

Rainbow triangles in arc-colored tournaments ^{*}

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Abstract

Let T_n be an arc-colored tournament of order n . The maximum monochromatic indegree $\Delta^{-mon}(T_n)$ (resp. outdegree $\Delta^{+mon}(T_n)$) of T_n is the maximum number of in-arcs (resp. out-arcs) of a same color incident to a vertex of T_n . The irregularity $i(T_n)$ of T_n is the maximum difference between the indegree and outdegree of a vertex of T_n . A subdigraph H of an arc-colored digraph D is called rainbow if each pair of arcs in H have distinct colors. In this paper, we show that each vertex v in an arc-colored tournament T_n with $\Delta^{-mon}(T_n) \leq \Delta^{+mon}(T_n)$ is contained in at least $\frac{\delta(v)(n-\delta(v)-i(T_n))}{2} - [\Delta^{-mon}(T_n)(n-1) + \Delta^{+mon}(T_n)d^+(v)]$ rainbow triangles, where $\delta(v) = \min\{d^+(v), d^-(v)\}$. We also give some maximum monochromatic degree conditions for T_n to contain rainbow triangles, and to contain rainbow triangles passing through a given vertex. Finally, we present some examples showing that some of the conditions in our results are best possible.

Keywords: arc-colored tournament, rainbow triangle, maximum monochromatic indegree (outdegree), irregularity

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1 Introduction

In this paper we only consider finite and simple graphs and digraphs, i.e. without loops or multiple edges (arcs). A cycle in a digraph always means a *directed cycle*. We use [6] and [8] for terminology and notations not defined here.

Let $G = (V(G), E(G))$ be a graph. An *edge-coloring* of G is a mapping $C : E(G) \rightarrow \mathbb{N}$, where \mathbb{N} is the set of natural numbers. We call G an *edge-colored graph*, if it has an edge-coloring. We use $C(G)$ to denote the set of colors appearing on the edges of G . The *maximum monochromatic degree* $\Delta^{mon}(G)$ of G is the maximum number of edges of a same color incident to a vertex of G . For a vertex v of G , the *color degree* $d_G^c(v)$ of v is the number of colors assigned on the edges incident to v . The *minimum color degree* $\delta^c(G)$ is the minimum $d_G^c(v)$ over all vertices v of G . A subgraph H of G is called *rainbow* if all edges of H have distinct colors.

The existence of rainbow subgraphs has been widely studied, readers can see the survey papers [14, 20]. In particular, the existence of rainbow triangles attracts much attention during the past decades. For an edge-colored complete graph K_n , Gallai [16] and Fox et al. [11] characterized the coloring structure of K_n without containing rainbow triangles. Balogh et al. [5] obtained the maximum number of rainbow triangles in 3-edge-colored complete graphs. Gyarfas and Simonyi [17] proved that each edge-colored K_n with $\Delta^{mon}(K_n) < \frac{2n}{5}$ contains a rainbow triangle and this bound is tight. Fujita et al. [13] proved that each edge-colored K_n with $\delta^c(K_n) > \log_2 n$ contains a rainbow triangle and this bound is tight. For a general edge-colored graph G of order n , Li and Wang [23] proved that if $\delta^c(G) \geq \frac{\sqrt{7}+1}{6}n$, then G has a rainbow triangle. Li [22] and Li et al. [21] improved the condition to $\delta^c(G) > \frac{n}{2}$ independently, and showed that this bound is tight. Li et al. [24] proved that if G is an edge-colored graph of order n satisfying $d^c(u) + d^c(v) \geq n + 1$ for every edge $uv \in E(G)$, then it contains a rainbow triangle. Li et al. [21] proved that if G is an edge-colored graph of order n with $|E(G)| + |C(G)| \geq \frac{n(n+1)}{2}$, then it contains a rainbow triangle. Fujita et al. [15] characterized all graphs G satisfying $|E(G)| + |C(G)| \geq \frac{n(n+1)}{2} - 1$ but containing no rainbow triangles. Ehard et al. [9] proved that if G is an edge-colored graph of order n with $|E(G)| + |C(G)| \geq \frac{n(n+1)}{2} + k - 1$, then it contains at least k rainbow triangles. Hoppen et al. [19] characterized the graphs with the largest number of edge-colorings avoiding a rainbow triangle. Aharoni et al. [1] determined the maximum number of edges in an n -vertex edge-colored graph where all color classes have size at most k and there is no rainbow triangle. Jin et al. [25] studied rainbow triangles in edge-colored Kneser graphs. For more results on rainbow cycles, we

recommend [2, 10, 12, 18].

Motivated by the fruitful results on the existence of rainbow triangles in undirected graphs, we propose the problem on the existence of rainbow triangles in digraphs. Before proceeding, we first give some terminology and notations in digraphs.

Let $D = (V(D), A(D))$ be a digraph. If $uv \in A(D)$, then we say that u *dominates* v (or v is *dominated* by u). For a vertex v of D , the *in-neighborhood* $N_D^-(v)$ of v is the set of vertices dominating v , and the *out-neighborhood* $N_D^+(v)$ of v is the set of vertices dominated by v . The *indegree* $d_D^-(v)$ and *outdegree* $d_D^+(v)$ of v are the cardinality of $N_D^-(v)$ and $N_D^+(v)$, respectively. Let $\delta_D(v) = \min\{d_D^-(v), d_D^+(v)\}$. The *maximum indegree* $\Delta^-(D)$ (resp. *maximum outdegree* $\Delta^+(D)$) and the *minimum indegree* $\delta^-(D)$ (resp. *minimum outdegree* $\delta^+(D)$) of D , is the maximum $d_D^-(v)$ (resp. $d_D^+(v)$) and the minimum $d_D^-(v)$ (resp. $d_D^+(v)$) over all vertices v of D , respectively. The digraph D is *strongly connected* if for every pair of distinct vertices u, v in D , there exists a (u, v) -path. The subdigraph of D induced by $S \subseteq V(D)$ is denoted by $D[S]$. For two disjoint subsets X and Y of $V(D)$, we use $A_D(X, Y)$ to denote the set of arcs from X to Y . An *arc-coloring* of D is a mapping $C: A(D) \rightarrow \mathbb{N}$, where \mathbb{N} is the natural number set. We call D an *arc-colored* digraph if it has an arc-coloring. We use $C(D)$ and $c(D)$ to denote the set and the number of colors appearing on the arcs of D , respectively. For a nonempty subset S of $V(D)$, the *maximum monochromatic indegree* (resp. *maximum monochromatic outdegree*) of S , denoted by $\Delta_D^{-mon}(S)$ (resp. $\Delta_D^{+mon}(S)$), is the maximum number of in-arcs (resp. out-arcs) of a same color incident to a vertex $v \in S$. We write $\Delta^{-mon}(D)$ for $\Delta_D^{-mon}(V(D))$ and $\Delta^{+mon}(D)$ for $\Delta_D^{+mon}(V(D))$, respectively. If there is no ambiguity, we often omit the subscript D in the above notations. When a set contains only one element s , we often write s instead of $\{s\}$. A digraph is called *rainbow* if all of its arcs have distinct colors. A digraph, which is not rainbow, is called *non-rainbow*.

