# A SIZE CONDITION FOR DIAMETER TWO ORIENTABLE GRAPHS

GARNER COCHRAN, ÉVA CZABARKA, PETER DANKELMANN, AND LÁSZLÓ SZÉKELY

ABSTRACT. It was conjectured by Koh and Tay [Graphs Combin. 18(4) (2002), 745–756] that for  $n \geq 5$  every simple graph of order n and size at least  $\binom{n}{2} - n + 5$  has an orientation of diameter two. We prove this conjecture and hence determine for every  $n \geq 5$  the minimum value of m such that every graph of order n and size m has an orientation of diameter two.

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#### 1. INTRODUCTION

This paper is concerned with the diameter of orientations of graphs. Given a graph G, an orientation  $O_G$  of G is a digraph obtained from G by assigning a direction to every edge of G. The *distance* between two vertices u and v in a graph or digraph H, denoted by  $d_H(u, v)$ , is the minimum length of a (u, v)-path in H; it is infinite if there is no such path. The *diameter* of H is the largest of the distances between all pairs of vertices, it is denoted by  $\operatorname{diam}(H)$ . The well-known Robbin's Theorem [10] states that a connected graph has an orientation of finite diameter if and only if it is bridgeless. The *oriented diameter* of a graph is the minimum diameter of an orientation of G. Chvátal and Thomassen [2] showed that there is a function f such that every bridgeless graph of diameter d has an orientation of diameter at most f(d). The determination of the exact values of this function appears extremely difficult. Chvátal and Thomassen [2] showed that every bridgeless graph of diameter two has an orientation of diameter at most six, and that this value is attained by the Petersen graph, so f(2) = 6. Already the value f(3) is not known. Egawa and Iida [4] and, independently, Kwok, Liu and West [9] showed that the oriented diameter of a bridgeless graph of diameter three is at most 11. In [9] an example of a graph of diameter 3 and oriented diameter 9 was given. Hence  $9 \le f(3) \le 11$ . It was shown by Bau and Dankelmann [1] that every bridgeless graph of order n and minimum degree  $\delta$  has an orientation of diameter at most  $\frac{11n}{\delta+1} + O(1)$ . Surmacs [11] improved this bound to  $\frac{7n}{\delta+1} + O(1)$ . An upper bound on the oriented diameter terms of maximum degree was given by Dankelmann, Guo and Surmacs [3].

Chvátal and Thomassen [2] further showed that the problem of deciding whether a given graph has an orientation of diameter two is NP-complete. Even for complete multipartite graphs the problem which such graphs have an orientation of diameter

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two has not been solved completely, see [6, 7]. Hence it is desirable to have sufficient conditions that guarantee that a given graph has an orientation of diameter two.

In this paper we relate the existence of an orientation of diameter two of a graph of given order to its size. Füredi, Horák, Pareek and Zhu [5] gave an asymptotically sharp lower bound on the number of edges in a graph of given order that admits an orientation of diameter two. The purpose of this paper is to determine for every  $n \ge 5$  the minimum value m(n) such that every simple graph of order n and size at least m(n) has an orientation of diameter two.

For  $n \geq 5$ , the graph  $G_n$ , obtained from a complete graph on n-1 vertices by adding a new vertex v and edges joining v to three vertices in the complete graph, does not have an orientation of diameter two. Indeed, suppose to the contrary that  $G_n$  has an orientation  $O_n$  of diameter two. Then v has either two in-neighbors and one out-neighbor, or vice versa. We may assume the former. Let u be the out-neighbor and  $y_1, y_2$  be the two in-neighbors of v in  $O_n$ . Since every vertex is at distance at most two from v in  $O_n$ , for every vertex  $w \in V(G_n) - \{u, v\}$  the edge uw is oriented from u to w. Hence, if  $x \in V(G_n) - \{u, v, y_1, y_2\}$  any (x, u)-path in  $O_n$  goes through v and has thus length at least three, a contradiction to  $O_n$ having diameter two. Hence  $G_n$  has no orientation of diameter two. It follows that  $m(n) \geq m(G_n) + 1 = {n \choose 2} - n + 5$  for  $n \geq 5$ . This was observed by Koh and Tay [8], who conjectured that this construction is best possible, and so  $m(n) = {n \choose 2} - n + 5$ for  $n \geq 5$ . It is the aim of this paper to show that this conjecture is true by proving the following theorem.

**Theorem 1.1.** Let G be a simple graph of order n, where  $n \ge 5$ , and size at least  $\binom{n}{2} - n + 5$ . Then G has an orientation of diameter two.

Our proof of Theorem 1.1 consists of a sequence of lemmata. An outline of the proof is as follows. We suppose to the contrary that the theorem is false and that G is a counterexample of minimum order, and among those, minimum size. Our proof focuses on the complement  $\overline{G}$  of G, defined as the graph on the same vertex set as G, where two vertices are adjacent in  $\overline{G}$  if and only if they are not adjacent in G.

In Section 3 we give some sufficient conditions for graphs to have an orientation of diameter two, and we present several graphs that have an orientation of diameter two. In Section 4 we present some properties of the graph  $\overline{G}$  that will be useful later; in particular we show that each component of  $\overline{G}$  contains neither three independent vertices nor two non-adjacent vertices that share more than one neighbour. These results, together with some results in Section 5 on the components of  $\overline{G}$  that are trees, will be used in Section 6 to show that the components of  $\overline{G}$  are short paths, and possibly an additional component that is one of four types of graphs on at most 6 vertices. In Section 7 we complete the proof by showing that the presence of any of these four types of graphs either allows us to apply certain reductions to the graph G to obtain a smaller counterexample G', or that G is one of the graphs in the list of graphs with an orientation of diameter two presented in Section 3, so G is not a counterexample. Finally, we conclude the proof by dealing with the case that all components of  $\overline{G}$  are trees.

#### 2. NOTATION

All graphs and digraphs in this paper have neither loops nor multiple edges, i.e. they are unoriented or oriented simple graphs. Let G be a graph of order n = n(G)and size m = m(G). We define  $G_1 = (V_1, E_1)$  to be a subgraph of  $G_2 = (V_2, E_2)$ when  $V_1 \subseteq V_2$  and  $E_1 \subseteq E_2$ . We denote this as  $G_1 \leq G_2$ . We define the excess of Gby ex(G) = m(G) - n(G). We find it convenient to consider G and  $\overline{G}$  as obtained by colouring the edges of a complete graph on n vertices either red or blue, with the edges of G being the red, and the edges of  $\overline{G}$  as blue edges. Accordingly, we usually denote G as R, and  $\overline{G}$  as B. We denote the vertex set common to R and B by V. If  $W \subseteq V$ , then the red and blue subgraph induced by W in R and B, respectively, is denoted by R[W] and B[W].

Let u, v be vertices of a graph G or digraph  $O_G$ . If  $uv \in E(G)$  then we say that u and v are adjacent in G and that u is a neighbor of v. The set of all neighbors of v is the neighborhood of v in G, denoted by  $N_G(v)$ . The closed neighborhood  $N_G[v]$  of v in G is defined as  $N_G(v) \cup \{v\}$ . If  $\overline{uv}$  is a directed edge of  $O_G$ , then we say that v is an out-neighbor of u and that u is an in-neighbor of v. The degree of vertex v in G is the number of neighbors of v, it is denoted by  $\deg_G(v)$ .

