) \geq 10$,

Figure 1. Orientations of graphs where either $n(G)<8$ or $a=4$,
$b=0, c=0$, and $d=1$.

so (since $j \geq k \geq \ell) \frac{10-2 i}{3} \leq j \leq 4$ and $k+\ell \leq 2 j$. We have two cases.
CASE 1: $i=1$
As $\frac{8}{3} \leq j \leq 4$, we have $j \in\{3,4\}$ and $j \leq 4 \leq 10-j-2 \leq k+\ell \leq 2 j$. Take the quadruple $\left(P_{1}, P_{1}, P_{j}, P_{k} \cup P_{\ell}\right)$; Lemmata 3.2 and 3.3 give the required orientation.
CASE 2: $i \geq 2$
By the definition of $i$ we must have $\max (k, \ell)>1$, so $k+\ell \geq 3$. If $j=2$, this gives $i=j=k=\ell=2$ and $G=K_{10}-M$, which has the required orientation by using Corollary 3.1 and Lemma 3.3 with the partition $\left(K_{1}, K_{1}, 2 P_{2}, 2 P_{2}\right)$, so assume $j \geq 3$. Now either $3 \leq k+\ell \leq j \leq 4$ or $3 \leq j \leq k+\ell \leq 2 j$ and in both cases the quadruple ( $P_{i}, P_{i}, P_{j}, P_{k} \cup P_{\ell}$ ) with Lemmata 3.2 and 3.3 give the required orientation.

Definition 3.3. Let $W \subseteq V$ such that $B[W]$ is the union of one or more components of $B$. We say that $W$ is a reducible unit if $R[W]$ has a good orientation. We say that $W$ is a reduction if $R[W]$ has a non-trivial good orientation and $\operatorname{ex}(B[W]) \geq-1$.

## 4. Properties of $B$

From now on we assume that $G$ is a minimal counterexample, that is, $G$ is a graph on $n$ vertices, $n \geq 5$, and at least $\binom{n}{2}-(n-5)$ edges that has no orientation of diameter two, and among those graphs let $G$ be a graph of minimum order and of minimum size. Clearly, if $G$ has $n$ vertices, then $G$ has exactly $\binom{n}{2}-(n-5)$ edges. Hence the corresponding graph $B$ has order $n$ and size $n-5$. Moreover, $n \geq 8$ by Lemma 3.4.

In this section we show that a minimal counterexample cannot have a reduction. We also show that no component of $B$ contains three independent vertices, and that no component has two independent vertices that have at least two common neighbors.

Lemma 4.1. Let $G$ be a minimal counterexample. Then $B$ has no reduction.
Proof. Suppose to the contrary that $B$ has a reduction $W$. Then $|W|>2$ and, by $m(B[W]) \geq|W|-1$, also $W \neq V$. Let $O_{W}$ be a non-trivial good orientation of $R[W]$ and let $U_{1}$ and $V_{1}$ be the partition classes of $O_{W}$. Create $B^{\star}$ from $B$ by removing the vertices of $W$ and adding two new vertines $u_{1}, v_{1}$ with a blue edge $u_{1} v_{1}$ connecting them. As $B[W]$ is a union of components of $B, B$ contains no edges joining vertices in $W$ to vertices in $V-W$. Then $n\left(B^{\star}\right)=n+2-|W|<n$ and since $m(B[W]) \geq|W|-1$,

$$
1 \leq m\left(B^{\star}\right)=(n-5)-m(B[W])+1 \leq n-3-|W|<n\left(B^{\star}\right)-5 .
$$

In particular, $5<n\left(B^{\star}\right)$. Since $B$ was a minimal a counterexample, the red graph $R^{\star}$ corresponding to $B^{\star}$ has an orientation $O^{\star}$ of diameter 2.

We now make use of $O_{W}$ and $O^{\star}$ to obtain an orientation $O_{R}$ of diameter 2 of $R$. Let $x, y \in V$. If $x, y \in W$ then orient $x y$ as in $O_{W}$. If $x, y \in V-W$ then orient $x y$ as in $R^{\star}$. The remaining edges, joining a vertex in $x \in V-W$ to a vertex in $y \in W$ are oriented as follows. If $x u_{1}$ has received the orientation $\overrightarrow{x u_{1}}$ in $O^{\star}$ then we orient $x \rightarrow U_{1}$, and if $x u_{1}$ has received the orientation $\overrightarrow{u_{1} \vec{x}}$ in $O^{\star}$ then we orient
$U_{1} \rightarrow x$. Similarly, if $x v_{1}$ has received the orientation $\overrightarrow{x v_{1}}$ in $O^{\star}$ then we orient $x \rightarrow V_{1}$, and if $x v_{1}$ has received the orientation $\overrightarrow{v_{1} x}$ in $O^{\star}$ then we orient $V_{1} \rightarrow x$.