A *tournament* is a digraph such that each pair of vertices are joined by precisely one arc. We use T_n to denote a tournament of order n . The *irregularity* $i(T_n)$ of T_n is the maximum difference between the indegree and outdegree of a vertex. A tournament T_n is said to be *regular* if $i(T_n) = 0$, and *almost regular* if $i(T_n) = 1$. Since $d^+(v) + d^-(v) = n - 1$ for each vertex $v \in V(T_n)$, we have $\max\{\Delta^-(T_n), \Delta^+(T_n)\} \leq \frac{n+i(T_n)-1}{2}$ and $\min\{\delta^-(T_n), \delta^+(T_n)\} \geq \frac{n-i(T_n)-1}{2}$.

For research on arc-colored tournaments, see [3, 4, 7]. In this paper we focus on the existence and enumeration of rainbow triangles in strongly connected arc-colored tournaments. By symmetry, throughout this paper we can assume that $\Delta^{-mon}(T_n) \leq \Delta^{+mon}(T_n)$.

We first consider the number of rainbow triangles passing through a given vertex in an arc-colored tournament.

Theorem 1. *Let T_n be a strongly connected arc-colored tournament with $\Delta^{-\text{mon}}(T_n) \leq \Delta^{+\text{mon}}(T_n)$. Then for each vertex v of T_n , the number of rainbow triangles containing v is at least*

$$\frac{\delta(v)(n - \delta(v) - i(T_n))}{2} - [\Delta^{-\text{mon}}(T_n)(n - 1) + \Delta^{+\text{mon}}(T_n)d^+(v)].$$

Remark 1. The bound in Theorem 1 is tight. Let T_n be an arc-colored tournament of order $n = 4k - 1 + 2i(T_n)$, where k is a positive integer. Let v be a vertex of T_n with $d^+(v) = d^-(v) = 2k - 1 + i(T_n)$, $W = N^+(v) = \{w_1, w_2, \dots, w_{2k-1+i(T_n)}\}$ and $U = N^-(v) = \{u_1, u_2, \dots, u_{2k-1+i(T_n)}\}$. Let $T_n[W]$ and $T_n[U]$ be two regular tournaments of order $2k - 1 + i(T_n)$. For each vertex $w_j \in W$, let $N^+(w_j) \cap U = \{u_j, \dots, u_{j+k-1}\}$. Then T_n is strongly connected. For each w_j , $1 \leq j \leq 2k - 1 + i(T_n)$, let $C(vw_j) = j$, $C(u_jv) = j$, $C(w_ju_{j+1}) = j$ and $C(w_ju_{j+2}) = j + 2$. Finally, color the remaining arcs with distinct new colors. Then $\Delta^{+\text{mon}}(T_n) = \Delta^{-\text{mon}}(T_n) = 1$, and the number of rainbow triangles containing v is

$$(k-3)(2k - 1 + i(T_n)) = \frac{\delta(v)(n - \delta(v) - i(T_n))}{2} - [\Delta^{-\text{mon}}(T_n)(n - 1) + \Delta^{+\text{mon}}(T_n)d^+(v)].$$

Based on Theorem 1, we give some maximum monochromatic degree conditions so that every vertex in T_n is contained in a rainbow triangle.

Theorem 2. *Let T_n be a strongly connected arc-colored tournament with $\Delta^{-\text{mon}}(T_n) \leq \Delta^{+\text{mon}}(T_n)$ and*

$$2\Delta^{-\text{mon}}(T_n) + \Delta^{+\text{mon}}(T_n) \leq \begin{cases} \frac{(n - 1 - i(T_n))(n + 1 - i(T_n))}{4(n - 1 + i(T_n))}, & 1 \leq i(T_n) < \frac{n + 3}{3}; \\ \frac{n - 2i(T_n)}{4}, & \text{otherwise.} \end{cases}$$

Then every vertex of T_n is contained in a rainbow triangle

Remark 2. When $i(T_n) = 0$ or $\frac{n+3}{3} \leq i(T_n) \leq n - 3$, the bounds in Theorem 2 are tight. When $1 \leq i(T_n) < \frac{n+3}{3}$, the gap between the bound in Theorem 2 and the best possible bound is at most 1 (See Section 3 for examples).

We also investigate the existence of rainbow triangles in T_n .

Theorem 3. *Let T_n be a strongly connected arc-colored tournament with $\Delta^{-\text{mon}}(T_n) \leq \Delta^{+\text{mon}}(T_n)$. If n is odd and $\Delta^{-\text{mon}}(T_n) < \frac{n^2 + n - 3i(T_n)^2}{12n}$, or n is even and $\Delta^{-\text{mon}}(T_n) < \frac{n^2 - 1 - 3i(T_n)^2}{12(n-1)}$, then there exists a rainbow triangle in T_n .*

Remark 3. The bounds in Theorem 3 may not be tight. However, for regular tournaments the best possible bound can not be larger than $\frac{n}{11}$ (See Section 3 for examples).

We prove Theorems 1, 2 and 3 in Section 2. In Section 3, we give some examples to analyze the tightness of the bounds in Theorems 2 and 3. In Section 4, we propose some further research problems.

2 The proofs

Let v be a vertex in an arc-colored tournament T_n . We use $n(\vec{C}_3, v)$ to denote the number of triangles containing v , and $n(R\vec{C}_3, v)$ and $n(NR\vec{C}_3, v)$ to denote the number of rainbow and non-rainbow triangles containing v , respectively. Then

$$n(R\vec{C}_3, v) = n(\vec{C}_3, v) - n(NR\vec{C}_3, v).$$

So by estimating the minimum number of triangles and the maximum number of non-rainbow triangles passing through the vertex v , we will get a lower bound for $n(\vec{C}_3, v)$.

