ON EXPLICIT RANDOM-LIKE TOURNAMENTS

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Abstract. We give a new theorem describing a relation between the quasi-random property of regular tournaments and their spectra. This provides many solutions to a constructing problem mentioned by Erdős and Moon (1965) and Spencer (1985).

1. Introduction

A *tournament* is an oriented complete graph. *Random tournaments* $\mathcal{T}_n$ with n vertices are obtained by choosing a direction of each edge of a complete graph with n vertices with probability $1/2$, independently. We say that random tournaments *asymptotically almost surely* (*a.a.s.*) satisfy a property $\mathcal{P}$ if the probability of the event that tournaments satisfy $\mathcal{P}$ tends to 1 when n goes to infinity. In graph theory, there have been many problems focusing on deterministic tournaments satisfying properties which random tournaments a.a.s satisfy; see e.g. [1], [4], [8], [9], [19].

In this paper, as such a property, we mainly focus on the *quasi-random property* proposed by Chung-Graham [8]. Our main result is to give a new theorem describing a relation between the quasi-random property and spectra of regular tournaments. This result also provides many solutions to a problem, proposed by Erdős-Moon [14] and Spencer [31] (see also [1, Section 9.1]), on explicit constructions of tournaments with a small number of consistent edges. It is well-known that Paley tournaments have the quasi-random property (e.g. [8]). Moreover, by proving that Paley tournaments have a property stronger than the quasi-random property, Alon-Spencer [1] showed that they provide solutions to the problem by Erdős, Moon and Spencer. We note that the proof in [1] contains a part (Lemma 9.1.2 in [1]) depending on the definition of Paley tournaments. Remarkably, we generalize their discussion to all regular tournaments by using a digraph-version of the *expander-mixing lemma* proved by Vu [33].

The rest of this paper is organized as follows. In Section 2, we recap the quasi-random property and introduce some related known facts. In Section 3, we introduce our main result and give its proof. In Section 4, we

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provide some examples of regular tournaments satisfying the quasi-random property which are also solutions to the problem by Erdős, Moon and Spencer. At last, in Section 5, we discuss another random-like property defined as an adjacency property.

2. The quasi-random property and related facts

In this section, we review the quasi-random property and some related known facts. For a digraph D, let $V(D)$ and $E(D)$ be the vertex and the edge set of D, respectively. For two distinct vertices x and y, let the ordered pair (x, y) denote the edge directed from x to y.

First, we give the definition of the quasi-random property of tournaments which was formulated by Chung-Graham [8].

Definition 2.1 (The quasi-random property, [8]). Let T be a tournament with n vertices. Let σ be a bijection from $V(T)$ to $\{1, 2, \ldots, n\}$. An edge (x, y) of T is called *consistent* with σ if $\sigma(x) < \sigma(y)$. Let $C(T, \sigma)$ be the number of consistent edges with σ and $C(T) = \max_\sigma C(T, \sigma)$. Then, T has the *quasi-random property* if T satisfies

$$C(T) \leq (1 + o(1))\frac{n^2}{4}. \tag{2.1}$$

Surprisingly, Chung-Graham [8] gave some other properties which are seemingly unrelated, but actually equivalent with (2.1). The interested reader is referred to [8].

Consistent edges of tournaments was originally investigated by Erdős-Moon [14]. Their work was from paired comparisons (e.g. [18]). It is reasonable to find suitable rankings, that is, bijections with many consistent edges. First observe that for every tournament T with n vertices,

$$\frac{1}{2}\binom{n}{2} \leq C(T) \leq \binom{n}{2}. \tag{2.2}$$

The lower bound of $C(T)$ is obtained by the following simple fact:

$$C(T, \sigma) + C(T, \sigma') = \binom{n}{2}, \tag{2.3}$$

where σ' is the reversed ranking of σ which is defined as $\sigma'(v) = n+1-\sigma(v)$ for each $v \in V(T)$. For the upper bound of $C(T)$, the equality holds if and only if T is a transitive tournament. On the other hand, it is non-trivial to check the tightness of the lower bound of $C(T)$. In [14], it was proved that there exist tournaments T such that $C(T) \leq (1 + o(1))\binom{n}{2}/2$ by a probabilistic argument. Moreover Spencer [29], [30] and de la Vega [11] proved that random tournament $\mathcal{T}_n$ a.a.s satisfies the following property which is stronger than the quasi-random property:

$$C(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2} + O(n^{\frac{3}{2}}). \tag{2.4}$$