If $x, y \in V$, then either both vertices are in the same set $U_{1}$ (or $V_{1}$ ), in which case there is a path of length at most two in $O_{W}$, or they are in different sets, for example $x \in U_{1}$ and $y \in V_{1}$, in which case the $\left(u_{1}, v_{1}\right)$-path in $O^{\star}$ gives rise to an $(x, y)$-path in $O_{R}$.
Lemma 4.2. Let $G$ be a minimal counterexample. If $X$ is an independent set of order 3 in $B$, and $N_{i}$ is the set of vertices in $v \in V-X$ having exactly $i$ neighbors (in $B$ ) in $X$, then

$$
\begin{equation*}
\left|N_{2}\right| \leq 1 \quad \text { and } \quad N_{3}=\emptyset \tag{1}
\end{equation*}
$$

Proof. Suppose that $X=\left\{x_{1}, x_{2}, x_{3}\right\}$ is an independent set $B$ such that (1) does not hold. Create a new blue graph $B^{\star}$ by identifying the vertices of $X$ to a new vertex $x$ and removing multiple edges. Then $n\left(B^{\star}\right)=n-2 \geq 5$ and

$$
m\left(B^{\star}\right)=m(B)-\left|N_{2}\right|-2\left|N_{3}\right| \leq m(B)-2=n-7=n\left(B^{\star}\right)-5
$$

Therefore, since $G$ is a minimal counterexample, the red graph $R^{\star}$ corresponding to $B^{\star}$ has an orientation $O^{\star}$ of diameter 2.

We will now orient $R$. Orient every edge $u v$ with $u, v \notin\left\{x_{1}, x_{2}, x_{3}\right\}$ as in $O^{\star}$. Orient $R[X]$ as $\overrightarrow{x_{1} \overrightarrow{x_{2}}}, \overrightarrow{x_{2} x_{3}}$ and $\overrightarrow{x_{3} x_{1}}$, . If an edge $u x$ is present in $R^{\star}$, then all edges $u x_{i}, i=1,2,3$ are present in $R$, and depending on whether $u x$ is oriented as $\overrightarrow{u x}$ or as $\overrightarrow{x u}$ in $O^{\star}$, we orient them $u \rightarrow\left\{x_{1}, x_{2}, x_{3}\right\}$ or $\left\{x_{1}, x_{2}, x_{3}\right\} \rightarrow u$. Orient any remaining edges in $R$ arbitrarily. to obtain the orientation $O_{R}$

Let $u, v \in V(G)$. If $u, v \in\left\{x_{1}, x_{2}, x_{3}\right\}$, then clearly there exists a $(u, v)$-path of length at most two in $O_{R}$. If $u \in X$ and $v \in V-X$ or vice versa then the $(x, v)$-path of length at most two in $O^{\star}$ gives rise to a $(u, v)$-path of the same length in $O_{R}$. If $u, v \in V-X$ then the $(u, v)$-path of length at most two in $O^{\star}$ gives rise to a $(u, v)$-path of the same length in $O_{R}$. This shows that $O_{R}$ is an orientation of $R$ of diameter 2 , a contradiction to $G$ being a counterexample.

Lemma 4.3. Let $G$ be a minimal counterexample. Then no component of $B$ has three independent vertices.

Proof. Suppose to the contrary that $B$ has a component which contains three independent vertices $x_{1}, x_{2}$ and $x_{3}$. We may assume that

$$
\begin{equation*}
d_{B}\left(x_{1},\left\{x_{2}, x_{3}\right\}\right)=2 \tag{2}
\end{equation*}
$$

Indeed, if $d_{B}\left(x_{1},\left\{x_{2}, x_{3}\right\}\right) \geq 3$ then let $x_{1}^{\prime}$ be a vertex on a shortest path in $B$ from $x_{1}$ to $\left\{x_{2}, x_{3}\right\}$ that is at distance two from $\left\{x_{2}, x_{3}\right\}$. The new set $\left\{x_{1}^{\prime}, x_{2}, x_{3}\right\}$ is independent and satisfies (2).
By (2) we may assume, possibly after renaming vertices, that $d_{B}\left(x_{1}, x_{2}\right)=2$. A similar argument as above now yields that we can choose $x_{3}$ such that also

$$
d_{B}\left(x_{3},\left\{x_{1}, x_{2}\right\}\right)=2
$$

Hence we can choose $\left\{x_{1}, x_{2}, x_{3}\right\}$ such that it contains at least two pairs of vertices at distance two in $B$. Hence, possibly after renaming the vertices, we have

$$
\begin{equation*}
d_{B}\left(x_{1}, x_{2}\right)=d_{B}\left(x_{2}, x_{3}\right)=2 . \tag{3}
\end{equation*}
$$

Now (3) implies that there exists a common neighbor $y_{12}$ of $x_{1}$ and $x_{2}$, and a common neighbor $y_{23}$ of $x_{2}$ and $x_{3}$ in $B$. If $y_{12}=y_{23}$, then the set $N_{3}$ of vertices
with exactly three neighbors in $\left\{x_{1}, x_{2}, x_{3}\right\}$ contains $y_{12}$ and is thus not empty, a contradiction to Lemma 4.2. If $y_{12} \neq y_{23}$, then the set $N_{2}$ of vertices with exactly two neighbors in $\left\{x_{1}, x_{2}, x_{3}\right\}$ contains $y_{12}$ and $y_{23}$, again a contradiction to Lemma 4.2 .