Lemma 1. *Let T_n be a strongly connected tournament of order n and irregularity $i(T_n)$. Then for each vertex v of T_n , we have*

$$n(\vec{C}_3, v) \geq \frac{\delta(v)(n - \delta(v) - i(T_n))}{2}.$$

Proof. Let v be a vertex of T_n . Let $W = N^+(v)$ and $U = N^-(v)$. Then $V(T_n) = W \cup U \cup \{v\}$ and $n(\vec{C}_3, v) = |A(W, U)|$. By the definition of irregularity, we have

$$-|W|i(T_n) \leq \sum_{w \in W} d^+(w) - \sum_{w \in W} d^-(w) = |A(W, U)| - |A(U, W)| - |W| \leq |W|i(T_n).$$

Note that

$$|A(W, U)| + |A(U, W)| = |W||U| = |W|(n - 1 - |W|).$$

We have

$$|A(W, U)| \geq \frac{|W|(n - |W| - i(T_n))}{2} = \frac{d^+(v)(n - d^+(v) - i(T_n))}{2}. \quad (1)$$

Similarly, since

$$-|U|i(T_n) \leq \sum_{u \in U} d^+(u) - \sum_{u \in U} d^-(u) = |A(U, W)| + |U| - |A(W, U)| \leq |U|i(T_n),$$

we have

$$|A(W, U)| \geq \frac{(n - 1 - |W|)(|W| + 1 - i(T_n))}{2} = \frac{d^-(v)(n - d^-(v) - i(T_n))}{2}. \quad (2)$$

Combining Inequalities (1) and (2), we get

$$|A(W, U)| \geq \frac{1}{2} \max \{d^+(v)(n - d^+(v) - i(T_n)), d^-(v)(n - d^-(v) - i(T_n))\}.$$

By easy calculation, we can get

$$n(\vec{C}_3, v) = |A(W, U)| \geq \frac{\delta(v)(n - \delta(v) - i(T_n))}{2}.$$

This completes the proof of Lemma 1. \square

Next we will estimate the maximum number of non-rainbow triangles passing through a given vertex v . A triangle with vertex set $\{v, w, u\}$ and arc set $\{vw, wu, uv\}$ is denoted by T_{vwu} .

Lemma 2. *Let T_n be a strongly connected arc-colored tournament of order n . Then for each vertex v of T_n , we have*

$$n(NR\vec{C}_3, v) \leq \Delta^{-mon}(T_n)(n - 1) + \Delta^{+mon}(T_n)d^+(v).$$

Proof. Let v be a vertex of T_n and set $W = N^+(v)$ and $U = N^-(v)$. Define $W_i = \{w \in W | C(vw) = i\}$ and $U_j = \{u \in U | C(uv) = j\}$. Then $\sum_{i \in C(T_n)} |W_i| = d^+(v)$ and $\sum_{j \in C(T_n)} |U_j| = d^-(v)$. For a triangle T_{vwu} , if it is not rainbow, then it must belong to at least one of the sets $S_1(v) = \{T_{vwu} | C(vw) = C(wu)\}$, $S_2(v) = \{T_{vwu} | C(wu) = C(uv)\}$ and $S_3(v) = \{T_{vwu} | C(uv) = C(vw)\}$. Denote the cardinality of $S_i(v)$ by $t_i(v)$, $i = 1, 2, 3$. Then

$$n(NR\vec{C}_3, v) \leq t_1(v) + t_2(v) + t_3(v).$$

Let $X_i = \{wu \in A(W_i, U) : C(wu) = i\}$. Then we have

$$t_1(v) = \sum_{i \in C(T_n)} |X_i| \leq \sum_{i \in C(T_n)} |W_i| \Delta^{+mon}(T_n) = \Delta^{+mon}(T_n)d^+(v).$$

Let $Y_j = \{wu \in A(W, U_j) : C(wu) = j\}$. Then we have

$$t_2(v) = \sum_{j \in C(T_n)} |Y_j| \leq \sum_{j \in C(T_n)} |U_j| \Delta^{-mon}(T_n) = \Delta^{-mon}(T_n)d^-(v).$$

It is not hard to see that

$$t_3(v) \leq \sum_{k \in C(T_n)} |W_k| |U_k|.$$

Note that $|U_k| \leq \Delta^{-mon}(T_n)$ for all $k \in C(T_n)$. We have

$$\sum_{k \in C(T_n)} |W_k| |U_k| \leq \Delta^{-mon}(T_n) \sum_{k \in C(T_n)} |W_k| = \Delta^{-mon}(T_n)d^+(v).$$

Then

$$\begin{aligned}
n(NR\vec{C}_3, v) &\leq t_1(v) + t_2(v) + t_3(v) \\
&\leq \Delta^{+mon}(T_n)d^+(v) + \Delta^{-mon}(T_n)d^-(v) + \Delta^{-mon}(T_n)d^+(v) \\
&= \Delta^{-mon}(T_n)(n-1) + \Delta^{+mon}(T_n)d^+(v).
\end{aligned}$$

This completes the proof of Lemma 2. \square

Proof of Theorem 1. Since $n(R\vec{C}_3, v) = n(\vec{C}_3, v) - n(NR\vec{C}_3, v)$, Theorem 1 follows immediately from Lemmas 1 and 2. \square

Note that if $n(R\vec{C}_3, v) > 0$, then v is contained in a rainbow triangle.

Proof of Theorem 2. Let v be a vertex of T_n . The lower bound of $n(\vec{C}_3, v)$ in Lemma 1 is related to $\delta(v)$. Now, we will give a new lower bound of $n(\vec{C}_3, v)$ without using $\delta(v)$.

Claim 1.

$$n(\vec{C}_3, v) \geq \begin{cases} \frac{n(n-2)}{8}, & \text{if } i(T_n) = 1; \\ \frac{(n-1)(n+1-2i(T_n))}{8}, & \text{otherwise.} \end{cases}$$

Proof. Let $f(\delta(v)) = \frac{\delta(v)(n-\delta(v)-i(T_n))}{2}$. Since $f'(\delta(v)) = \frac{n-2\delta(v)-i(T_n)}{2}$, we can see that $f(\delta(v))$ increases when $\delta(v) \leq \frac{n-i(T_n)}{2}$ and decreases when $\delta(v) \geq \frac{n-i(T_n)}{2}$. Note that $\frac{n-1-i(T_n)}{2} \leq \delta(v) \leq \frac{n-1}{2}$. So the minimum value of $f(\delta(v))$ can only be obtained when $\delta(v) = \frac{n-1-i(T_n)}{2}$ or $\frac{n-1}{2}$. Comparing $f(\frac{n-1-i(T_n)}{2})$ and $f(\frac{n-1}{2})$, we can prove Claim 1. \square

We divide the rest of the proof into three cases according to the irregularity of T_n .

