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ORIGINAL PAPER

Exponential Lower Bound for Berge-Ramsey Problems

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Abstract

We give an exponential lower bound for the smallest N such that no matter how we c-color the edges of a complete r-uniform hypergraph on N vertices, we can always find a monochromatic Berge-K_n.

Keywords Hypergraphs · Ramsey theory · Berge-graph

Gerbner and Palmer [5], generalizing the definition of hypergraph cycles due to Berge, introduced the following notion. A hypergraph H contains a *Berge copy* of a graph G, if there are injections $\Psi_1 : V(G) \to V(H)$ and $\Psi_2 : E(G) \to E(H)$ such that for every edge $uv \in E(G)$ the containment $\Psi_1(u), \Psi_1(v) \in \Psi_2(uv)$ holds, i.e., each graph edge can be mapped into a distinct hyperedge containing it to create a copy of G. If $|E(H)| = |E(G)|$, then we say that H is a *Berge-*G, and we denote such hypergraphs by $\mathcal{B}G$.

The study of Ramsey problems for such hypergraphs started independently in 2018 by three groups of authors [1, 4, 6]. Denote by $R_r(\mathcal{B}G; c)$ the size of the smallest N such that no matter how we c-color the r-edges of K_N^r, the complete r-uniform hypergraph, we can always find a monochromatic $\mathcal{B}G$. In [1] $R_r(\mathcal{B}K_n; c)$ was studied for $n = 3, 4$. In [4] it was conjectured that $R_r(\mathcal{B}K_n; c)$ is bounded by a polynomial of n (depending on r and c), and they showed that $R_r(\mathcal{B}K_n; c) = n$ if $r > 2c$ and $R_r(\mathcal{B}K_n; c) = n + 1$ if $r = 2c$, while $R_3(\mathcal{B}K_n; 2) < 2n$ (also proved in [6]). In [6] a superlinear lower bound was shown for $r = c = 3$ and for every other r for large enough c. This was improved in [3] to $R_r(\mathcal{B}K_n; c) = \Omega(n^d)$ if $c > (d-1)\binom{r}{2}$ and $R_r(\mathcal{B}K_n; c) = \Omega(n^{1+1/(r-2)}/\log n)$. We further improve these to disprove the conjecture of [4].

Theorem $R_r(\mathcal{B}K_n; c) > \left(1 + \frac{1}{r^2}\right)^{n-1}$ *if* $c > \binom{r}{2}$.

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Proof It is enough to prove the statement for $c = \binom{r}{2} + 1$. For $r = 2$ this reduces to the classical Ramsey's theorem, so we can assume $r \geq 3$. We can also suppose $n \geq \binom{r}{2} + 1 = c$, or the lower bound becomes trivial. Suppose $N \leq (1 + \frac{1}{r^2})^{n-1}$. Assign randomly (uniformly and independently) a forbidden color to every pair of vertices in K_N^r. Color the r-edges of K_N^r arbitrarily, respecting the following rule: if $\{u, v\} \subset E$, then the color of E cannot be the forbidden color of $\{u, v\}$. Since $c > \binom{r}{2}$, this leaves at least one choice for each edge. Following the classic proof of the lower bound of the Ramsey's theorem, now we calculate the probability of having a monochromatic $\mathcal{B}K_n$. The chance of a monochromatic $\mathcal{B}K_n$ on a fixed set of n vertices for a fixed color is at most $(\frac{c-1}{c})^{\binom{n}{2}}$, as the fixed color cannot be the forbidden one on any of the pairs of vertices. Thus the expected number of monochromatic $\mathcal{B}K_n$'s is at most $c\binom{N}{n}(\frac{c-1}{c})^{\binom{n}{2}}$. If this quantity is less than 1, then we know that a suitable coloring exists. Since $c \leq n \leq n!$, it is enough to show that $N < (\frac{c}{c-1})^{\frac{n-1}{2}}$, but this is true using $c = \binom{r}{2} + 1$ and $r \geq 3$. □

1 Remarks and Acknowledgment

As was brought to my attention by an anonymous referee, my construction for $r = 3$ and $c = 4$ is essentially the same as the one used in the proof of Theorem 1(ii) in [2] for a different problem, the 4-color Ramsey number of the so-called *hedgehog*. A hedgehog with body of order n is a 3-uniform hypergraph on $n + \binom{n}{2}$ vertices such that n vertices form its body, and any pair of vertices from its body are contained in exactly one hyperedge, whose third vertex is one of the other $\binom{n}{2}$ vertices, a different one for each hypderedge. It is easy to see that such a hypergraph is a Berge copy of K_n, and while their result, an exponential lower bound for the 4-color Ramsey number of the hedgehog, does not directly imply mine, their construction is such that it also avoids a monochromatic $\mathcal{B}K_n$.

It is an interesting problem to determine how $R_r(\mathcal{B}K_n; c)$ behaves if $c \leq \binom{r}{2}$. The first open case is $r = c = 3$, just like for hedgehogs.

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