By  $K_n$ ,  $P_n$ ,  $C_n$ , and  $K_{a,b}$  we mean the complete graph on n vertices, the path on n vertices, the cycle on n vertices, and the complete bipartite graph whose partite sets have a and b vertices, respectively. If G and H are graphs, then  $G \cup H$  is the disjoint union of G and H. If a is a positive integer, then aG is the disjoint union of a copies of G, so the edgeless graph on n vertices is denoted by  $nK_1$ .

If U and W are disjoint subsets of V then  $U \to W$  indicates that for all  $x \in U$ and  $y \in W$  that are adjacent in R we orient the edge xy as  $\overrightarrow{xy}$ , i.e., from x to y. We write  $u \to W$  instead of  $\{u\} \to W$ , and similarly  $U \to w$  and  $u \to w$  instead of  $U \to \{w\}$  and  $\{u\} \to \{w\}$ .

If A, B are sets of vertices in H, then their distance,  $d_H(A, B)$ , is defined as the Hausdorff distance  $\min_{u \in A, v \in B} d_H(u, v)$ .  $d_H(u, B)$  and  $d_H(A, v)$  are defined analogously.

As usual,  $[n] = \{1, 2, 3, ..., n\}$  and for a set A and  $k \in \mathbb{N}$ ,  $\binom{A}{k}$  is the collection of k-element subsets of A.

**Definition 2.1.** Let  $k, \ell \in \mathbb{Z}^+$ . A  $(k, \ell)$ -dumbbell, denoted by  $D_{k,\ell}$ , is a graph of order  $k + \ell$  obtained from the disjoint union of two complete graphs  $K_k$  and  $K_\ell$  by adding an edge joining a vertex of  $K_k$  to a vertex of  $K_\ell$ . A short  $(k, \ell)$ -dumbbell, denoted by  $S_{k,\ell}$ , is a graph of order  $k + \ell - 1$  obtained from the disjoint union of two complete graphs  $K_k$  and  $K_\ell$  by identifying a vertex of  $K_k$  and a vertex of  $K_\ell$ . A  $(k, \ell)$ -dumbbell is proper if it not a tree, i.e., if  $\max(k, \ell) \geq 3$ . A short  $(k, \ell)$ -dumbbell is proper if it is neither complete, nor a tree, nor a dumbbell, i.e., if  $\min(k, \ell) \geq 3$ .

Note that a  $(k, \ell)$ -dumbbell is a tree if and only if  $\max(k, \ell) \leq 2$ . The dumbbells that are trees are paths  $P_i$  on  $2 \leq i \leq 4$  vertices. A short  $(k, \ell)$ -dumbbell is a dumbbell or a complete graph if and only if  $\min(k, \ell) \leq 2$ .

#### 3. Sufficient conditions for a diameter two orientation

In this section we present a few sufficient conditions for the existence of a diameter two orientation of a graph. Using these conditions we obtain a list of several graphs that have diameter two orientations. This list will be used extensively in later sections.

**Definition 3.1.** Let  $W \subseteq V$ . An orientation  $O_W$  of R[W] is good if there exists a partition of W into two sets  $U_1$  and  $V_1$ , which we call the partition classes of W (or of  $O_W$ ), such that

(i)  $d_{O_W}(x,y) \leq 2$  whenever x and y are both in  $U_1$  or both in  $V_1$ . If in addition

(ii) every vertex in  $U_1$  has an in-neighbor and an out-neighbor in  $V_1$  and vice versa, then  $O_W$  is a non-trivial good orientation. If R[W] has a (non-trivial) good orientation, then we sometimes say simply that W has a (non-trivial) good orientation.

The following lemma is based on a construction of digraphs of diameter two with no 2-cycles having close to the minimum number or edges by Füredi, Horák, Pareek and Zhu [5].

**Lemma 3.1.** Let  $a, b \in \mathbb{N}$  with  $2 \leq a \leq b \leq {\binom{a}{\lfloor a/2 \rfloor}}$ . If R[W] contains  $K_{a,b}$  as a spanning subgraph, then R[W] has a non-trivial good orientation. If R[W] is isomorphic to  $K_{1,1}$ , then R[W] has a good orientation.

Let the partite classes of  $K_{a,b}$  be  $U_1 = \{x_1, \ldots, x_a\}$  and  $V_1 = \{y_1, \ldots, y_b\}$  and set  $c = \lfloor \frac{a}{2} \rfloor - 1$ . Consider an injection  $f : [b] \to {\binom{[a]}{c+1}}$  such that for  $i \in [a] \subseteq [b]$ we have  $f(i) = \{i, \ldots, i+c\}$ , where numbers in f(i) are taken modulo a. Such an injection exists by the conditions on a, b and c. Orient the edge  $y_i x_j$  as  $\overline{y_i x_j}$  if  $j \in f(i)$ , and as  $\overline{x_j y_i}$  otherwise. For  $i \neq k, i, k \in [b]$ , both  $f(i) \setminus f(k)$  and  $f(k) \setminus f(i)$ are nonempty, ensuring a directed path of length 2 in both directions between  $y_i$ and  $y_k$ .

Now take i, k such that  $1 \leq i < k \leq a$ . If  $k - i \leq c$ , let  $\ell \in [a]$  such that  $\ell \equiv k + c$ mod a; we have that  $i \in f(i) \setminus f(k)$  an  $\ell \in f(k) \setminus f(i)$ . If k - i > c, let  $\ell = i + c$ ; we have that  $k \in f(k) \setminus f(i)$  and  $\ell \in f(i) \setminus f(k)$ . This ensures a directed path of length 2 in both directions between  $x_i$  and  $x_k$ . So  $K_{a,b}$  has a good orientation.

As every vertex  $y_i \in V_1$  has  $\lfloor \frac{a}{2} \rfloor$  in-neighbors and  $\lfloor \frac{a}{2} \rfloor$  out-neighbors in  $U_1$ , it has at least one of each. For each  $x_i \in U_1$ , the arc  $\overline{y_i x_i}$  exists, and the arc  $\overline{x_i y_{i-1}}$  exists. Hence  $K_{a,b}$  has a non-trivial good orientation.

**Definition 3.2.** Let  $\ell \geq k$  be positive integers. We define  $K_{\ell} \boxplus K_k$  as the disjoint union of  $K_{\ell}$  and  $K_k$  together with a set of edges  $M^*$  that match every vertex of  $K_k$  to a vertex of  $K_{\ell}$ .

**Lemma 3.2.** Let  $a, b \in \mathbb{N}$  with  $3 \leq a \leq b \leq 2a$ . If R[W] contains  $K_{a,b}$  as a spanning subgraph with partite sets X and Y such that  $B[Y] \subseteq K_a \boxplus K_{b-a}$ , then R[W] has a non-trivial good orientation.

*Proof.* Let  $W = X \cup Y$  where  $X = \{x_1, \ldots, x_a\}$ ,  $Y = \{y_1, \ldots, y_b\}$ . It suffices to prove that R[W] has a non-trivial good orientation when the edges of B are the union of the edges of the complete graphs on X,  $\{y_1, \ldots, y_a\}$  and  $\{y_{a+1}, \ldots, y_b\}$  together with the edges  $\{y_i y_{a+i} : i \in [b-a]\}$ .

We will provide an appropriate orientation of the red edges.