Case 1. $i(T_n) = 0$.

In this case, T_n is a regular tournament. Since $2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) \leq \frac{n}{4}$, by Lemma 2 and Claim 1, we have

$$n(NR\vec{C}_3, v) \leq \Delta^{-mon}(T_n)(n-1) + \Delta^{+mon}(T_n)d^+(v) \leq \frac{n(n-1)}{8} < \frac{n^2-1}{8} \leq n(\vec{C}_3, v).$$

Thus, $n(R\vec{C}_3, v) > 0$, namely, v is contained in a rainbow triangle.

Case 2. $\frac{n+3}{3} \leq i(T_n) \leq n-3$.

If $d^+(v) < \frac{n-1}{2}$, then by Lemma 2 we have

$$\begin{aligned}
n(NR\vec{C}_3, v) &\leq \Delta^{-mon}(T_n)(n-1) + \Delta^{+mon}(T_n)d^+(v) \\
&\leq \frac{(n-1)(n-2i(T_n))}{8} \\
&< \frac{(n-1)(n+1-2i(T_n))}{8} \leq n(\vec{C}_3, v).
\end{aligned}$$

So v is contained in a rainbow triangle.

If $d^+(v) \geq \frac{n-1}{2}$, then $d^+(v) = n - 1 - \delta(v)$. By Lemma 2 we have

$$\begin{aligned} n(NRC_3^{\vec{v}}, v) &\leq \Delta^{-mon}(T_n)(n-1) + \Delta^{+mon}(T_n)d^+(v) \\ &\leq d^+(v)(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)). \end{aligned}$$

Now we will show

$$d^+(v)(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)) < \frac{\delta(v)(n - \delta(v) - i(T_n))}{2}.$$

It suffices to prove that

$$\frac{\delta(v)(n - \delta(v) - i(T_n))}{n - 1 - \delta(v)} > 4\Delta^{-mon}(T_n) + 2\Delta^{+mon}(T_n).$$

Define

$$g(\delta(v)) = \frac{\delta(v)(n - \delta(v) - i(T_n))}{n - 1 - \delta(v)}.$$

Since

$$g''(\delta(v)) = -\frac{2(n-1)(i(T_n)-1)}{(n-1-\delta(v))^3} < 0,$$

we can see that $g(\delta(v))$ is convex when $\frac{n-1-i(T_n)}{2} \leq \delta(v) \leq \frac{n-1}{2}$. So the minimum value of $g(\delta(v))$ can only be obtained when $\delta(v) = \frac{n-1-i(T_n)}{2}$ or $\frac{n-1}{2}$. Comparing $g(\frac{n-1-i(T_n)}{2})$ and $g(\frac{n-1}{2})$, we have

$$\min_{\frac{n-1-i(T_n)}{2} \leq \delta(v) \leq \frac{n-1}{2}} g(\delta(v)) = g\left(\frac{n-1}{2}\right) = \frac{n+1-2i(T_n)}{2},$$

and thus

$$2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) \leq \frac{n-2i(T_n)}{4} < \frac{n+1-2i(T_n)}{4} \leq \frac{\delta(v)(n - \delta(v) - i(T_n))}{2(n-1-\delta(v))}.$$

Therefore,

$$n(NRC_3^{\vec{v}}, v) \leq d^+(v)(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)) < \frac{\delta(v)(n - \delta(v) - i(T_n))}{2} \leq n(\vec{C}_3, v).$$

This implies that v is contained in a rainbow triangle.

Case 3. $1 \leq i(T_n) < \frac{n+3}{3}$.

The proof of Case 3 is similar to that of Case 2. Note that in this case we have

$$\frac{(n-1-i(T_n))(n+1-i(T_n))}{2(n-1+i(T_n))} < \frac{n+1-2i(T_n)}{2}.$$

If $d^+(v) \leq \frac{n-1}{2}$, then we have

$$\begin{aligned} n(NRC_3^{\vec{}} , v) &\leq \frac{n-1}{2}(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)) \\ &\leq \frac{n-1}{2} \frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} \\ &< \frac{(n-1)(n+1-2i(T_n))}{8} \leq n(\vec{C}_3, v) \text{ (for } 1 < i(T_n) < \frac{n+3}{3}\text{)} \end{aligned}$$

and

$$\begin{aligned} n(NRC_3^{\vec{}} , v) &\leq \frac{n-1}{2}(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)) \\ &\leq \frac{n-1}{2} \frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} \\ &= \frac{(n-1)(n-2)}{8} < \frac{n(n-2)}{8} \leq n(\vec{C}_3, v) \text{ (for } i(T_n) = 1\text{)}. \end{aligned}$$

This implies that v is contained in a rainbow triangle.

If $d^+(v) > \frac{n-1}{2}$, then we have

$$n(NRC_3^{\vec{}} , v) < d^+(v)(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)).$$

Since $i(T_n) < \frac{n+3}{3}$, we have

$$\min_{\frac{n-1-i(T_n)}{2} \leq \delta(v) \leq \frac{n-1}{2}} g(\delta(v)) = g\left(\frac{n-1-i(T_n)}{2}\right) = \frac{(n-1-i(T_n))(n+1-i(T_n))}{2(n-1+i(T_n))}$$

and

$$2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) \leq \frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} \leq \frac{\delta(v)(n-\delta(v)-i(T_n))}{2(n-1-\delta(v))}.$$

Thus,

$$n(NRC_3^{\vec{}} , v) < d^+(v)(2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)) \leq \frac{\delta(v)(n-\delta(v)-i(T_n))}{2} \leq n(\vec{C}_3, v).$$

This implies that v is contained in a rainbow triangle, completing the proof of Theorem 2. \square

To prove Theorem 3, we will estimate the minimum number of triangles and the maximum number of non-rainbow triangles in T_n , respectively. We use $n(\vec{C}_3)$, $n(RC_3^{\vec{}})$ and $n(NRC_3^{\vec{}})$ to denote the number of triangles, rainbow triangles and non-rainbow triangles in T_n , respectively. Here we give Lemmas 3 and 4.