Lemma 4.4. Let $G$ be a minimal counterexample. If $x_{1}, x_{2}$ are independent vertices in $B$, then $x_{1}$ and $x_{2}$ have at most one common blue neighbor.

Proof. Suppose to the contrary that $B$ has two vertices $x_{1}$ and $x_{2}$ that share at least two neighbors. Then $x_{1}$ and $x_{2}$ are in the same component of $B$. As $n(B)>1$ and $m(B)<n(B)-1, B$ is not connected. Choose a vertex $x_{3}$ from another component. Then $x_{1}, x_{2}, x_{3}$ are independent vertices, for which the set $N_{2}$ of vertices having exactly two neighbors in $\left\{x_{1}, x_{2}, x_{3}\right\}$ has at least two elements, a contradiction to Lemma 4.2

## 5. On tree components of $B$

Since $B$ has $n$ vertices and $n-5$ edges, $B$ is not connected. In this section we give useful lower bounds on the number of components of $B$ that are trees, and we show that for a given order $t$ we can find a union $F_{t}$ of tree components of $B$ whose order is close to $t$ and excess is at most $-t$. This will be useful in finding reductions and further restricting the possible structure of $B$ for a minimal counterexample. Recall that the excess of a graph $H$ is defined as $\operatorname{ex}(H)=m(H)-n(H)$.
Lemma 5.1. If $B$ contains a component $B_{1}$ that is not a tree, then $B$ has at least $\operatorname{ex}\left(B_{1}\right)+5 \geq 5$ components that are trees. If $B$ has only tree components, it has exactly five components.

Proof. Let $T_{1}, T_{2}, \ldots, T_{k}$ be the components of $B$ that are trees, and $B_{1}, B_{2}, \ldots, B_{\ell}$ be the components that are not trees. Then $\operatorname{ex}\left(T_{i}\right)=-1$ for all $i \in\{1,2, \ldots, k\}$ and $\operatorname{ex}\left(B_{i}\right) \geq 0$ for all $i \in\{1,2, \ldots, \ell\}$. Since $m(B)=n-5$, we have $\operatorname{ex}(B)=-5$, and so

$$
-5=\operatorname{ex}(B)=\sum_{i=1}^{k} \operatorname{ex}\left(T_{i}\right)+\sum_{i=1}^{\ell} \operatorname{ex}\left(B_{i}\right)=-k+\sum_{i=1}^{\ell} \operatorname{ex}\left(B_{i}\right)
$$

If $B$ has no tree component (i.e. $\ell=0$ ), this gives $k=5$. Hence, $B$ has exactly five components. If $B$ contains a component that is not a tree, $B_{1}$ say, then this yields

$$
-5=-k+\sum_{i=1}^{\ell} \operatorname{ex}\left(B_{i}\right) \geq-k+\operatorname{ex}\left(B_{1}\right)
$$

and so $k \geq 5+\operatorname{ex}\left(B_{1}\right) \geq 5$, as claimed.
Lemma 5.2. Assume $B$ contains at least $t$ tree components whose size does not exceed $m_{0}$. Then there exists $t_{0}$ with $t \leq t_{0} \leq t+m_{0}$ such that some subset of the tree components in $B$ forms a forest $F_{t}$ satisfying $n\left(F_{t}\right)=t_{0}$ and $\operatorname{ex}\left(F_{t}\right) \geq-t$. If $B$ contains a tree of size $m_{0}$ where $t>m_{0}$, then we can choose $F_{t}$ such that $\operatorname{ex}\left(F_{t}\right) \geq-t+m_{0}$.
Proof. Let $T_{1}, T_{2}, \ldots, T_{t}$ be the $t$ largest tree components of $B$ whose size does not exceed $m_{0}$. Clearly $T_{1} \cup T_{2} \cup \cdots \cup T_{t}$ contains at least $t$ vertices. Let $j$ be the smallest positive integer such that $T_{1} \cup T_{2} \cup \cdots \cup T_{j}$ contains $t$ or more vertices. Let $F_{t}=T_{1} \cup T_{2} \cup \cdots T_{j}$ and let $t_{0}=n\left(F_{t}\right)$. Since $T_{j}$ has size at most $m_{0}$ and thus
order at most $m_{0}+1$, we have $t \leq t_{0} \leq t+m_{0}$. Moreover, since $T_{1} \cup T_{2} \cup \cdots \cup T_{j-1}$ has less than $t$ vertices, it follows that $T_{j}$ has at least $t_{0}-t+1$ vertices and at least $t_{0}-t$ edges. Hence $m\left(F_{t}\right) \geq m\left(T_{j}\right) \geq t_{0}-t$, and thus ex $\left(F_{t}\right) \geq-t$.
If $t>m_{0}$, we have that $j \geq 2$ and $T_{1}$ has size $m_{0}$. The same argument as above yields that $m\left(F_{t}\right) \geq m\left(T_{1}\right)+m\left(T_{j}\right)=m_{0}+t_{0}-t$ and thus $\operatorname{ex}\left(F_{t}\right) \geq-t+m_{0}$, as desired.