For  $i \in [a]$ , orient the edges  $x_i y_i$  as  $\overrightarrow{x_i y_i}$ . For  $i, j \in [a]$ , where  $i \neq j$ , orient the edges  $x_i y_j$  as  $\overrightarrow{y_j x_i}$ . Note that, as a > 2, this already ensures that for all  $i, j \in [a]$ ,

there is a path of length at most two from  $x_i$  to  $x_j$  and from  $y_i$  to  $y_j$ , and vertices in  $\{x_1, \ldots, x_1, y_1, \ldots, y_a\}$  have both an in-neighbor and an out-neighbor in R.

For  $i \in [b-a]$ , orient the edges  $x_i y_{a+i}$  as  $\overline{y_{a+i}x_i}$ . For  $i, j \in [b-a]$ ,  $i \neq j$ , orient the edges  $x_i y_{a+j}$  as  $\overrightarrow{x_i y_{a+j}}$ . This ensures that for all  $i, j \in [b-a]$  and  $j \in [a] \setminus \{i\}$  there is an oriented path of length at most two from  $y_{a+i}$  to  $y_{a+j}$  and from  $y_{a+i}$  to  $y_i$  (through  $x_i$ ); and all vertices of W have an in-neighbor and an out-neighbor in R.

For  $i \in [a] \setminus [b-a]$  and  $j \in [b-a]$ , orient the edges  $x_i y_{a+j}$  as  $\overrightarrow{x_i y_{a+j}}$ . This ensures that for all  $j \in [b-a]$  and  $k \in [a]$  there is an oriented path from  $y_k$  to  $y_{a+j}$  (through an  $x_\ell$  where  $\ell \in [a] \setminus \{k, j\}$ ).

Finally, for  $i, j \in [b - a]$ , with  $i \neq j$ , orient the edges  $y_{a+i}y_j$  as  $\overline{y_{a+i}y_j}$ . The resulting orientation of R[W] is non-trivially good.

**Corollary 3.1.** For a vertex set  $W \subseteq V$ , if B[W] is a disjoint union of paths and the components of B[W] can be partitioned into sets X and Y such that |X| = a and |Y| = b for some  $3 \le a \le b \le 2a$ , then R[W] has a non-trivial good orientation.

*Proof.* Let B[W] be the disjoint union of paths which can be partitioned into sets X and Y such that |X| = a and |Y| = b where  $3 \le a \le b \le 2a$ . Then R[W] has  $K_{a,b}$  as spanning subgraph with partite sets X and Y. Moreover, Y can be partitioned into two sets  $Y_a$  and  $Y_{b-a}$  of cardinality a and b-a respectively, such that B[Y] contains at most one edge joining a vertex in  $Y_a$  to a vertex in  $Y_{b-a}$ . Hence,  $B[Y] \le P_b \le K_a \boxplus K_{b-a}$ .

**Lemma 3.3.** Assume that V can be partitioned into two disjoint sets W and Z so that there is no edge in B joining a vertex in W to a vertex in Z. Furthermore, assume that R[W] has a non-trivial good orientation, and one of the following holds for Z:

(i) Z has a non-trivial good orientation, or

(ii) |Z| = 3 and the vertices in Z are isolated in B, or

(iii) |Z| = 2,

then R has an orientation of diameter 2.

*Proof.* Let  $O_W$  be a non-trivial good orientation of R[W] with a corresponding partition of W into sets  $U_1$  and  $V_1$ . We will extend it to a non-trivial good orientation of V.

Proof of (i): Let  $O_Z$  be a non-trivial good orientation of R[Z] with a corresponding partition of Z into sets  $U_2$  and  $V_2$ . We assign the orientation  $U_1 \to U_2$ ,  $U_2 \to V_1, V_1 \to V_2$ , and  $V_2 \to U_1$ . We also include  $O_W$  and  $O_Z$  in the orientation. It is easy to verify that this in indeed a non-trivial orientation of diameter 2.

Proof of (ii) and (iii): Let  $Z = \{y_1, \ldots, y_k\}$   $(k \in \{2, 3\})$ . If k = 3, orient R[Z] as  $y_1 \to y_2 \to y_3 \to y_1$ . For the remaining red edges, orient  $U_1 \to y_1$  and  $y_1 \to V_1$ , and for  $j \in [k] \setminus \{1\}$  orient  $y_j \to U_1$  and  $V_1 \to y_j$ . Orient any remaining red edges arbitrarily. It is easy to verify that this is indeed a non-trivial orientation of diameter two.

Lemma 3.4. The following graphs have an orientation of diameter two:

(1)  $\overline{Q \cup 7K_1}$ , where  $Q \in \{K_4, D_{4,2}, D_{4,1}\}$ 

(3)  $\overline{Q \cup 6K_1}$  and  $\overline{Q \cup K_2 \cup 5K_1}$ , where  $Q \in \{D_{3,3}, S_{3,3}\}$ 

(4)  $\overline{Q \cup aP_1 \cup bP_2}$ , with  $a, b \ge 0$  and a + b = 5, where  $Q \in \{D_{3,2}, C_5, D_{3,1}, K_3\}$ 

<sup>(2)</sup>  $D_{4,3} \cup 8K_1$ ,

- (5)  $\overline{aP_1 \cup bP_2 \cup cP_3 \cup dP_4}$ , with  $a, b, c, d \ge 0$  and a + b + c + d = 5.
- In particular by case (5) Theorem 1.1 holds for  $5 \le n \le 7$ .

Proof. We either directly give the orientation (for small graphs in case (5)) or find a partition of V into two disjoint sets W and Z for which the conditions of Lemma 3.3 hold. We will do the latter by exhibiting a quadruple  $(U_1, V_1, U_2, V_2)$  of subgraphs of B whose vertices partition V. This signifies that  $Z = V(U_1) \cup V(V_1)$ ,  $B[W] = U_2 \cup V_2$ , all edges between Z and W are red, R[W] has a non-trivial good orientation with partition classes  $U_2$  and  $V_2$ , and either |Z| = 2 (i.e. both  $U_1$  and  $V_1$  are the singleton  $K_1$  and  $B[Z]) \in \{K_2, 2K_1\}$ ), or |Z| = 3 and the vertices in Z are isolated in B, or R[Z] has a non-trivial good orientation with partition classes  $U_1$  and  $V_1$  (and consequently  $B[Z] = U_1 \cup V_1$ ).

The proofs of each case in the theorem follow.

- (1)  $B = Q \cup 7K_1$ , where  $Q \in \{K_4, D_{4,2}, D_{4,1}\}$ . As  $4 \le n(Q) \le 6$ , the quadruple  $(K_1, K_1, Q, 5K_1)$  gives an orientation of diameter two by Lemmata 3.1 and 3.3.
- (2)  $B = D_{4,3} \cup 8K_1$ . We use quadruple  $(K_1, K_1, 6K_1, D_{4,3})$ . Since  $6K_1$  and  $D_{4,3}$  form a partition of B into two graphs  $U_2$  and  $V_2$ , with  $n(U_2) = 6$  and  $n(V_2) = 7$ , Lemma 3.1 gives that W has a non-trivial good orientation. Since |Z| = 2, Lemma 3.3 gives a diameter two orientation of R.
- (3)  $B \in \{Q \cup 6K_1, Q \cup K_2 \cup 5K_1\}$  where  $Q \in \{D_{3,3}, S_{3,3}\}$ . In both cases quadruple  $(K_1, K_1, 4K_1, Q)$  gives the required orientation by Lemmata 3.1 and 3.3.