Lemma 3. *Let T_n be a strongly connected tournament of order n and irregularity $i(T_n)$.*

If n is odd, then

$$n(\vec{C}_3) \geq \frac{(n-1)(n^2 + n - 3i(T_n)^2)}{24}.$$

If n is even, then

$$n(\vec{C}_3) \geq \frac{n(n^2 - 1 - 3i(T_n)^2)}{24}.$$

Proof. Let v be a vertex of T_n and set $W = N^+(v)$ and $U = N^-(v)$. Then $n(\vec{C}_3, v) = |A(W, U)|$. We can partition the out-neighborhood of a vertex $w \in W$ into two parts. Let $P_1(w) = N^+(w) \cap W$ and $P_2(w) = N^+(w) \cap U$. Then $d^+(w) = |P_1(w)| + |P_2(w)|$. Thus,

$$\begin{aligned} |A(W, U)| &= \sum_{w \in W} |P_2(w)| = \sum_{w \in W} d^+(w) - \sum_{w \in W} |P_1(w)| \\ &= \sum_{w \in N^+(v)} d^+(w) - \frac{d^+(v)(d^+(v) - 1)}{2}. \end{aligned}$$

Since a triangle contains three vertices, we have

$$n(\vec{C}_3) = \frac{1}{3} \sum_{v \in V(T_n)} n(\vec{C}_3, v) = \frac{1}{3} \sum_{v \in V(T_n)} \left(\sum_{w \in N^+(v)} d^+(w) - \frac{d^+(v)(d^+(v) - 1)}{2} \right).$$

For each vertex w , we can see that w is an out-neighbor of exactly $d^-(w)$ vertices. So

$$\begin{aligned} & \sum_{v \in V(T_n)} \left(\sum_{w \in N^+(v)} d^+(w) - \frac{d^+(v)(d^+(v) - 1)}{2} \right) \\ &= \sum_{v \in V(T_n)} \left(d^+(v)d^-(v) - \frac{(d^+(v))^2}{2} + \frac{d^+(v)}{2} \right) \\ &= \frac{n(n-1)(2n-1)}{4} - \frac{3}{2} \sum_{v \in V(T_n)} (d^+(v))^2. \end{aligned}$$

It suffices to calculate

$$\begin{aligned} & \max \sum_{i=1}^n (d^+(v_i))^2 \\ \text{s.t. } & \begin{cases} \frac{n-1-i(T_n)}{2} \leq d^+(v_i) \leq \frac{n-1+i(T_n)}{2}; \\ \sum_{i=1}^n d^+(v_i) = \frac{n(n-1)}{2}. \end{cases} \end{aligned}$$

We claim that if $\sum_{i=1}^n (d^+(v_i))^2$ attains the maximum value then the number of vertices with outdegree $\frac{n-1+i(T_n)}{2}$ is maximum, and subject to this, the number of vertices with outdegree $\frac{n-1-i(T_n)}{2}$ is maximum. In other words, there exist no two vertices x, y with $\frac{n-1-i(T_n)}{2} < d^+(x) \leq d^+(y) < \frac{n-1+i(T_n)}{2}$. Otherwise, we can get a larger $\sum_{i=1}^n (d^+(v_i))^2$ by changing $d^+(x), d^+(y)$ to $d^+(x) - 1, d^+(y) + 1$. If n is odd, then

$$\begin{aligned} \max \sum_{v \in V(T_n)} (d^+(v))^2 &= \frac{n-1}{2} \left(\frac{n-1+i(T_n)}{2} \right)^2 + \left(\frac{n-1}{2} \right)^2 + \frac{n-1}{2} \left(\frac{n-1-i(T_n)}{2} \right)^2 \\ &= \frac{(n-1) [(n-1)^2 + i(T_n)^2 + n-1]}{4}. \end{aligned}$$

So we have

$$n(\vec{C}_3) \geq \frac{(n-1)(n^2 + n - 3i(T_n)^2)}{24}.$$

If n is even, then

$$\begin{aligned} \max_{v \in V(T_n)} \sum (d^+(v))^2 &= \frac{n}{2} \left(\frac{n-1+i(T_n)}{2} \right)^2 + \frac{n}{2} \left(\frac{n-1-i(T_n)}{2} \right)^2 \\ &= \frac{n [(n-1)^2 + i(T_n)^2]}{4}. \end{aligned}$$

So we have

$$n(\vec{C}_3) \geq \frac{n(n^2 - 1 - 3i(T_n)^2)}{24},$$

completing the proof. \square

Lemma 4. *Let T_n be a strongly connected arc-colored tournament of order n . Then*

$$n(NRC_3^{\vec{}}) \leq \frac{n(n-1)\Delta^{-mon}(T_n)}{2}.$$

Proof. Let P be the set of all monochromatic directed path of length 2 in T_n . For each non-rainbow triangle, there must be a monochromatic directed path of length 2 in it. For each two distinct non-rainbow triangles, the corresponding monochromatic directed paths of length 2 are also distinct. So we have $n(NRC_3^{\vec{}}) \leq |P|$. Let v be a vertex of T_n . Let $W = N^+(v)$ and $U = N^-(v)$. Define $W_i = \{w \in W | C(vw) = i\}$ and $U_j = \{u \in U | C(uv) = j\}$. Then $\sum_{i \in C(T_n)} |W_i| = d^+(v)$ and $\sum_{j \in C(T_n)} |U_j| = d^-(v)$. The number of monochromatic directed paths of length 2 which contain v as the center is

$$\sum_{k \in C(T_n)} |W_k||U_k|.$$

By the proof of Lemma 2, we know that

$$\sum_{k \in C(T_n)} |W_k||U_k| \leq \Delta^{-mon}(T_n)d^+(v).$$

Thus,

$$n(NRC_3^{\vec{}}) \leq |P| \leq \sum_{v \in V(T_n)} \Delta^{-mon}(T_n)d^+(v) = \frac{n(n-1)\Delta^{-mon}(T_n)}{2}.$$

The proof is complete. \square

Proof of Theorem 3. If $n(NRC_3^{\vec{}}) < n(\vec{C}_3)$, then there must be a rainbow triangle in T_n . So Theorem 3 follows from Lemmas 3 and 4 immediately. \square

3 Tightness analysis of the bounds in Theorems 2 and 3

In this section, we will give some examples to analyze the tightness of the bounds in Theorems 2 and 3. Examples 1, 2 and 3 are for Theorem 2. Examples 4 and 5 are for Theorem 3.

Before we give the examples, we will prove an easy but useful result first.