## 6. Describing the components of $B$

In this section further restrict the structure of $B$ in a minimal counterexample. We show that each component of $B$ is either a a path on at most four vertices, a complete graph, a proper dumbbell, a proper short dumbbell, or a 5 -cycle, and none of these components have order more than six.

Lemma 6.1. Let $G$ be a minimal counterexample and $B_{1}$ a component of $B$.
(a) If $B_{1}$ is a tree, then $B_{1}$ is a path $P_{i}$ with $1 \leq i \leq 4$.
(b) If $B_{1}$ is not a tree, then $B_{1}$ is one of the following:
(i) a complete graph $K_{i}$ with $i \geq 3$,
(ii) a proper dumbbell,
(iii) a proper short dumbbell, or
(iv) a 5-cycle.

Proof. As any tree that is not a path on at most 4 vertices is not a complete graph, a dumbbell or a short dumbblell, it is enough to show that $B_{1}$ is a complete graph, a dumbbell, a short dumbbell or a 5 cycle.
If $B_{1}$ is complete, then the lemma holds, so assume that $B_{1}$ is not complete. Let $x_{1}$ and $x_{2}$ be two vertices of $B_{1}$ with $d_{B}\left(x_{1}, x_{2}\right)=\operatorname{diam}\left(B_{1}\right) \geq 2$. By Lemma 4.3 $B_{1}$ does not have three independent vertices, so $d_{B}\left(x_{1}, x_{2}\right)=\operatorname{diam}\left(B_{1}\right) \leq 3$ and $V\left(B_{1}\right)=N_{B}\left(x_{1}\right) \cup N_{B}\left(x_{2}\right)$, and $\left|N_{B}\left(x_{1}\right) \cap N_{B}\left(x_{2}\right)\right| \leq 1$ by Lemma 4.4.
CASE 1: $\operatorname{diam}\left(B_{1}\right)=3$ (consequently $N_{B}\left(x_{1}\right) \cap N_{B}\left(x_{2}\right)=\emptyset$ ).
Since $B_{1}$ does not have three independent vertices by Lemma 4.3, we conclude that each $N_{B}\left[x_{i}\right]$ forms a clique.
Since $B_{1}$ is connected, $B_{1}$ has an edge joining a vertex $y_{1} \in N_{B}\left(x_{1}\right)$ to a vertex $y_{2} \in N_{B}\left(x_{2}\right)$. We show that $B_{1}$ does not contain a further edge joining a vertex $z_{1} \in N_{B}\left(x_{1}\right)$ to a vertex $z_{2} \in N_{B}\left(x_{2}\right)$ by using that Lemma 4.4 gives that two independent vertices share at most one neighbor. Indeed, if $y_{1}=z_{1}$, then $\left\{y_{1}, x_{2}\right\}$ would be a set of two independent vertices that share two neighbors. If $y_{2}=z_{2}$, then $\left\{y_{2}, x_{1}\right\}$ would be a set of two independent vertices that share two neighbors. Lastly, if $y_{1} \neq z_{1}$ and $y_{2} \neq z_{2}$, then $\left\{y_{1}, z_{2}\right\}$ would be a set of two independent vertices that share two neighbors. It follows that $B_{1}$ is a dumbbell.
CASE 2: $\operatorname{diam}\left(B_{1}\right)=2$ (consequently $\left.N_{B}\left[x_{1}\right] \cap N_{B}\left[x_{2}\right]=\{y\}\right)$.
We consider two subcases:
If $\min \left(\operatorname{deg}_{B}\left(x_{1}\right), \operatorname{deg}_{B}\left(x_{2}\right)\right)=1$, then without loss of generality $\operatorname{deg}_{B}\left(x_{1}\right)=1$ and $N_{B}\left(x_{1}\right)=\{y\}$. Since $\operatorname{diam}\left(B_{1}\right)=2$, every vertex in $V\left(B_{1}\right)-\left\{x_{1}, y\right\}$ is adjacent to $y$ in $B_{1}$. Since $B_{1}$ does not contain three independent vertices, $V\left(B_{1}\right)-\left\{x_{1}, y\right\}$ induces a complete graph in $B_{1}$. Therefore $B_{1}$ is a short dumbbell.
If $\min \left(\operatorname{deg}_{B}\left(x_{1}\right), \operatorname{deg}_{B}\left(x_{2}\right)\right) \geq 2$, then, since $B_{1}$ does not contain three independent vertices, $N_{B}\left[x_{i}\right] \backslash\{y\}$ induces a complete graph in $B$ for $i \in\{1,2\}$. If $y$ is adjacent to all vertices in $B_{1}$, then $B_{1}$ is a short dumbbell and we are done. Assume without
loss of generality that there is a vertex $z_{1} \in N_{B}\left[x_{1}\right]$ to which $y$ is non-adjacent in $B_{1}$. Then $d_{B}\left(z_{1}, x_{2}\right)=2$, so $z_{1}$ and $x_{2}$ have a common blue neighbor $z_{2}$. Since $x_{1}$ and $z_{2}$ are non-adjacent in $B$ and thus cannot have two common neighbors, $z_{2}$ and $y$ are non-adjacent in $B$. Since also the edges $x_{1} x_{2}, x_{1} z_{2}$ and $x_{2} z_{1}$ are not present in $B$, we conclude that $x_{1}, y, x_{2}, z_{2}, z_{1}, x_{1}$ form an induced 5 -cycle in $B_{1}$. Hence $B_{1}$ contains an induced 5 -cycle.
Rename the vertices of the 5 -cycle so the cycle is $v_{0} v_{1} v_{2} v_{3} v_{4} v_{0}$. Suppose there is a sixth vertex $w$ adjacent to a vertex in $\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right\}$ in $B_{1}$. If $\mid N_{B}(w) \cap$ $\left\{v_{0}, \ldots, v_{4}\right\} \mid \leq 2$, it is easy to see that $v$ together with two suitably chosen vertices in $\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right\}$ forms an independent set of cardinality three, which is impossible. Hence $v$ is adjacent to at least three vertices in $\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right\}$. But then $v$ has two neighbors among these vertices that are not adjacent, without loss of generality $v_{1}$ and $v_{3}$, so that $v_{1}$ and $v_{3}$ are non-adjacent vertices with two common neighbors, a contradiction to Lemma 4.4. This proves that $B_{1}$ contains only $\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right\}$, and so $B_{1}$ is a 5 -cycle.