(4)  $B = Q \cup aP_1 \cup bP_2$ , with  $a, b \ge 0$  and a + b = 5, where  $Q \in \{D_{3,2}, C_5, D_{3,1}, K_3\}$ . Then  $Q = K_3$  or  $n(Q) \in \{4, 5\}$ . As  $\max(a, b) \ge 3$ , there are two paths of the same size. Choose a pair of such paths of minimum order i (so  $i \in \{1, 2\}$ ), and let H be the union of the remaining three paths. Clearly  $3 \le n(H) \le 6$ .

Consider the quadruple  $(P_i, P_i, H, Q)$ .

If n(Q) = n(H) = 3 or  $n(Q) \neq 3 \neq n(H)$ , then by Lemmata 3.1 and 3.3 we have the required orientation.

If  $n(H) = 3 \neq n(Q)$ , notice that  $D_{3,2} \leq K_3 \boxplus K_2$ ,  $C_5 \leq K_3 \boxplus K_2$  and  $D_{3,1} = K_3 \boxplus K_1$  and use Lemmata 3.2 and 3.3 to find an orientation of diameter two. If  $n(H) \neq 3 = n(Q)$ , then the fact that H is the disjoint union of paths gives that  $H \leq K_3 \boxplus K_{n(H)-3}$ . Lemmata 3.2 and 3.3 give the required orientation. (5)  $B = aP_1 \cup bP_2 \cup cP_3 \cup dP_4$ , with  $a, b, c, d \geq 0$  and a + b + c + d = 5.

All cases where n(G) < 8 (i.e. when  $a + 2b + 3c + 4d \le 7$ ) and the case where a = 4, b = 0, c = 0, and d = 1 were done by computer search. See Figure 1 for the orientations of these graphs.

For  $8 \leq n(G) \leq 9$  and we are not in the case  $B = P_4 \cup 4P_1$ , we will consider partitions which use Corollary 3.1 and Lemma 3.3. If  $B = P_3 \cup P_2 \cup$  $3P_1$ , consider the partition  $(K_1, K_1, 3P_1, P_3)$ . If  $B = 2P_1 \cup 3P_2$ , consider the partition  $(K_1, K_1, P_2 \cup P_1, P_2 \cup P_1)$ . If  $B = P_4 \cup P_2 \cup 3P_1$ , consider the partition  $(K_1, K_1, 3P_1, P_4)$ . If  $B = 2P_3 \cup 3P_1$ , consider the partition  $(P_1, P_1, P_3, P_3 \cup P_1)$ . If  $B = P_3 \cup 2P_2 \cup 2P_1$ , consider the partition  $(P_1, P_1, P_3, 2P_2)$ . If  $B = 4P_2 \cup P_1$ , consider the partition  $(K_1, K_1, 2P_2, 2P_2 \cup P_1)$ . This considers all cases where  $n(G) \leq 9$ .

Let  $n(G) \geq 10$ . As  $\max(a, b, c, d) \geq 2$ , we again have two paths of the same length. Let H be the union of two paths  $P_i$  of the same length where i is chosen to be minimum possible, and the remaining three paths be  $P_j, P_k, P_\ell$ where without loss of generality  $k \leq \ell \leq j$ . We have  $2i + j + k + \ell = n(G) \geq 10$ ,

so (since  $j \ge k \ge \ell$ )  $\frac{10-2i}{3} \le j \le 4$  and  $k + \ell \le 2j$ . We have two cases. CASE 1: i = 1

As  $\frac{8}{3} \leq j \leq 4$ , we have  $j \in \{3,4\}$  and  $j \leq 4 \leq 10 - j - 2 \leq k + \ell \leq 2j$ . Take the quadruple  $(P_1, P_1, P_j, P_k \cup P_\ell)$ ; Lemmata 3.2 and 3.3 give the required orientation.

CASE 2:  $i \ge 2$ 

By the definition of i we must have  $\max(k, \ell) > 1$ , so  $k + \ell \ge 3$ . If j = 2, this gives  $i = j = k = \ell = 2$  and  $G = K_{10} - M$ , which has the required orientation by using Corollary 3.1 and Lemma 3.3 with the partition  $(K_1, K_1, 2P_2, 2P_2)$ , so assume  $j \ge 3$ . Now either  $3 \le k + \ell \le j \le 4$  or  $3 \le j \le k + \ell \le 2j$  and in both cases the quadruple  $(P_i, P_i, P_j, P_k \cup P_\ell)$  with Lemmata 3.2 and 3.3 give the required orientation.

**Definition 3.3.** Let  $W \subseteq V$  such that B[W] is the union of one or more components of B. We say that W is a reducible unit if R[W] has a good orientation. We say that W is a reduction if R[W] has a non-trivial good orientation and  $ex(B[W]) \geq -1$ .

#### 4. Properties of B

From now on we assume that G is a minimal counterexample, that is, G is a graph on n vertices,  $n \ge 5$ , and at least  $\binom{n}{2} - (n-5)$  edges that has no orientation of diameter two, and among those graphs let G be a graph of minimum order and of minimum size. Clearly, if G has n vertices, then G has exactly  $\binom{n}{2} - (n-5)$  edges. Hence the corresponding graph B has order n and size n-5. Moreover,  $n \ge 8$  by Lemma 3.4.

In this section we show that a minimal counterexample cannot have a reduction. We also show that no component of B contains three independent vertices, and that no component has two independent vertices that have at least two common neighbors.

**Lemma 4.1.** Let G be a minimal counterexample. Then B has no reduction.

Proof. Suppose to the contrary that B has a reduction W. Then |W| > 2 and, by  $m(B[W]) \ge |W| - 1$ , also  $W \ne V$ . Let  $O_W$  be a non-trivial good orientation of R[W] and let  $U_1$  and  $V_1$  be the partition classes of  $O_W$ . Create  $B^*$  from B by removing the vertices of W and adding two new vertines  $u_1, v_1$  with a blue edge  $u_1v_1$  connecting them. As B[W] is a union of components of B, B contains no edges joining vertices in W to vertices in V - W. Then  $n(B^*) = n + 2 - |W| < n$ and since  $m(B[W]) \ge |W| - 1$ ,

$$1 \le m(B^{\star}) = (n-5) - m(B[W]) + 1 \le n-3 - |W| < n(B^{\star}) - 5.$$

In particular,  $5 < n(B^*)$ . Since B was a minimal a counterexample, the red graph  $R^*$  corresponding to  $B^*$  has an orientation  $O^*$  of diameter 2.

We now make use of  $O_W$  and  $O^*$  to obtain an orientation  $O_R$  of diameter 2 of R. Let  $x, y \in V$ . If  $x, y \in W$  then orient xy as in  $O_W$ . If  $x, y \in V - W$  then orient xy as in  $R^*$ . The remaining edges, joining a vertex in  $x \in V - W$  to a vertex in  $y \in W$  are oriented as follows. If  $xu_1$  has received the orientation  $\overline{xu_1}$  in  $O^*$  then we orient  $x \to U_1$ , and if  $xu_1$  has received the orientation  $\overline{u_1x}$  in  $O^*$  then we orient

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 $U_1 \to x$ . Similarly, if  $xv_1$  has received the orientation  $\overrightarrow{xv_1}$  in  $O^*$  then we orient  $x \to V_1$ , and if  $xv_1$  has received the orientation  $\overrightarrow{v_1x}$  in  $O^*$  then we orient  $V_1 \to x$ .