Erdős-Moon [14] and Spencer [31] mentioned the problem on explicit constructions of tournaments T such that $C(T)$ is close to the lower bound. At present, such a construction of tournaments T giving the best known "constructive" upper bound of $C(T)$ is obtained by Alon-Spencer [1]. For a prime $p \equiv 3 \pmod 4$, the *Paley tournament* T_p is the tournament with vertex set $\mathbb{F}_p$, the finite field of p elements, and edge set formed by all edges (x, y) such that $x - y$ is a non-zero square of $\mathbb{F}_p$. In [1, Theorem 9.1.1], it was proved that

$$C(T_p) \leq \frac{1}{2}\binom{p}{2} + O(p^{\frac{3}{2}} \log p). \tag{2.5}$$

In Section 4, by applying the main theorem proved in the next section, we give some new explicit constructions of regular tournaments T with n vertices such that $C(T)$ is close to the lower bound.

3. Main theorem

In this section, we prove our main theorem. We first give the definition of regular digraphs and the adjacency matrix of a digraph. A digraph is said to be *d-regular* if in-degree and out-degree of each vertex is d. Especially a tournament with n vertices is simply said to be *regular* if it is $(n-1)/2$-regular. The *adjacency matrix* M_D of a digraph D with vertices is the $\{0, 1\}$-square matrix of size n whose rows and columns are indexed by the vertices of D and the (x, y)-entry is equal to 1 if and only if $(x, y) \in E(D)$.

The following is our main theorem.

Theorem 3.1. *Let T be a regular tournament with n vertices. Suppose that the adjacency matrix M_T of T has eigenvalues such that $(n-1)/2 = \lambda_1, \lambda_2, \cdots, \lambda_n$. Let $\lambda(T) = \max_{2 \leq i \leq n} |\lambda_i|$. Then,*

$$C(T) \leq \frac{1}{2}\binom{n}{2} + \lambda(T) \cdot n \log_2(2n). \tag{3.1}$$

Remark 3.2. Theorem 3.1 implies that every regular tournament T with n vertices such that $\lambda(T) = o(n/\log n)$ has the quasi-random property. It should be remarked that Kalyanasundaram-Shapira [19] shows a stronger result; a proof of Lemma 2.3 and the first concluding remark in [19] implies that a regular tournament T with n vertices has the quasi-random property if and only if T satisfies that $\lambda(T) = o(n)$. (In [19], the authors considered the eigenvalues of the $\{0, \pm 1\}$-matrix $2M_T - J_n + I_n$, but these eigenvalues can be directly computed from ones of M_T.)

On the other hand, Theorem 3.1 not only gives a spectral condition for the quasi-random property, but also implies that estimating eigenvalues of M_T provides better upper bounds of $C(T)$ than the bound (2.1). Thus, considering (2.4), Theorem 3.1 provides a spectral condition for a property, which random tournaments a.a.s. satisfy, stronger than the quasi-random property; for example, if T satisfies $\lambda(T) = o(n/\log n)$, then Theorem 3.1

implies that $C(T) \leq \binom{n}{2}/2 + o(n^2)$, which immediately implies the quasi-random property.

In the proof of Theorem 3.1, we use the *expander-mixing lemma* for normal regular digraphs proved by Vu [33]. A digraph D is said to be *normal* if M_D and its transpose M_D^t are commutative. In other word, D is normal if $|N^+(x,y)| = |N^-(x,y)|$ for any two distinct vertices x and y where $N^+(x,y)$ (resp. $N^-(x,y)$) is the set of vertices z such that $(x,z),(y,z) \in E(D)$ (resp. $(z,x),(z,y) \in E(D)$).