Lemma 6.2. In a minimal counterexample all components of $B$ are of order at most six.

Proof. Suppose to the contrary that $B$ contains a component $B_{1}$ with more than six vertices. Let $n_{1} \geq 7$ and $m_{1}$ be the order and size, respectively, of $B_{1}$. By Lemma 6.1. $B_{1}$ is a complete graph, a dumbbell, or a short dumbbell. It is easy to see that among all such graphs of order $n_{1}$ the dumbbell $D_{\left\lceil n_{1} / 2\right\rceil,\left\lfloor n_{1} / 2\right\rfloor}$ has minimum size, and every other graph has bigger size. A simple calculation shows that

$$
\begin{equation*}
m_{1} \geq m\left(D_{\left\lceil n_{1} / 2\right\rceil,\left\lfloor n_{1} / 2\right\rfloor}\right) \geq\left\lceil\frac{1}{4} n_{1}^{2}-\frac{1}{2} n_{1}+1\right\rceil \tag{4}
\end{equation*}
$$

and consequently

$$
\begin{equation*}
\operatorname{ex}\left(B_{1}\right)=m_{1}-n_{1} \geq\left\lceil\frac{\left(n_{1}-3\right)^{2}-5}{4}\right\rceil \tag{5}
\end{equation*}
$$