If  $x, y \in V$ , then either both vertices are in the same set  $U_1$  (or  $V_1$ ), in which case there is a path of length at most two in  $O_W$ , or they are in different sets, for example  $x \in U_1$  and  $y \in V_1$ , in which case the  $(u_1, v_1)$ -path in  $O^*$  gives rise to an (x, y)-path in  $O_R$ .

**Lemma 4.2.** Let G be a minimal counterexample. If X is an independent set of order 3 in B, and  $N_i$  is the set of vertices in  $v \in V - X$  having exactly i neighbors (in B) in X, then

$$(1) |N_2| \le 1 \quad and \quad N_3 = \emptyset.$$

*Proof.* Suppose that  $X = \{x_1, x_2, x_3\}$  is an independent set B such that (1) does not hold. Create a new blue graph  $B^*$  by identifying the vertices of X to a new vertex x and removing multiple edges. Then  $n(B^*) = n - 2 \ge 5$  and

$$m(B^{\star}) = m(B) - |N_2| - 2|N_3| \le m(B) - 2 = n - 7 = n(B^{\star}) - 5.$$

Therefore, since G is a minimal counterexample, the red graph  $R^*$  corresponding to  $B^*$  has an orientation  $O^*$  of diameter 2.

We will now orient R. Orient every edge uv with  $u, v \notin \{x_1, x_2, x_3\}$  as in  $O^*$ . Orient R[X] as  $\overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}$  and  $\overrightarrow{x_3x_1}$ . If an edge ux is present in  $R^*$ , then all edges  $ux_i$ , i = 1, 2, 3 are present in R, and depending on whether ux is oriented as  $\overrightarrow{ux}$  or as  $\overrightarrow{xu}$  in  $O^*$ , we orient them  $u \to \{x_1, x_2, x_3\}$  or  $\{x_1, x_2, x_3\} \to u$ . Orient any remaining edges in R arbitrarily. to obtain the orientation  $O_R$ 

Let  $u, v \in V(G)$ . If  $u, v \in \{x_1, x_2, x_3\}$ , then clearly there exists a (u, v)-path of length at most two in  $O_R$ . If  $u \in X$  and  $v \in V - X$  or vice versa then the (x, v)-path of length at most two in  $O^*$  gives rise to a (u, v)-path of the same length in  $O_R$ . If  $u, v \in V - X$  then the (u, v)-path of length at most two in  $O^*$  gives rise to a (u, v)-path of the same length in  $O_R$ . This shows that  $O_R$  is an orientation of R of diameter 2, a contradiction to G being a counterexample.

**Lemma 4.3.** Let G be a minimal counterexample. Then no component of B has three independent vertices.

*Proof.* Suppose to the contrary that B has a component which contains three independent vertices  $x_1$ ,  $x_2$  and  $x_3$ . We may assume that

(2) 
$$d_B(x_1, \{x_2, x_3\}) = 2.$$

Indeed, if  $d_B(x_1, \{x_2, x_3\}) \ge 3$  then let  $x'_1$  be a vertex on a shortest path in B from  $x_1$  to  $\{x_2, x_3\}$  that is at distance two from  $\{x_2, x_3\}$ . The new set  $\{x'_1, x_2, x_3\}$  is independent and satisfies (2).

By (2) we may assume, possibly after renaming vertices, that  $d_B(x_1, x_2) = 2$ . A similar argument as above now yields that we can choose  $x_3$  such that also

$$d_B(x_3, \{x_1, x_2\}) = 2$$

Hence we can choose  $\{x_1, x_2, x_3\}$  such that it contains at least two pairs of vertices at distance two in *B*. Hence, possibly after renaming the vertices, we have

(3) 
$$d_B(x_1, x_2) = d_B(x_2, x_3) = 2.$$

Now (3) implies that there exists a common neighbor  $y_{12}$  of  $x_1$  and  $x_2$ , and a common neighbor  $y_{23}$  of  $x_2$  and  $x_3$  in B. If  $y_{12} = y_{23}$ , then the set  $N_3$  of vertices

with exactly three neighbors in  $\{x_1, x_2, x_3\}$  contains  $y_{12}$  and is thus not empty, a contradiction to Lemma 4.2. If  $y_{12} \neq y_{23}$ , then the set  $N_2$  of vertices with exactly two neighbors in  $\{x_1, x_2, x_3\}$  contains  $y_{12}$  and  $y_{23}$ , again a contradiction to Lemma 4.2.

**Lemma 4.4.** Let G be a minimal counterexample. If  $x_1, x_2$  are independent vertices in B, then  $x_1$  and  $x_2$  have at most one common blue neighbor.

*Proof.* Suppose to the contrary that B has two vertices  $x_1$  and  $x_2$  that share at least two neighbors. Then  $x_1$  and  $x_2$  are in the same component of B. As n(B) > 1 and m(B) < n(B) - 1, B is not connected. Choose a vertex  $x_3$  from another component. Then  $x_1, x_2, x_3$  are independent vertices, for which the set  $N_2$  of vertices having exactly two neighbors in  $\{x_1, x_2, x_3\}$  has at least two elements, a contradiction to Lemma 4.2.

#### 5. On tree components of B

Since B has n vertices and n-5 edges, B is not connected. In this section we give useful lower bounds on the number of components of B that are trees, and we show that for a given order t we can find a union  $F_t$  of tree components of B whose order is close to t and excess is at most -t. This will be useful in finding reductions and further restricting the possible structure of B for a minimal counterexample. Recall that the *excess* of a graph H is defined as ex(H) = m(H) - n(H).

**Lemma 5.1.** If B contains a component  $B_1$  that is not a tree, then B has at least  $ex(B_1) + 5 \ge 5$  components that are trees. If B has only tree components, it has exactly five components.

*Proof.* Let  $T_1, T_2, \ldots, T_k$  be the components of B that are trees, and  $B_1, B_2, \ldots, B_\ell$  be the components that are not trees. Then  $ex(T_i) = -1$  for all  $i \in \{1, 2, \ldots, k\}$  and  $ex(B_i) \ge 0$  for all  $i \in \{1, 2, \ldots, \ell\}$ . Since m(B) = n - 5, we have ex(B) = -5, and so

$$-5 = \exp(B) = \sum_{i=1}^{k} \exp(T_i) + \sum_{i=1}^{\ell} \exp(B_i) = -k + \sum_{i=1}^{\ell} \exp(B_i)$$

If B has no tree component (i.e.  $\ell = 0$ ), this gives k = 5. Hence, B has exactly five components. If B contains a component that is not a tree,  $B_1$  say, then this yields

$$-5 = -k + \sum_{i=1}^{\ell} \exp(B_i) \ge -k + \exp(B_1),$$

and so  $k \ge 5 + ex(B_1) \ge 5$ , as claimed.