Now we are ready to introduce the expander-mixing lemma for normal regular digraphs.

Lemma 3.3 (Expander-mixing lemma, [33]). *Let D be a normal d-regular digraph with n vertices and $\lambda(D) = \max_{2\leq i\leq n} |\lambda_i|$. For two disjoint subsets $A, B \subset V(D)$, let*

$$e(A,B) := \left|\{(a,b) \in E(D) \mid a \in A,\ b \in B\}\right|.$$

Then for every pair of two disjoint subsets $A, B \subset V(D)$, it holds that

$$\left|e(A,B) - \frac{d}{n} \cdot |A| \cdot |B|\right| \leq \lambda(D)\sqrt{|A| \cdot |B|}. \tag{3.2}$$

From this lemma, we can easily obtain the following corollary.

Corollary 3.4. *Let D be a normal d-regular digraph with n vertices. Then for every pair of two disjoint subsets $A, B \subset V(D)$,*

$$|e(A,B) - e(B,A)| \leq 2\lambda(D)\sqrt{|A| \cdot |B|}. \tag{3.3}$$

Proof. From the triangle inequality, we see that

$$\begin{aligned} |e(A,B) - e(B,A)| &= \left|\left(e(A,B) - \frac{d}{n} \cdot |A| \cdot |B|\right) - \left(e(B,A) - \frac{d}{n} \cdot |B| \cdot |A|\right)\right| \\ &\leq \left|e(A,B) - \frac{d}{n} \cdot |A| \cdot |B|\right| + \left|e(B,A) - \frac{d}{n} \cdot |B| \cdot |A|\right|. \end{aligned}$$

Thus, by Lemma 3.3, we get the corollary. □

By Corollary 3.4, we get the following lemma.

Lemma 3.5. *Let T be a regular tournament with n vertices and let σ be a bijection from $V(T)$ to $\{1, 2, \dots, n\}$. Then*

$$C(T,\sigma) - C(T,\sigma') \leq 2\lambda(T) \cdot n \log_2(2n). \tag{3.4}$$

Proof of Lemma 3.5. The lemma follows by combining Corollary 3.4 and the argument in [1, pp.150-151] to prove the bound (2.5) for Paley tournaments. It should be noted (see also [6]) that every regular tournament T with n vertices is normal since it holds that $M_T^t = J_n - I_n - M_T$, where I_n and J_n are the identity matrix and the all-one matrix of order n, respectively.

Fix a bijection σ. Let r be the smallest integer such that $n \leq 2^r$. Let $n = a_1 + a_2$, where a_1 and a_2 are positive integers with $a_1, a_2 \leq 2^{r-1}$. Consider a partition of $V(T)$, say A_1 and A_2, such that A_1 is the set of "highly ranked" a_1 vertices in σ and A_2 is the remaining a_2 vertices. It follows from Corollary 3.4 that

$$e(A_1, A_2) - e(A_2, A_1) \leq 2\lambda(T)\sqrt{a_1 a_2} \leq 2\lambda(T) \cdot 2^{r-1}. \tag{3.5}$$

Next, let $a_1 = a_{11} + a_{12}$, where a_{11} and a_{12} are positive integers with $a_{11}, a_{12} \leq 2^{r-2}$, and similarly for $a_2 = a_{21} + a_{22}$. As above, divide A_1 into two subsets, say A_{11} and A_{12}, where A_{11} is the set of "highly ranked" a_{11} vertices of A_1 in σ and A_{12} is the remaining a_{12} vertices of A_1. For a_{21} and a_{22}, two subsets A_{21} and A_{22} of A_2 are defined in the same way as A_{11}, A_{12}. It then follows from Corollary 3.4 that

$$\begin{aligned}
&e(A_{11}, A_{12}) - e(A_{12}, A_{11}) + e(A_{21}, A_{22}) - e(A_{22}, A_{21}) \\
&\leq 2\lambda(T)\sqrt{a_{11}a_{12}} + 2\lambda(T)\sqrt{a_{21}a_{22}} \\
&\leq 2 \cdot 2\lambda(T) \cdot 2^{r-2}.
\end{aligned}$$