where equality holds only when $B=D_{\left\lceil n_{1} / 2\right\rceil,\left\lfloor n_{1} / 2\right\rfloor}$.
Assume first that $B_{1} \neq D_{3,4}$. If $n_{1} \geq 8$, equation (5) easily gives $\operatorname{ex}\left(B_{1}\right) \geq n_{1}-3$. If $n_{1}=7$, then, as the lower bound in (5) is only sharp when $B_{1}=D_{3,4}$, it follows that $\operatorname{ex}\left(B_{1}\right) \geq 4=n_{1}-3$. By Lemma 5.1, $B$ contains at least $\operatorname{ex}\left(B_{1}\right)+5 \geq n_{1}+2$ tree components. Set $t=n_{1}-2$. By Lemma 5.2, for some $t_{0}$ with $n_{1}-1 \leq t_{0} \leq n_{1}+2$, $B$ contains a forest $F_{t}$ of order $t_{0}$ and excess at least $-t=-n_{1}+2$ that is the union of the tree components of $B$. Let $W:=V\left(B_{1}\right) \cup V\left(F_{t}\right)$. We show that $W$ is a reduction. Clearly the graph $R[W]$ contains a spanning subgraph $K_{n_{1}, t_{0}}$. Since $n_{1}-1 \leq t_{0} \leq n_{1}+2$, it is easy to verify that either $n_{1} \leq t_{0} \leq\binom{ n_{1}}{2}$ or $t_{0}<n_{1} \leq\binom{ t_{0}}{2}$. So $R[W]$ has a non-trivial good orientation by Lemma 3.1 and $\operatorname{ex}(B[W])=\operatorname{ex}\left(B_{1}\right)+\operatorname{ex}\left(F_{n_{1}-2}\right) \geq n_{1}-3+\left(-n_{1}+2\right)=-1$. Hence, $W$ is a reduction, a contradiction to Lemma 4.1. So we must have that $B_{1}=D_{3,4}$
If $B_{1}=D_{3,4}, \operatorname{ex}\left(B_{1}\right)=3$ by equation (5), and $B$ has at least 8 tree components. Set $m_{0}$ be the size of the largest tree component. If $1 \leq m_{0}$, Lemma 5.2 with $t=5$ and $1 \leq m_{0} \leq 3$ yields that there exists a forest $F_{5}$ in $B$ of order $t_{0}$, where $5 \leq t_{0} \leq 8$, and excess at least $-5+1=-4$. Let $W=V\left(B_{1}\right) \cup V\left(F_{5}\right)$, then $\operatorname{ex}(B[W])=\operatorname{ex}\left(B_{1}\right)+\operatorname{ex}\left(F_{5}\right) \geq 3+(-4)=-1$, and $R[W]$ has a non-trivial good orientation by Lemma 3.1. Hence $W$ is a reduction, a contradiction to Lemma 4.1 So all tree components of $B$ are singletons. If all $k$ components of $B-B_{1}$ are $P_{1}$,
then $-5=\operatorname{ex}(B)=3-k$ gives $B=D_{3,4} \cup 8 P_{1}$. But by Lemma 3.4, $\overline{D_{3,4} \cup 8 P_{1}}$ has an orientation of diameter two, which is a contradiction. Therefore $B$ contains another non-tree component $B_{2}$ with at least one edge, so it has at least 3 (and by our proof so far, at most 7) vertices. Set $W=B_{1} \cup B_{2} \cup 2 P_{1}$. By Lemma 3.1, $W$ has a non-trivial good orientation with partition classes $B_{1}$ and $B_{2} \cup 2 P_{1}$ and $\operatorname{ex}(B[W]) \geq 3-2>0$. So $W$ is a reduction, which is a contradiction.

Lemma 6.3. If a minimal counterexample $B$ contains a component $B_{1}$ that is not a tree, then $B-B_{1}$ has exactly $\operatorname{ex}\left(B_{1}\right)+5$ components, all of which are trees.

Proof. Suppose to the contrary that $B$ contains two non-tree components $B_{1}$ and $B_{2}$ with $3 \leq n\left(B_{1}\right) \leq n\left(B_{1}\right)$. Then ex $\left(B_{1}\right) \geq 0$ and ex $\left(B_{2}\right) \geq 0$, and by Lemma 6.2 $n\left(B_{1}\right) \leq 6$. If $n\left(B_{1}\right)=n\left(B_{2}\right)=3$ or $n\left(B_{1}\right), n\left(B_{2}\right) \in\{4,5,6\}$, then $V\left(B_{1}\right) \cup V\left(B_{2}\right)$ has a non-trivial good orientation by Lemma 3.1 and is thus a reduction, since $\operatorname{ex}\left(B_{1} \cup B_{2}\right)=\operatorname{ex}\left(B_{1}\right)+\operatorname{ex}\left(B_{2}\right) \geq 0$. So we have $n_{1} \in\{4,5,6\}$ and $n_{2}=3$. As $V\left(B_{1}\right) \cup V\left(P_{4}\right)$ or $V\left(B_{2}\right) \cup V\left(P_{3}\right)$ would form a reduction, all tree components in $B$ are $P_{1}$ or $P_{2}$. Since $V\left(B_{1}\right) \cup V\left(B_{2}\right) \cup V\left(P_{i}\right)$ forms a reduction for $i \in\{1,2\}$, this is a contradiction to Lemma 4.1. Hence all $k$ components of $B-B_{1}$ are trees. As $-5=\operatorname{ex}(B)=\operatorname{ex}\left(B_{1}\right)-k$ we are done.