**Lemma 5.2.** Assume B contains at least t tree components whose size does not exceed  $m_0$ . Then there exists  $t_0$  with  $t \le t_0 \le t + m_0$  such that some subset of the tree components in B forms a forest  $F_t$  satisfying  $n(F_t) = t_0$  and  $ex(F_t) \ge -t$ . If B contains a tree of size  $m_0$  where  $t > m_0$ , then we can choose  $F_t$  such that  $ex(F_t) \ge -t + m_0$ .

*Proof.* Let  $T_1, T_2, \ldots, T_t$  be the *t* largest tree components of *B* whose size does not exceed  $m_0$ . Clearly  $T_1 \cup T_2 \cup \cdots \cup T_t$  contains at least *t* vertices. Let *j* be the smallest positive integer such that  $T_1 \cup T_2 \cup \cdots \cup T_j$  contains *t* or more vertices. Let  $F_t = T_1 \cup T_2 \cup \cdots \cup T_j$  and let  $t_0 = n(F_t)$ . Since  $T_j$  has size at most  $m_0$  and thus

order at most  $m_0 + 1$ , we have  $t \le t_0 \le t + m_0$ . Moreover, since  $T_1 \cup T_2 \cup \cdots \cup T_{j-1}$  has less than t vertices, it follows that  $T_j$  has at least  $t_0 - t + 1$  vertices and at least  $t_0 - t$  edges. Hence  $m(F_t) \ge m(T_j) \ge t_0 - t$ , and thus  $\exp(F_t) \ge -t$ .

If  $t > m_0$ , we have that  $j \ge 2$  and  $T_1$  has size  $m_0$ . The same argument as above yields that  $m(F_t) \ge m(T_1) + m(T_j) = m_0 + t_0 - t$  and thus  $ex(F_t) \ge -t + m_0$ , as desired.

### 6. Describing the components of B

In this section further restrict the structure of B in a minimal counterexample. We show that each component of B is either a path on at most four vertices, a complete graph, a proper dumbbell, a proper short dumbbell, or a 5-cycle, and none of these components have order more than six.

**Lemma 6.1.** Let G be a minimal counterexample and  $B_1$  a component of B.

- (a) If  $B_1$  is a tree, then  $B_1$  is a path  $P_i$  with  $1 \le i \le 4$ .
- (b) If  $B_1$  is not a tree, then  $B_1$  is one of the following:
  - (i) a complete graph  $K_i$  with  $i \geq 3$ ,
  - (ii) a proper dumbbell,
  - (iii) a proper short dumbbell, or
  - (iv) a 5-cycle.

*Proof.* As any tree that is not a path on at most 4 vertices is not a complete graph, a dumbbell or a short dumbbell, it is enough to show that  $B_1$  is a complete graph, a dumbbell, a short dumbbell or a 5 cycle.

If  $B_1$  is complete, then the lemma holds, so assume that  $B_1$  is not complete. Let  $x_1$  and  $x_2$  be two vertices of  $B_1$  with  $d_B(x_1, x_2) = \operatorname{diam}(B_1) \ge 2$ . By Lemma 4.3  $B_1$  does not have three independent vertices, so  $d_B(x_1, x_2) = \operatorname{diam}(B_1) \le 3$  and  $V(B_1) = N_B(x_1) \cup N_B(x_2)$ , and  $|N_B(x_1) \cap N_B(x_2)| \le 1$  by Lemma 4.4.

CASE 1: diam $(B_1) = 3$  (consequently  $N_B(x_1) \cap N_B(x_2) = \emptyset$ ).

Since  $B_1$  does not have three independent vertices by Lemma 4.3, we conclude that each  $N_B[x_i]$  forms a clique.

Since  $B_1$  is connected,  $B_1$  has an edge joining a vertex  $y_1 \in N_B(x_1)$  to a vertex  $y_2 \in N_B(x_2)$ . We show that  $B_1$  does not contain a further edge joining a vertex  $z_1 \in N_B(x_1)$  to a vertex  $z_2 \in N_B(x_2)$  by using that Lemma 4.4 gives that two independent vertices share at most one neighbor. Indeed, if  $y_1 = z_1$ , then  $\{y_1, x_2\}$  would be a set of two independent vertices that share two neighbors. If  $y_2 = z_2$ , then  $\{y_2, x_1\}$  would be a set of two independent vertices that share two neighbors. Lastly, if  $y_1 \neq z_1$  and  $y_2 \neq z_2$ , then  $\{y_1, z_2\}$  would be a set of two independent vertices that share two neighbors.

CASE 2: diam $(B_1) = 2$  (consequently  $N_B[x_1] \cap N_B[x_2] = \{y\}$ ). We consider two subcases:

If  $\min(\deg_B(x_1), \deg_B(x_2)) = 1$ , then without loss of generality  $\deg_B(x_1) = 1$  and  $N_B(x_1) = \{y\}$ . Since  $\dim(B_1) = 2$ , every vertex in  $V(B_1) - \{x_1, y\}$  is adjacent to y in  $B_1$ . Since  $B_1$  does not contain three independent vertices,  $V(B_1) - \{x_1, y\}$  induces a complete graph in  $B_1$ . Therefore  $B_1$  is a short dumbbell.

If  $\min(\deg_B(x_1), \deg_B(x_2)) \ge 2$ , then, since  $B_1$  does not contain three independent vertices,  $N_B[x_i] \setminus \{y\}$  induces a complete graph in B for  $i \in \{1, 2\}$ . If y is adjacent to all vertices in  $B_1$ , then  $B_1$  is a short dumbbell and we are done. Assume without

loss of generality that there is a vertex  $z_1 \in N_B[x_1]$  to which y is non-adjacent in  $B_1$ . Then  $d_B(z_1, x_2) = 2$ , so  $z_1$  and  $x_2$  have a common blue neighbor  $z_2$ . Since  $x_1$  and  $z_2$  are non-adjacent in B and thus cannot have two common neighbors,  $z_2$  and y are non-adjacent in B. Since also the edges  $x_1x_2$ ,  $x_1z_2$  and  $x_2z_1$  are not present in B, we conclude that  $x_1, y, x_2, z_2, z_1, x_1$  form an induced 5-cycle in  $B_1$ . Hence  $B_1$  contains an induced 5-cycle.

Rename the vertices of the 5-cycle so the cycle is  $v_0v_1v_2v_3v_4v_0$ . Suppose there is a sixth vertex w adjacent to a vertex in  $\{v_0, v_1, v_2, v_3, v_4\}$  in  $B_1$ . If  $|N_B(w) \cap$  $\{v_0, \ldots, v_4\}| \leq 2$ , it is easy to see that v together with two suitably chosen vertices in  $\{v_0, v_1, v_2, v_3, v_4\}$  forms an independent set of cardinality three, which is impossible. Hence v is adjacent to at least three vertices in  $\{v_0, v_1, v_2, v_3, v_4\}$ . But then v has two neighbors among these vertices that are not adjacent, without loss of generality  $v_1$  and  $v_3$ , so that  $v_1$  and  $v_3$  are non-adjacent vertices with two common neighbors, a contradiction to Lemma 4.4. This proves that  $B_1$  contains only  $\{v_0, v_1, v_2, v_3, v_4\}$ , and so  $B_1$  is a 5-cycle.