Then iterate such estimation from the first to the r-th step. In the i-th step, $V(T)$ is partitioned into 2^i subsets, say $A_{\varepsilon 1}$ and $A_{\varepsilon 2}$ ($\varepsilon \in \{1,2\}^i$), such that each $A_{\varepsilon j}$ ($j = 1, 2$) contains at most 2^{r-i} vertices which are consecutive in σ. It follows from Corollary 3.4 that

$$\sum_{\varepsilon \in \{1,2\}^{i-1}} \{e(A_{\varepsilon 1}, A_{\varepsilon 2}) - e(A_{\varepsilon 2}, A_{\varepsilon 1})\} \leq 2^{i-1} \cdot 2\lambda(T) \cdot 2^{r-i} = 2\lambda(T) \cdot 2^{r-1}. \tag{3.6}$$

On the other hand, it turns out from the construction of partitions that

$$\sum_{1 \leq i \leq r} \sum_{\varepsilon \in \{1,2\}^{i-1}} \{e(A_{\varepsilon 1}, A_{\varepsilon 2}) - e(A_{\varepsilon 2}, A_{\varepsilon 1})\} = C(T, \sigma) - C(T, \sigma'). \tag{3.7}$$

Thus by combining (3.6) and (3.7), it follows that

$$C(T, \sigma) - C(T, \sigma') \leq r \cdot 2\lambda(T) \cdot 2^{r-1} \leq 2\lambda(T) \cdot n \log_2(2n).$$

□

Proof of Theorem 3.1. The theorem is a direct consequence of the equality (2.3) and Lemma 3.5. □

Remark 3.6. It should be noted that for every regular tournament T with n vertices, $\lambda(T) \cdot n \log_2(2n)$ cannot be less than $\sqrt{n^3 + n}\log_2(2n)/2$. In fact, for every such tournament T, it holds that

$$\lambda(T) \geq \frac{\sqrt{n+1}}{2}. \tag{3.8}$$

Indeed, for every strongly-connected normal d-regular digraph D with n vertices, it holds that

$$nd = E(D) = Tr(M_D M_D^t) = \sum_{i=1}^{n} |\lambda_i|^2 \leq d^2 + (n-1)\lambda(D)^2,$$

which follows from the hand shaking lemma and the Perron-Frobenius theorem (see e.g. [21]). The idea of the above inequality can be found in [20, p.217]. Also note that every regular tournament T is strongly connected, which follows from the Perron-Frobenius theorem and facts that T is normal and every eigenvalue of M_T corresponding to eigenvectors distinct to the all-one vector has the real part equal to $-1/2$ (see also [5]).

4. Examples of quasi-random regular tournaments

In this section, we give some examples of regular tournaments T with n vertices and $\lambda(T) = o(n/\log n)$. As will be shown below, we can construct such tournaments for almost all positive integers n.

First we consider the following tournaments constructed from finite fields which are variants of cyclotomic tournaments (see e.g. [24] and reference therein). Let m be a positive even integer and $p \equiv m+1 \pmod{2m}$ be a prime. Note that there exist infinitely many such primes by the Dirichlet's theorem on arithmetic progressions and the fact that $m+1$ and $2m$ are coprime when m is even. Recall that $\mathbb{F}_p$ is the finite field of order p. Let g be a primitive element of $\mathbb{F}_p$. For even m, the multiplicative group of $\mathbb{F}_p$, which is denoted by $\mathbb{F}_p^*$, is divided into m cosets $S_0, S_1, \dots, S_{m-1}$ where $S_i := \{g^t \mid t \equiv i \pmod{m}\}$ for each $0 \le i \le m-1$. Note that $S_j = -S_i$ if $j \equiv -i \pmod{m}$.