Lemma 6.4. Assume $B$ contains a non-tree component $B_{1}$. Let $F$ be a forest that is the union of the smallest number of tree components of $B$ such that $\min \left(4, n\left(B_{1}\right)\right) \leq n(F) \leq 6$ and $k_{0}$ be the number of tree components that make up $F$. Then $k_{0} \geq \operatorname{ex}\left(B_{1}\right)+2$, $\operatorname{ex}\left(B_{1}\right) \leq 2$, and the tree components of $B$ contain at most $\min \left(3, n\left(B_{1}\right)-1\right)$ vertices.

Proof. If $B_{1}$ is a component that is not a tree, by Lemma $6.2 \operatorname{ex}\left(B_{1}\right) \geq 0$ and $3 \leq n\left(B_{1}\right) \leq 6$. $B$ does not contain a $P_{4}$ component, otherwise $W=V\left(B_{1}\right) \cup V\left(P_{4}\right)$ would form a reduction by Lemma 3.1 and $\operatorname{ex}(B[W]) \geq-1$.

Let $F$ and $k_{0}$ be given as in the conditions of the lemma. Clearly, $k_{0} \leq 4$ and $\operatorname{ex}(F)=-k_{0}$. Consider $W=V\left(B_{1}\right) \cup V(F)$. If $n\left(B_{1}\right) \neq 3$ or $n\left(B_{1}\right)=n(F)$, then $R[W]$ has a non-trivial good orientation by Lemma 3.1. If $n\left(B_{1}\right)=3$ and $4 \leq n(F) \leq 6$, then $B_{1}=K_{3}, F \unlhd K_{3} \boxplus K_{n(F)-3}$, and $R[W]$ has a non-trivial good orientation by Lemma 3.2. As $W$ is not a reduction, we must have $-2 \geq$ $\operatorname{ex}(B[W])=\operatorname{ex}\left(B_{1}\right)-k$, giving $k \geq \operatorname{ex}\left(B_{1}\right)+2$. $\operatorname{ex}\left(B_{1}\right) \leq 2$ follows from $k \leq 4$. As $k \geq 2$, no tree component has size $n\left(B_{1}\right)$.

## 7. Proof of the main result

We start by eliminating the possibility of a non-tree component from a minimal counterexample.
Lemma 7.1. In a minimal counterexample no component of $B$ is a complete graph on three or more vertices.

Proof. Suppose to the contrary that $B$ contains a component $B_{1}$ that is a complete graph of order $n_{1} \geq 3$. By Lemma 6.4 we have $\operatorname{ex}\left(B_{1}\right) \leq 2$ and consequently $n_{1} \in\{3,4\}$.
If $B_{1}=K_{4}$, then ex $\left(B_{1}\right)=2$ and $B$ contains exactly 7 tree components by Lemma 6.3. By Lemma 6.4 all tree components must be $P_{1}$ (otherwise $k_{0}<4$ in the lemma, which is a contradiction). By Lemma 3.4, the graph $\overline{K_{4} \cup 7 K_{1}}$ has an orientation
of diameter two, so it is not a counterexample, which is a contradiction.
If $B_{1}=K_{3}$, then ex $\left(B_{1}\right)=0$ and $B$ contains exactly 5 tree components by Lemma 6.3. By Lemma 6.4 all these tree components must be $P_{1}$ or $P_{2}$, so we have $B=$ $K_{3} \cup a K_{1} \cup b K_{2}$ for some nonnegative integers $a, b$ with $a+b=5$. But by Lemma 3.4 all such graphs have an orientation of diameter two. So $G$ is not a counterexample, a contradiction.

Lemma 7.2. In a minimal counterexample no component of $B$ is a proper dumbbell.
Proof. Assume that $B_{1}$ is a component of $B$ that is a proper dumbbell; then $n\left(B_{1}\right) \geq 4$, and by Lemmata 6.2 and 6.4 we have $n\left(B_{1}\right) \leq 6$ and $\operatorname{ex}\left(B_{1}\right) \leq 2$, and $B-B_{1}$ has no $P_{4}$ component. Hence $B_{1} \in\left\{D_{3,1}, D_{4,1}, D_{3,2}, D_{4,2}, D_{3,3}\right\}$. By Lemma 6.3, $B-B_{1}$ has exactly $\operatorname{ex}\left(B_{1}\right)+5$ other components that are all paths on at most three vertices. We will examine each case grouped by ex $\left(B_{1}\right)$.
(1) $B_{1} \in\left\{D_{4,1}, D_{4,2}\right\}$. Then $\operatorname{ex}\left(B_{1}\right)=2$ and $B-B_{1}$ has exactly 7 tree components by Lemma 6.3, By Lemma 3.4, the graph $\overline{B_{1} \cup 7 K_{1}}$ has an orientation of diameter two, therefore not all tree components of $B$ are singletons. We get $k_{0} \leq 3$ and a contradiction in Lemma 6.4.
(2) $B_{1} \in\left\{D_{3,1}, D_{3,2}\right\}$. Then $\operatorname{ex}\left(B_{1}\right)=0$ and $B-B_{1}$ has exactly five tree components by Lemma 6.3. Lemma 3.4 gives that $\overline{B_{1} \cup a K_{1} \cup b K_{2}}$ has a diameter two orientation for all $a+b=5$, so at least one of the tree components is a $P_{3}$. For $j \in\{1,2\}, D_{3, j} \unlhd K_{3} \boxplus K_{j}$, and by Lemma 3.2 $V\left(P_{3}\right) \cup V\left(B_{1}\right)$ is a reduction, which is again a contradiction.
(3) $B_{1}=D_{3,3}$. Then $\operatorname{ex}\left(B_{1}\right)=1$. By Lemma 6.3, $B-B_{1}$ contains exactly 6 components which are trees. By Lemma 3.4 the graphs $\overline{D_{3,3} \cup 6 K_{1}}$ and $\overline{D_{3,3} \cup K_{2} \cup 5 K_{1}}$ have an orientation of diameter two. Hence $B-B_{1}$ contains a $P_{3}$ or two components that are $P_{2}$. We get $k_{0} \leq 2$ and a contradiction in Lemma 6.4.