**Lemma 6.2.** In a minimal counterexample all components of B are of order at most six.

*Proof.* Suppose to the contrary that B contains a component  $B_1$  with more than six vertices. Let  $n_1 \ge 7$  and  $m_1$  be the order and size, respectively, of  $B_1$ . By Lemma 6.1,  $B_1$  is a complete graph, a dumbbell, or a short dumbbell. It is easy to see that among all such graphs of order  $n_1$  the dumbbell  $D_{\lceil n_1/2 \rceil, \lfloor n_1/2 \rfloor}$  has minimum size, and every other graph has bigger size. A simple calculation shows that

(4) 
$$m_1 \ge m(D_{\lceil n_1/2 \rceil, \lfloor n_1/2 \rfloor}) \ge \left\lceil \frac{1}{4}n_1^2 - \frac{1}{2}n_1 + 1 \right\rceil,$$

and consequently

(5) 
$$\operatorname{ex}(B_1) = m_1 - n_1 \ge \left\lceil \frac{(n_1 - 3)^2 - 5}{4} \right\rceil,$$

where equality holds only when  $B = D_{\lceil n_1/2 \rceil, \lfloor n_1/2 \rfloor}$ .

Assume first that  $B_1 \neq D_{3,4}$ . If  $n_1 \geq 8$ , equation (5) easily gives  $\operatorname{ex}(B_1) \geq n_1 - 3$ . If  $n_1 = 7$ , then, as the lower bound in (5) is only sharp when  $B_1 = D_{3,4}$ , it follows that  $\operatorname{ex}(B_1) \geq 4 = n_1 - 3$ . By Lemma 5.1, B contains at least  $\operatorname{ex}(B_1) + 5 \geq n_1 + 2$  tree components. Set  $t = n_1 - 2$ . By Lemma 5.2, for some  $t_0$  with  $n_1 - 1 \leq t_0 \leq n_1 + 2$ , B contains a forest  $F_t$  of order  $t_0$  and excess at least  $-t = -n_1 + 2$  that is the union of the tree components of B. Let  $W := V(B_1) \cup V(F_t)$ . We show that W is a reduction. Clearly the graph R[W] contains a spanning subgraph  $K_{n_1,t_0}$ . Since  $n_1 - 1 \leq t_0 \leq n_1 + 2$ , it is easy to verify that either  $n_1 \leq t_0 \leq \binom{n_1}{2}$  or  $t_0 < n_1 \leq \binom{t_0}{2}$ . So R[W] has a non-trivial good orientation by Lemma 3.1, and  $\operatorname{ex}(B[W]) = \operatorname{ex}(B_1) + \operatorname{ex}(F_{n_1-2}) \geq n_1 - 3 + (-n_1 + 2) = -1$ . Hence, W is a reduction, a contradiction to Lemma 4.1. So we must have that  $B_1 = D_{3,4}$ 

If  $B_1 = D_{3,4}$ ,  $ex(B_1) = 3$  by equation (5), and *B* has at least 8 tree components. Set  $m_0$  be the size of the largest tree component. If  $1 \le m_0$ , Lemma 5.2 with t = 5 and  $1 \le m_0 \le 3$  yields that there exists a forest  $F_5$  in *B* of order  $t_0$ , where  $5 \le t_0 \le 8$ , and excess at least -5 + 1 = -4. Let  $W = V(B_1) \cup V(F_5)$ , then  $ex(B[W]) = ex(B_1) + ex(F_5) \ge 3 + (-4) = -1$ , and R[W] has a non-trivial good orientation by Lemma 3.1. Hence *W* is a reduction, a contradiction to Lemma 4.1. So all tree components of *B* are singletons. If all *k* components of  $B - B_1$  are  $P_1$ , then  $-5 = \exp(B) = 3 - k$  gives  $B = D_{3,4} \cup 8P_1$ . But by Lemma 3.4,  $\overline{D_{3,4} \cup 8P_1}$  has an orientation of diameter two, which is a contradiction. Therefore B contains another non-tree component  $B_2$  with at least one edge, so it has at least 3 (and by our proof so far, at most 7) vertices. Set  $W = B_1 \cup B_2 \cup 2P_1$ . By Lemma 3.1, W has a non-trivial good orientation with partition classes  $B_1$  and  $B_2 \cup 2P_1$  and  $\exp(B[W]) \geq 3 - 2 > 0$ . So W is a reduction, which is a contradiction.

**Lemma 6.3.** If a minimal counterexample B contains a component  $B_1$  that is not a tree, then  $B - B_1$  has exactly  $ex(B_1) + 5$  components, all of which are trees.

Proof. Suppose to the contrary that B contains two non-tree components  $B_1$  and  $B_2$  with  $3 \le n(B_1) \le n(B_1)$ . Then  $\operatorname{ex}(B_1) \ge 0$  and  $\operatorname{ex}(B_2) \ge 0$ , and by Lemma 6.2  $n(B_1) \le 6$ . If  $n(B_1) = n(B_2) = 3$  or  $n(B_1), n(B_2) \in \{4, 5, 6\}$ , then  $V(B_1) \cup V(B_2)$  has a non-trivial good orientation by Lemma 3.1 and is thus a reduction, since  $\operatorname{ex}(B_1 \cup B_2) = \operatorname{ex}(B_1) + \operatorname{ex}(B_2) \ge 0$ . So we have  $n_1 \in \{4, 5, 6\}$  and  $n_2 = 3$ . As  $V(B_1) \cup V(P_4)$  or  $V(B_2) \cup V(P_3)$  would form a reduction, all tree components in B are  $P_1$  or  $P_2$ . Since  $V(B_1) \cup V(B_2) \cup V(P_i)$  forms a reduction for  $i \in \{1, 2\}$ , this is a contradiction to Lemma 4.1. Hence all k components of  $B - B_1$  are trees. As  $-5 = \operatorname{ex}(B) = \operatorname{ex}(B_1) - k$  we are done.  $\Box$ 

**Lemma 6.4.** Assume B contains a non-tree component  $B_1$ . Let F be a forest that is the union of the smallest number of tree components of B such that  $\min(4, n(B_1)) \leq n(F) \leq 6$  and  $k_0$  be the number of tree components that make up F. Then  $k_0 \geq \exp(B_1) + 2$ ,  $\exp(B_1) \leq 2$ , and the tree components of B contain at most  $\min(3, n(B_1) - 1)$  vertices.

*Proof.* If  $B_1$  is a component that is not a tree, by Lemma 6.2  $ex(B_1) \ge 0$  and  $3 \le n(B_1) \le 6$ . B does not contain a  $P_4$  component, otherwise  $W = V(B_1) \cup V(P_4)$  would form a reduction by Lemma 3.1 and  $ex(B[W]) \ge -1$ .

Let F and  $k_0$  be given as in the conditions of the lemma. Clearly,  $k_0 \leq 4$  and  $ex(F) = -k_0$ . Consider  $W = V(B_1) \cup V(F)$ . If  $n(B_1) \neq 3$  or  $n(B_1) = n(F)$ , then R[W] has a non-trivial good orientation by Lemma 3.1. If  $n(B_1) = 3$  and  $4 \leq n(F) \leq 6$ , then  $B_1 = K_3$ ,  $F \leq K_3 \boxplus K_{n(F)-3}$ , and R[W] has a non-trivial good orientation by Lemma 3.2. As W is not a reduction, we must have  $-2 \geq ex(B[W]) = ex(B_1) - k$ , giving  $k \geq ex(B_1) + 2$ .  $ex(B_1) \leq 2$  follows from  $k \leq 4$ . As  $k \geq 2$ , no tree component has size  $n(B_1)$ .