Definition 4.1. Let $\boldsymbol{i} = (i_1, i_2, \dots, i_{m/2}) \in \{0, 1, \dots, m-1\}^{m/2}$ such that $S_{\boldsymbol{i}} = S_{i_1} \cup \cdots \cup S_{i_{m/2}}$ and $\mathbb{F}_p^* \setminus S = -S$. Then the tournament $T_p^m(S_{\boldsymbol{i}})$ is defined as follows:

$$\begin{aligned} V(T_p^m(S_{\boldsymbol{i}})) &= \mathbb{F}_p, \\ E(T_p^m(S_{\boldsymbol{i}})) &= \{(x,y) \in \mathbb{F}_p^2 \mid x - y \in S_{\boldsymbol{i}}\}. \end{aligned} \tag{4.1}$$

This is a direct generalization of Paley tournament since $T_p^m(S_{\boldsymbol{i}})$ is exactly T_p in the case of $m = 2$. Moreover from the definition, it is not so hard to see that $T_p^m(S_{\boldsymbol{i}})$ is a regular tournament with p vertices.

Now we obtain the following corollary.

Corollary 4.2.

$$C\big(T_p^m(S_{\boldsymbol{i}})\big) \le \frac{1}{2}\binom{p}{2} + O(p^{\frac{3}{2}} \log p). \tag{4.2}$$

Corollary 4.2 is proved by combining Lemma 3.5 and the following evaluation of $\lambda(T_p^m(S_{\boldsymbol{i}}))$.

Lemma 4.3.

$$\lambda\big(T_p^m(S_{\boldsymbol{i}})\big) \le \frac{m\sqrt{p}}{2}. \tag{4.3}$$

Proof. First, by a simple calculation, it can be shown that the set of eigenvalue of $M_{T_p^m(S_{\boldsymbol{i}})}$ is

$$\Big\{ \sum_{s \in S_{\boldsymbol{i}}} \psi(s) \mid \psi \text{ is an additive character of } \mathbb{F}_p \Big\}.$$

Since $S_i = g^i S_0$ for each $1 \leq i \leq m-1$, we see that

$$\sum_{s \in S_i} \psi(s) = \sum_{s \in g^i S_0} \psi(s) = \sum_{s \in S_0} \psi(g^i s). \tag{4.4}$$

Since S_0 is the set of non-zero m-th power elements and each non-zero m-th power residue appears exactly m times in the sequence $(x^m)_{x \in \mathbb{F}_p^*}$,

$$\sum_{s \in S_0} \psi(g^i s) = \frac{1}{m} \sum_{x \in \mathbb{F}_p^*} \psi(g^i x^m). \tag{4.5}$$

At last, we use the following known estimation (see e.g. [26, p.44]);

$$\Big| \sum_{x \in \mathbb{F}_p} \psi(a x^m) \Big| \leq (m-1)\sqrt{p}, \tag{4.6}$$

for any non-trivial additive character ψ and $a \neq 0$. By combining (4.4), (4.5) and (4.6),

$$\lambda\big(T_p^m(S_{\boldsymbol{i}})\big) \leq \frac{m}{2} \cdot \frac{1}{m} \cdot \{(m-1)\sqrt{p} + 1\} = \frac{(m-1)\sqrt{p}+1}{2} \leq \frac{m\sqrt{p}}{2}.$$

□

The second example is doubly regular tournament which has been extensively studied in algebraic combinatorics and related areas (e.g. [23]).

Definition 4.4. A tournament T with n vertices is called a *doubly regular tournament* if T is a regular tournament such that for any distinct two vertices x and y, $N^+(x,y) = N^-(x,y) = (n-3)/4$.

Let DRT_n denote a doubly regular tournament with n vertices.