Lemma 7.3. In a minimal counterexample no component of $B$ is a proper short dumbbell.

Proof. Assume that $B_{1}$ is a component of $B$ that is a proper short dumbbell. Then $5 \leq n\left(B_{1}\right)$. By Lemmata 6.2 and 6.4, $n\left(B_{1}\right) \leq 6, \operatorname{ex}\left(B_{1}\right) \leq 2$, and no tree component of $B$ is a $P_{4}$. This gives that $B_{1}=S_{3,3}, \operatorname{ex}\left(B_{1}\right)=1$, and $B-B_{1}$ has exactly 6 tree components. By Lemma 3.4, both $\overline{S_{3,3} \cup 6 K_{1}}$ and $\overline{S_{3,3} \cup K_{2} \cup 5 K_{1}}$ have diameter two orientations, so the components of $B$ include at least two $P_{2}$ or at least one $P_{3}$. This gives $k_{0}=2$ and a contradiction in Lemma 6.4.

Lemma 7.4. In a minimal counterexample no component of $B$ is a 5-cycle.
Proof. Assume that $B_{1}$ is a component of $B$ that is a 5 -cycle. Then $\operatorname{ex}\left(B_{1}\right)=0$ and, by Lemmata 6.3 and $6.4 B-B_{1}$ has exactly 5 components which are trees on at most three vertices. By Lemma $\sqrt[3.4]{,_{5} \cup a P_{2} \cup b P_{1}}$ has an orientation of diameter two for all non-negative integers $a, b$ with $a+b=5$, so at least one of these tree components is a $P_{3}$. As $P_{3} \leq K_{3}$ and $C_{5} \unlhd K_{3} \boxplus K_{2}$, by Lemma 3.2 $B_{1} \cup P_{3}$ forms a reduction, contradicting Lemma 4.1.

We are now ready to complete the proof of Theorem 1.1 .

Proof. Suppose to the contrary that Theorem 1.1 is false. Let $G$ be a minimal counterexample, that is a graph of minimum order and minimum size for which the theorem does not hold. By Lemma 3.4 $n(G) \geq 8$ and consequently $m(G)=$ $n(G)-5$. By Lemma 6.1, every component of $B$ that is not a tree is either a complete graph on at least three vertices, a proper dumbbell, a proper short dumbbell, or a 5 -cycle. By Lemmata 7.1, 7.2, 7.3, and 7.4, all components of $B$ must be trees, and by Lemmata 5.1 and $6.1 B=a P_{1} \cup b P_{2} \cup c P_{3} \cup d P_{4}$ for some $a+b+c+d=5$. But then Lemma 3.4 gives that $G$ has a diameter two orientation, a contradicton.

## 8. Open Problem

In Theorem 1.1 we show that in graph of given order $n$ we need at least $\binom{n}{2}-n+5$ edges to guarantee the existence of an orientation of diameter two. It is natural to ask the same question for any given value of $d$ : In a graph of order $n$, over all bridgeless graphs, how many edges do we need at least to guarantee the existence of an orientation of diameter at most $d$ ?

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Garner Cochran, Department of Mathematics and Computer Science, Berry College, 2277 Martha Berry Hwy NW, Mt Berry GA 30149, USA

E-mail address: gcochran@berry.edu
Éva Czabarka, Department of Mathematics, University of South Carolina, Columbia SC 29212, USA and Visiting Professor, Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

E-mail address: czabarka@math.sc.edu
Peter Dankelmann, Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

E-mail address: pdankelmann@uj.ac.za
László Székely, Department of Mathematics, University of South Carolina, Columbia SC 29212, USA and Visiting Professor, Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

E-mail address: szekely@math.sc.edu


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