Theorem 4. *If T_n is a regular tournament, then T_n must be strongly connected.*

Proof. Let T_n be a regular tournament. Then for every vertex v of T_n , we have $d^+(v) = d^-(v)$. By contradiction, assume that T_n is not strongly connected. Then the vertex set of T_n can be partitioned into two nonempty subsets V_1 and V_2 , such that all arcs between V_1 and V_2 have the same direction. Without loss of generality, we can assume that $|A(V_1, V_2)| = |V_1||V_2|$ and $|A(V_2, V_1)| = 0$. Since T_n is regular, we have

$$\sum_{v \in V_1} d^+(v) = \sum_{v \in V_1} d^-(v).$$

Note that

$$\begin{aligned} \sum_{v \in V_1} d^+(v) &= \sum_{v \in V_1} d_{T_n[V_1]}^+(v) + |A(V_1, V_2)|, \\ \sum_{v \in V_1} d^-(v) &= \sum_{v \in V_1} d_{T_n[V_1]}^-(v) + |A(V_2, V_1)| \end{aligned}$$

and

$$\sum_{v \in V_1} d_{T_n[V_1]}^+(v) = \sum_{v \in V_1} d_{T_n[V_1]}^-(v).$$

So we have $|A(V_1, V_2)| = |A(V_2, V_1)|$, a contradiction. \square

Example 1. This example shows that the upper bound $\frac{n-2i(T_n)}{4}$ in Theorem 2 is tight, for $i(T_n) = 0$ or $\frac{n+3}{3} \leq i(T_n) \leq \frac{n-3}{2}$.

We construct a tournament T_n with $n = 4m - 1$ vertices. Let v be a vertex of T_n with $d^+(v) = d^-(v) = 2m - 1$, $W = N^+(v) = \{w_1, w_2, \dots, w_{2m-1}\}$ and $U = N^-(v) = \{u_1, u_2, \dots, u_{2m-1}\}$. Let $T_n[W]$ and $T_n[U]$ be two regular tournaments. For each vertex $w_j \in W$, let $N^+(w_j) \cap U = \{u_j, \dots, u_{j+k-1}\}$ (indices are taken modulo $2m - 1$), where $1 \leq k \leq m = \frac{n+1}{4}$. By Theorem 4, we can see that T_n is strongly connected.

Now let us see the irregularity of T_n . For vertex v , we have $|d^+(v) - d^-(v)| = 0$. For every vertex $w \in W$, we have $d^+(w) = m - 1 + k$ and $d^-(w) = m - 1 + 1 + 2m - 1 - k = m - 1 + 2m - k$. Since $k \leq m$, we have $|d^+(w) - d^-(w)| = 2m - 2k$. Similarly, for every vertex $u \in U$, we have $d^+(u) = m - 1 + 2m - k$, $d^-(u) = m - 1 + k$ and $|d^+(u) - d^-(u)| = 2m - 2k$. So, $i(T_n) = 2m - 2k$. Since $1 \leq k \leq m = \frac{n+1}{4}$, we have $0 \leq i(T_n) \leq \frac{n-3}{2}$.

Next, we will assign colors to the arcs of T_n . For $1 \leq j \leq 2m - 1$, let $C(vw_j) = j$, $C(u_jv) = j$, $C(w_ju_{j+1}) = j+1$ and $C(w_ju_p) = j$ for $j+2 \leq p \leq j+k-1$. Finally, color the remaining arcs with distinct new colors. Then $\Delta^{+mon}(T_n) = k - 2$ and $\Delta^{-mon}(T_n) = 1$. Thus,

$$2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) = k = \frac{2m - i(T_n)}{2} = \frac{n + 1 - 2i(T_n)}{4},$$

but there is no rainbow triangle containing v .

Example 2. This example shows that if $i(T_n) \geq \frac{n-1}{2}$, then even the condition $\Delta^{+mon}(T_n) = \Delta^{-mon}(T_n) = 1$ can not guarantee every vertex in T_n is contained in a rainbow triangle.

We construct a tournament T_n with n vertices. Let v be a vertex of T_n with $d^+(v) = x$ and $d^-(v) = n - 1 - x$, where $\frac{n+1}{2} \leq x \leq n - 2$, x and $n - 1 - x$ are odd integers. Let $W = N^+(v) = \{w_1, w_2, \dots, w_x\}$ and $U = N^-(v) = \{u_1, u_2, \dots, u_{n-1-x}\}$. Let $T_n[W]$ and $T_n[U]$ be two regular tournaments. For a vertex $w_j \in W$, $1 \leq j \leq n - 1 - x$, let $N^+(w_j) \cap U = \{u_j\}$. For a vertex $w_i \in W$, $n - x \leq i \leq x$, let $N^+(w_j) \cap U = \emptyset$. By Theorem 4, we can see that T_n is strongly connected.

Now let us see the irregularity of T_n . For vertex v , since $\frac{n+1}{2} \leq x$, we have $|d^+(v) - d^-(v)| = 2x - n + 1$. For vertex $w_j \in W$, $1 \leq j \leq n - 1 - x$, we have $d^+(w_j) = \frac{x-1}{2} + 1$ and $d^-(w_j) = \frac{x-1}{2} + 1 + n - 1 - x - 1 = \frac{x-1}{2} + n - x - 1$. Since $x \leq n - 2$, we have $|d^+(w_j) - d^-(w_j)| = n - 2 - x$. Similarly, for vertex $w_i \in W$, $n - x \leq i \leq x$, we have $d^+(w_i) = \frac{x-1}{2}$, $d^-(w_i) = \frac{x-1}{2} + n - x$ and $|d^+(w_i) - d^-(w_i)| = n - x$. For every vertex $u \in U$, we have $d^+(u) = x + \frac{n-x-2}{2}$, $d^-(u) = 1 + \frac{n-x-2}{2}$ and $|d^+(u) - d^-(u)| = x - 1$. So, $i(T_n) = \max\{2x - n + 1, n - 2 - x, n - x, x - 1\}$. Since $\frac{n+1}{2} \leq x \leq n - 2$, we have $2x - n + 1 \leq x - 1$ and $n - x \leq \frac{n-1}{2} \leq x - 1$. So, $i(T_n) = x - 1 \geq \frac{n-1}{2}$.

Next, we will assign colors to the arcs of T_n . For $1 \leq j \leq x$ and $1 \leq i \leq n - 1 - x$, let $C(vw_j) = j$ and $C(u_i v) = i$. Finally, color the remaining arcs with distinct new colors. Then $\Delta^{+mon}(T_n) = \Delta^{-mon}(T_n) = 1$, but there is no rainbow triangle containing v .