Corollary 4.5.

$$C(DRT_n) \leq \frac{1}{2}\binom{n}{2} + O(n^{\frac{3}{2}} \log n). \tag{4.7}$$

Corollary 4.5 is proved by the following well-known evaluation of $\lambda(DRT_n)$ which also shows that the inequality (3.8) is tight.

Lemma 4.6 (e.g. [10])**.**

$$\lambda(DRT_n) = \frac{\sqrt{n+1}}{2}. \tag{4.8}$$

Proof. We give a proof for the reader's convenience. Let $M = M_{DRT_n}$. Then by the definition, it holds that

$$MM^t = \frac{n+1}{4} I_n + \frac{n-3}{4} J_n. \tag{4.9}$$

Since $M + M^t = J_n - I_n$, we obtain the following equality.

$$M^2 + M + \frac{n+1}{4} I_n - \frac{n+1}{4} J_n = O. \tag{4.10}$$

Since DRT_n is regular, we see that $(n-1)/2$ is an eigenvalue of M and a corresponding eigenvector is the all-one eigenvector $\mathbf{1}$. Since DRT_n is normal, each eigenvalue θ except for $(n-1)/2$ has an eigenvector $\boldsymbol{v}$ which is orthogonal to $\mathbf{1}$. Thus,

$$\Big(\theta^2+\theta+\frac{n+1}{4}\Big)\boldsymbol{v}=\mathbf{0}. \tag{4.11}$$

Since $\boldsymbol{v}\neq\mathbf{0}$, we get

$$\Big(\theta^2+\theta+\frac{n+1}{4}\Big)=0, \tag{4.12}$$

completing the proof. □

Remark 4.7. We remark that Corollary 4.5 is a generalization of the bound (2.5) because Paley tournaments are also doubly-regular tournaments. For other non-isomorphic examples of doubly regular tournaments, see e.g. [17] and [32]. As shown in, for example, [16] and [23], there are some known constructions of doubly regular tournaments such that the number of vertices is non-prime (and non-prime power). Especially, constructions of complex codebooks in [16] provide DRT_n for every integer n such that each prime factor f of n is the form of $f\equiv 3 \pmod 4$.

Remark 4.8. By the definition of DRT_n, n must be a positive integer of the form $n\equiv 3 \pmod 4$. On the other hand, as an analogue of DRT_n for integers n of the form $n\equiv 1 \pmod 4$, Savchenko [24] introduced the notion of a nearly-doubly-regular tournament $CNDR_n$ with n vertices which is a certain regular tournament with exactly four eigenvalues distinct to $(n-1)/2$ with multiplicity $(n-1)/4$. According to [24], it holds that $\lambda(CNDR_n)=(\sqrt{n}+1)/2$. Thus if there exists $CNDR_n$ for infinitely many $n\equiv 1 \pmod 4$, then it holds that

$$C(CNDR_n)\leq\frac{1}{2}\binom{n}{2}+O(n^{\frac{3}{2}}\log n).$$

It is conjectured in [24] (see also [25]) that there exists a $CNDR_n$ for every $n\equiv 1 \pmod 4$. Interestingly, Savchenko [24] also found examples of $CNDR_p$ for primes $p=5,13,29,53,173,229,293$ and 733 from the class of $T_p^4(S_{(0,1)})$ in the first example, and thus Lemma 4.3 can be improved for these examples. (It is shown in [24] that for every prime $p\equiv 5 \pmod 8$, $T_p^4(S_{(0,1)})$ has exactly four eigenvalues distinct to $(p-1)/2$ with multiplicity $(p-1)/4$.) It would be interesting to prove or disprove the existence of infinitely many primes $p\equiv 5 \pmod 8$ such that the tournament $T_p^4(S_{(0,1)})$ is in the class of $CNDR_p$.