#### 7. Proof of the main result

We start by eliminating the possibility of a non-tree component from a minimal counterexample.

**Lemma 7.1.** In a minimal counterexample no component of B is a complete graph on three or more vertices.

*Proof.* Suppose to the contrary that B contains a component  $B_1$  that is a complete graph of order  $n_1 \geq 3$ . By Lemma 6.4 we have  $ex(B_1) \leq 2$  and consequently  $n_1 \in \{3, 4\}$ .

If  $B_1 = K_4$ , then  $ex(B_1) = 2$  and B contains exactly 7 tree components by Lemma 6.3. By Lemma 6.4 all tree components must be  $P_1$  (otherwise  $k_0 < 4$  in the lemma, which is a contradiction). By Lemma 3.4, the graph  $\overline{K_4 \cup 7K_1}$  has an orientation

of diameter two, so it is not a counterexample, which is a contradiction.

If  $B_1 = K_3$ , then  $ex(B_1) = 0$  and B contains exactly 5 tree components by Lemma 6.3. By Lemma 6.4 all these tree components must be  $P_1$  or  $P_2$ , so we have  $B = K_3 \cup aK_1 \cup bK_2$  for some nonnegative integers a, b with a+b=5. But by Lemma 3.4 all such graphs have an orientation of diameter two. So G is not a counterexample, a contradiction.

#### **Lemma 7.2.** In a minimal counterexample no component of B is a proper dumbbell.

*Proof.* Assume that  $B_1$  is a component of B that is a proper dumbbell; then  $n(B_1) \geq 4$ , and by Lemmata 6.2 and 6.4 we have  $n(B_1) \leq 6$  and  $ex(B_1) \leq 2$ , and  $B - B_1$  has no  $P_4$  component. Hence  $B_1 \in \{D_{3,1}, D_{4,1}, D_{3,2}, D_{4,2}, D_{3,3}\}$ . By Lemma 6.3,  $B - B_1$  has exactly  $ex(B_1) + 5$  other components that are all paths on at most three vertices. We will examine each case grouped by  $ex(B_1)$ .

- (1)  $B_1 \in \{D_{4,1}, D_{4,2}\}$ . Then  $ex(B_1) = 2$  and  $B B_1$  has exactly 7 tree components by Lemma 6.3. By Lemma 3.4, the graph  $\overline{B_1 \cup 7K_1}$  has an orientation of diameter two, therefore not all tree components of B are singletons. We get  $k_0 \leq 3$  and a contradiction in Lemma 6.4.
- (2)  $B_1 \in \{D_{3,1}, D_{3,2}\}$ . Then  $ex(B_1) = 0$  and  $B B_1$  has exactly five tree components by Lemma 6.3. Lemma 3.4 gives that  $\overline{B_1 \cup aK_1 \cup bK_2}$  has a diameter two orientation for all a + b = 5, so at least one of the tree components is a  $P_3$ . For  $j \in \{1, 2\}, D_{3,j} \leq K_3 \boxplus K_j$ , and by Lemma 3.2  $V(P_3) \cup V(B_1)$  is a reduction, which is again a contradiction.
- (3)  $B_1 = D_{3,3}$ . Then  $ex(B_1) = 1$ . By Lemma 6.3,  $B B_1$  contains exactly 6 components which are trees. By Lemma 3.4, the graphs  $\overline{D_{3,3} \cup 6K_1}$  and  $\overline{D_{3,3} \cup K_2 \cup 5K_1}$  have an orientation of diameter two. Hence  $B B_1$  contains a  $P_3$  or two components that are  $P_2$ . We get  $k_0 \leq 2$  and a contradiction in Lemma 6.4.

# **Lemma 7.3.** In a minimal counterexample no component of B is a proper short dumbbell.

*Proof.* Assume that  $B_1$  is a component of B that is a proper short dumbbell. Then  $5 \leq n(B_1)$ . By Lemmata 6.2 and 6.4,  $n(B_1) \leq 6$ ,  $ex(B_1) \leq 2$ , and no tree component of B is a  $P_4$ . This gives that  $B_1 = S_{3,3}$ ,  $ex(B_1) = 1$ , and  $B - B_1$  has exactly 6 tree components. By Lemma 3.4, both  $\overline{S_{3,3} \cup 6K_1}$  and  $\overline{S_{3,3} \cup K_2 \cup 5K_1}$ have diameter two orientations, so the components of B include at least two  $P_2$  or at least one  $P_3$ . This gives  $k_0 = 2$  and a contradiction in Lemma 6.4.

## Lemma 7.4. In a minimal counterexample no component of B is a 5-cycle.

*Proof.* Assume that  $B_1$  is a component of B that is a 5-cycle. Then  $ex(B_1) = 0$  and, by Lemmata 6.3 and 6.4,  $B - B_1$  has exactly 5 components which are trees on at most three vertices. By Lemma 3.4,  $\overline{C_5 \cup aP_2 \cup bP_1}$  has an orientation of diameter two for all non-negative integers a, b with a + b = 5, so at least one of these tree components is a  $P_3$ . As  $P_3 \leq K_3$  and  $C_5 \leq K_3 \boxplus K_2$ , by Lemma 3.2  $B_1 \cup P_3$  forms a reduction, contradicting Lemma 4.1.

We are now ready to complete the proof of Theorem 1.1.

*Proof.* Suppose to the contrary that Theorem 1.1 is false. Let G be a minimal counterexample, that is a graph of minimum order and minimum size for which the theorem does not hold. By Lemma 3.4,  $n(G) \ge 8$  and consequently m(G) = n(G)-5. By Lemma 6.1, every component of B that is not a tree is either a complete graph on at least three vertices, a proper dumbbell, a proper short dumbbell, or a 5-cycle. By Lemmata 7.1, 7.2, 7.3, and 7.4, all components of B must be trees, and by Lemmata 5.1 and 6.1  $B = aP_1 \cup bP_2 \cup cP_3 \cup dP_4$  for some a + b + c + d = 5. But then Lemma 3.4 gives that G has a diameter two orientation, a contradicton.

#### 8. Open Problem

In Theorem 1.1, we show that in graph of given order n we need at least  $\binom{n}{2} - n + 5$  edges to guarantee the existence of an orientation of diameter two. It is natural to ask the same question for any given value of d: In a graph of order n, over all bridgeless graphs, how many edges do we need at least to guarantee the existence of an orientation of diameter at most d?

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GARNER COCHRAN, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BERRY COL-LEGE, 2277 MARTHA BERRY HWY NW, MT BERRY GA 30149, USA *E-mail address:* gcochran@berry.edu

ÉVA CZABARKA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA SC 29212, USA AND VISITING PROFESSOR, DEPARTMENT OF PURE AND APPLIED MATHEMATICS, UNIVERSITY OF JOHANNESBURG, SOUTH AFRICA

 $E\text{-}mail \ address: \verb"czabarka@math.sc.edu"$ 

Peter Dankelmann, Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

E-mail address: pdankelmann@uj.ac.za

László Székely, Department of Mathematics, University of South Carolina, Columbia SC 29212, USA and Visiting Professor, Department of Pure and Applied Mathematics, University of Johannesburg, South Africa

 $E\text{-}mail\ address:\ \texttt{szekelyQmath.sc.edu}$