Example 3. For $1 \leq i(T_n) < \frac{n+3}{3}$, denote $\lfloor \frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} \rfloor = m$. Let T_n be a tournament of order $n = 4k - 1 + i(T_n)$, where k is a positive integer, v be a vertex of T_n with $d^+(v) = 2k - 1 + i(T_n)$ and $d^-(v) = 2k - 1$, $W = N^+(v) = \{w_1, w_2, \dots, w_{2k-1+i(T_n)}\}$ and $U = N^-(v) = \{u_1, u_2, \dots, u_{2k-1}\}$. If $i(T_n)$ is even, then let $T_n[W]$ be a regular tournament, otherwise let $T_n[W]$ be an almost regular tournament. Let $T_n[U]$ be a regular tournament. Since $i(T_n) < \frac{n+3}{3}$, we have $3i(T_n) < n + 3 = 4k + 2 + i(T_n)$, namely, $i(T_n) \leq 2k$.

Case 1. $i(T_n) \leq 2k - 1$ and $(2k - 1)(m + 1) + i(T_n)m \geq k(2k - 1)$.

We can construct a tournament with $2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) = m + 1$ to show the bound is tight. For w_j , $1 \leq j \leq 2k - 1$, let $N^+(w_j) \cap U = \{u_j, \dots, u_{j+m}\}$, and for w_{2k-1+j} , $1 \leq j \leq i(T_n)$, let $N^+(w_{2k-1+j}) \cap U = \{u_j, u_{j+m+1}, \dots, u_{j+2m-1}\}$. Since $k \geq 1$

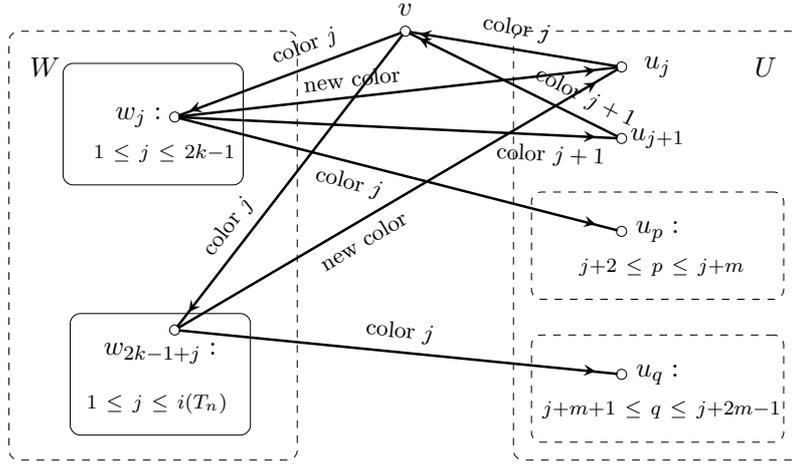


Figure 1: A digraph described in case 1 of Example 3.

and $i(T_n) \geq 1$, we have

$$\begin{aligned}
2m - 1 - (2k - 1) = 2m - 2k &\leq \frac{(n - 1 - i(T_n))(n + 1 - i(T_n))}{2(n - 1 + i(T_n))} - 2k \\
&= \frac{2k(4k - 2) - 2k(4k - 2 + 2i(T_n))}{4k - 2 + 2i(T_n)} \\
&= \frac{-4ki(T_n)}{4k - 2 + 2i(T_n)} \\
&< 0.
\end{aligned}$$

Namely, $2m - 1 < 2k - 1$. So

$$(N^+(w_j) \cap U) \cap (N^+(w_{2k-1+j}) \cap U) = \{u_j\},$$

and elements in $\{u_j, u_{j+m+1}, \dots, u_{j+2m-1}\}$ are pairwise distinct. We can see that T_n is strongly connected. Let $C(vw_j) = j$, $C(u_jv) = j$, $C(w_ju_{j+1}) = j + 1$ and $C(w_ju_p) = j$ for $1 \leq j \leq 2k - 1$ and $j + 2 \leq p \leq j + m$. Let $C(vw_{2k-1+j}) = j$, and $C(w_{2k-1+j}u_q) = j$ for $1 \leq j \leq i(T_n)$ and $j + m + 1 \leq q \leq j + 2m - 1$. Finally, color the remaining arcs with distinct new colors (See Figure 1). Then $\Delta^{+mon}(T_n) = m - 1$ and $\Delta^{-mon}(T_n) = 1$. Thus,

$$2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) = m + 1 = \lceil \frac{(n - 1 - i(T_n))(n + 1 - i(T_n))}{4(n - 1 + i(T_n))} \rceil,$$

but there is no rainbow triangle containing v (indices are taken modulo $2k - 1$). So the bound is tight in this case.

Note that if $i(T_n) = 1$, then $\frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} = \frac{n-2}{4} = \frac{4k-2}{4}$. Thus, $m = \lfloor \frac{4k-2}{4} \rfloor = k - 1$ and $(2k - 1)(m + 1) + i(T_n)m = k(2k - 1) + k - 1 > k(2k - 1)$.

Case 2. $i(T_n) = 2k$ or $(2k - 1)(m + 1) + i(T_n)m < k(2k - 1)$.

By the above argument, in this case we have $i(T_n) \geq 2$. We can construct a tournament with $2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) = m + 2$ and a vertex not contained in any rainbow

triangles. Since

$$\frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} < m+1$$

and

$$\begin{aligned} (2k-1+i(T_n))\frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))} &= \frac{k(4k-2)(2k-1+i(T_n))(4k-2)}{4k-2+2i(T_n)} \\ &= k(2k-1), \end{aligned}$$

we have $k(2k-1) < (2k-1+i(T_n))(m+1)$. Namely, $(2k-1+i(T_n))(m+1) - 1 \geq k(2k-1)$. For w_j , $1 \leq j \leq 2k-1$, let $N^+(w_j) \cap U = \{u_j, \dots, u_{j+m}\}$, and for w_{2k-1+j} , $1 \leq j \leq \min\{i(T_n), 2k-1\}$, let $N^+(w_{2k-1+j}) \cap U = \{u_j, u_{j+m+1}, \dots, u_{j+2m}\}$. If $i(T_n) = 2k$, let $N^+(w_{4k-1}) \cap U = \{u_1, \dots, u_m\}$. Since $k \geq 1$ and $i(T_n) \geq 2$, we have

$$\begin{aligned} 2m - (2k-1) = 2m - 2k + 1 &\leq \frac{(n-1-i(T_n))(n+1-i(T_n))}{2(n-1+i(T_n))} - 2k + 1 \\ &= \frac{2k(4k-2) - 2k(4k-2+2i(T_n)) + 4k-2+2i(T_n)}{4k-2+2i(T_n)} \\ &= \frac{4k-2+2i(T_n) - 4ki(T_n)}{4k-2+2i(T_n)} \\ &= \frac{(4k-2)(1-i(T_n))}{4k-2+2i(T_n)} \\ &< 0. \end{aligned}$$