The third example is based on a construction of pseudo-random graphs due to Shparlinski [27]. For related facts on eliptic curves, see [27, Section 2.1]. For a prime p, let $n\in[p+1-2\sqrt{p},p+1+2\sqrt{p}]$ be an odd integer. It is known (e.g. [7], [12]) that there exists an eliptic curve E over $\mathbb{F}_p$ such that the number of $\mathbb{F}_p$-rational points of E is n. It is also known (e.g. [28])

that all $\mathbb{F}_p$-rational points of E form an abelian group G of order n under an operation $\oplus$. Let 0_G be the identity of G. For an element $s \in G$ and a subset $S \subset G$, the inverse of s is denoted by $\ominus s$ and let $\ominus S = \{\ominus s \mid s \in S\}$.

Definition 4.9. Let $S \subset G$ be a subset such that $S \cup \ominus S \cup \{0_G\} = G$ and $|S| = (n-1)/2$. Then the tournament $T_{p,n}(S)$ is defined as follows.

$$\begin{aligned} V(T_{p,n}(S)) &= G, \\ E(T_{p,n}(S)) &= \{(x,y) \in G^2 \mid x \ominus y \in S\}. \end{aligned} \tag{4.13}$$

By the definition, $T_{p,n}(S)$ is a regular tournament with n vertices.

Corollary 4.10. *There exists a subset $S \subset G$ such that*

$$C(T_{p,n}(S)) \leq \frac{1}{2}\binom{n}{2} + O(n^{\frac{3}{2}} \log^2 n). \tag{4.14}$$

Corollary 4.10 is obtained by Lemma 3.5 and the following evaluation of $\lambda(T_{p,n}(S))$ which follows from [27, Theorem 1].

Lemma 4.11 ([27])**.** *There exists a subset $S \subset G$ such that*

$$\lambda(T_{p,n}(S)) = O(\sqrt{n} \log n). \tag{4.15}$$

For the details of a construction of such a subset S, see [27].

Remark 4.12. It is worth noting that as shown in [27], almost all positive integers are in the interval $[p+1-2\sqrt{p}, p+1+2\sqrt{p}]$ for some prime p. Indeed, it holds ([27]) that

$$\lim_{N\to\infty} \frac{|\{n \leq N \mid \exists \text{ prime } p \text{ s.t. } n \text{ is odd and } n \in [p+1-2\sqrt{p}, p+1+2\sqrt{p}]\}|}{\lceil \frac{N}{2} \rceil} = 1.$$

Thus the third example provides regular tournaments T with n vertices and small $\lambda(T)$ for almost all positive integers n.

5. Shütte's problem for tournaments

At last, in this section, we focus on another random-like property.

Definition 5.1. Let k be a positive integer. A tournament T has the property S_k if for every $A \subset V(T)$ of size k, there exists a vertex $z \notin A$ directing to all members of A.

The *Shütte's problem* asks the existence of tournaments satisfying this property (see [13] and [22]). As shown by Erdős [13], random tournaments a.a.s. satisfy S_k for any $k \geq 1$. On the other hand, the problem of explicit constructions has been considered in graph theory. For example, Graham-Spencer [15] showed that the Paley tournament T_p satisfies S_k if $p > k^2 2^{2k-2}$ for each $k \geq 1$. From the digraphs constructed in [3], we can also construct tournaments satisfying S_k for every k by adding some edges. At present, there seems to be almost no explicit constructions of tournaments satisfying both of the quasi-random property and S_k except for Paley tournaments. The following proposition and Corollary 4.2 show that the tournament $T_p^m(S_{\boldsymbol{i}})$ has the quasi-random property and S_k.

Proposition 5.2. *Let m be an even positive integer. Then for every $k \geq 1$, there exists a prime $p_m(k)$ such that for every prime $p > p_m(k)$, the tournament $T_p^m(S_{\boldsymbol{i}})$ has the property S_k.*

Proposition 5.2 is proved by a direct generalization of the discussion in [15] and [2], so we omit the proof here. Moreover, it is not so hard to prove that $T_p^m(S_{\boldsymbol{i}})$ has the existentially closed property (see e.g. [4]).

We also note that doubly regular tournaments constructed in [32] satisfy both of the quasi-random property and S_2, which follows from Corollary 4.5 and the corollary in [32, p.277].

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