Namely, $2m < 2k-1$. So

$$(N^+(w_j) \cap U) \cap (N^+(w_{2k-1+j}) \cap U) = \{u_j\}.$$

and elements in $\{u_j, u_{j+m+1}, \dots, u_{j+2m}\}$ are pairwise distinct. We can see that T_n is strongly connected. Let $C(vw_j) = j$, $C(u_jv) = j$, $C(w_ju_{j+1}) = j+1$ and $C(w_ju_p) = j$, for $1 \leq j \leq 2k-1$ and $j+2 \leq p \leq j+m$. Let $C(vw_{2k-1+j}) = j$, and $C(w_{2k-1+j}u_q) = j$, for $1 \leq j \leq \min\{i(T_n), 2k-1\}$ and $j+m+1 \leq q \leq j+2m$. If $i(T_n) = 2k$, let $C(vw_{4k-1}) = 2k$ and $C(w_{4k-1}u_s) = 2k$, for $1 \leq s \leq m$. Finally, color the remaining arcs with distinct new colors (See Figure 2). Then $\Delta^{+mon}(T_n) = m$ and $\Delta^{-mon}(T_n) = 1$. Thus,

$$2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n) = m+2,$$

but there is no rainbow triangle containing v (indices are taken modulo $2k-1$).

From this example we can see that the best bound of $2\Delta^{-mon}(T_n) + \Delta^{+mon}(T_n)$ is at most $m+1$. Namely, the gap between the bound in Theorem 2 in this case and the best possible bound is at most 1.

Example 4. This example shows that there is an arc-colored tournament T_n , in which n is odd and $\Delta^{-mon}(T_n) = \frac{n^2+n-3i(T_n)^2}{12n}$, but there is no rainbow triangle in T_n .

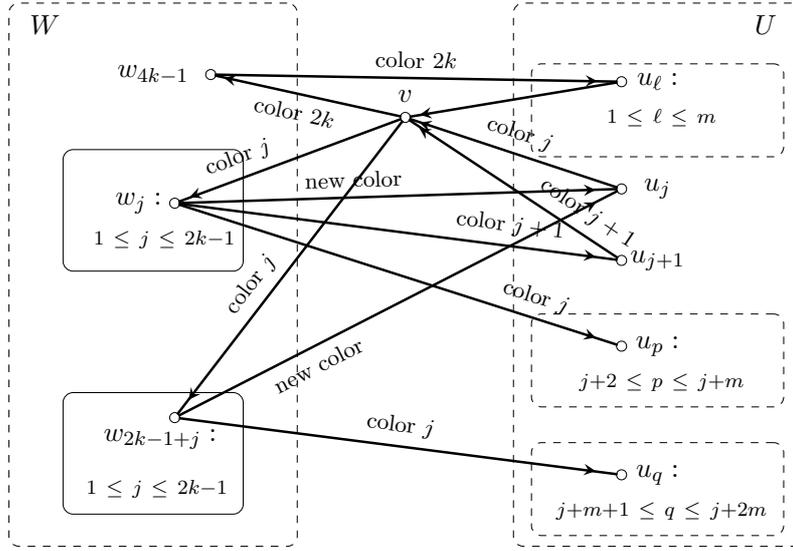
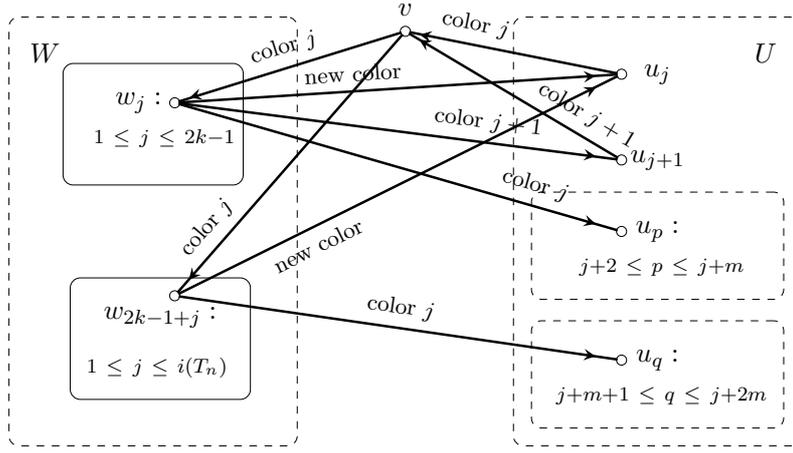


Figure 2: Two digraphs described in case 2 of Example 3.

Let T_n be a regular tournament of order 11 and $V(T_n) = \{v_0, \dots, v_{10}\}$. For each vertex $v_i \in V(T_n)$, let $N^+(v_i) = \{v_{i+1}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+9}\}$ (indices are taken modulo 11). Let $C(v_i v_{i+1}) = 1$, $C(v_i v_{i+3}) = 2$, $C(v_i v_{i+4}) = 3$, $C(v_i v_{i+5}) = 4$ and $C(v_i v_{i+9}) = 5$ for $0 \leq i \leq 10$. Then $\Delta^{-mon}(T_n) = 1 = \frac{n+1}{12}$, and there is no rainbow triangle in T_n .

But this example is too special. So we give the next example to show that for regular tournaments the best possible upper bound of $\Delta^{-mon}(T_n)$ can not be larger than $\frac{n}{11}$.

Example 5. Replace each vertex v_i of the tournament in Example 4 by a set V_i of k vertices. Let all arcs between V_i and V_j have the same directions and colors as the arc between v_i and v_j . Add arcs to each V_i to form a regular tournament H_i of order k and color all arcs in H_i with a same new color. Denote the resulting graph by D . Then D is a strongly connected regular tournament with $\Delta^{-mon}(D) = k = \frac{n}{11}$ and there is no rainbow triangle in D .

4 Concluding remarks

In Section 3, we show the gap between the bound $\frac{(n-1-i(T_n))(n+1-i(T_n))}{4(n-1+i(T_n))}$ in Theorem 2 and the best possible bound is at most 1. We wonder whether this bound can be improved or there exists some examples showing the tightness of this bound.

For the existence of rainbow triangles in arc-colored regular tournaments, we conjecture that $\Delta^{-mon}(T_n) < \frac{n}{11}$ is the best possible bound.

We also hope to make an improvement of the bound in Theorem 3 for the existence of rainbow triangles in arc-colored tournaments with $i(T_n) \neq 0$.

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